Math 403 Homework #4, Spring 2024

Instructor: Ezra Miller

Solutions by: ...your name...

Collaborators: ...list those with whom you worked on this assignment...
(1 point for each of up to 3 collaborators who also list you) /3

Due: 11:59pm Saturday 23 March 2024

Reading assignments /48

16–17. for Tue. 5 March and Thu. 7 March
- [Tapp, Chap.5] Lie algebras as tangent spaces to the identity
- [Lax, Chap.9: Matrix-valued Functions, through §Matrix Exponential]
- [Wikipedia, *Exponential map (Lie theory)*]

18. for Tue. 19 March
- [Lax, Chapter 16, p.237–240] entrywise positive matrices, Perron's theorem

19. for Thu. 21 March
- [Lax, Chapter 16, p.240–245] stochastic and nonnegative matrices, Frobenius thm

20. for Tue. 26 March
- [Treil, §8.5] multilinear algebra, tensor product
- [Wikipedia, *Tensor product*]

21–22. for Thu. 28 March and Tue. 2 April
- [Wikipedia, *Exterior algebra*]

Exercises

1. Show that $\|AB\|_2 \le \|A\|_2\|B\|_2$ whenever the product AB is defined. /3
2. Show that $\lim_{k\to\infty} A^k = 0$ if and only if its spectral radius satisfies $\rho(A) < 1$. /3
3. Show that if ν is a consistent norm on $\mathbb{C}^{n\times n}$, then $\lim_{k\to\infty} \nu(A^k)^{\frac{1}{k}} = \rho(A)$. /3
4. Prove or disprove, and salvage the statement as best you can in case you find it's false: /3
If $\|(\tilde\lambda I - A)^{-1}\| \ge \eta$ then there is an eigenvalue λ of $A \in \mathbb{C}^{n\times n}$ satisfying
$$|\tilde\lambda - \lambda| \le 2(\|A\| + \eta^{-1})\eta^{-\frac{1}{n}}.$$
5. For $A \in \mathbb{C}^{m\times n}$, set $\|A\|_\infty = \max\limits_{\|x\|_\infty=1} \|Ax\|_\infty$. Prove that $\|A\|_\infty = \max\limits_i \sum\limits_j |a_{ij}|$. /3

6. Assume that the union of m out of the n Gerschgorin disks $\mathcal{G}_i$ is disjoint from the other $n-m$ disks. Prove that the union of m disks contains precisely m of the eigenvalues. Hint: how do the eigenvalues move as A proceeds along a straight line to $\widetilde{A}$? /3

7. Let $G(A)$ be the union of the Gerschgorin disks of A. Show that the intersection $\bigcap_S G(S^{-1}AS)$ over all nonsingular matrices S equals the spectrum of A. /3

8. Prove that the spectrum of A is contained in $G(A) \cap G(A^\top)$. Illustrate with the 3×3 matrix with entries $a_{ij} = i/j$. /3

9. Fix the matrix

$$A = \begin{bmatrix} 7 & -16 & 8 \\ -16 & 7 & -8 \\ 8 & -8 & -5 \end{bmatrix}.$$

   (a) Use Gerschgorin's Theorem to say as much as you can about the locations of the eigenvalues of and the spectral radius of A. /3

   (b) Consider $D^{-1}AD$ for a diagonal matrix D to see if you can improve your estimates for the eigenvalue locations. /3

   (c) Compute the actual eigenvalues and comment on the quality of your estimates in (a) and (b). /3

10. Find the Lie algebra $\mathfrak{so}_n$ of the special orthogonal group $SO_n(\mathbb{R})$. /3

11. Fix a matrix group $G \subseteq M_n F$ over the field $F \in \{\mathbb{R}, \mathbb{C}\}$ and a matrix $A \in G$. Show that the tangent space $T_A(G)$ of G at A is $A\mathfrak{g}$, and show that this equals $\mathfrak{g}A$. /3

12. Prove that $\det(e^A) = e^{\mathrm{tr}(A)}$. /3

13. Explain why $\mathfrak{su}_n = \mathfrak{u}_n \cap \mathfrak{sl}_n(\mathbb{C})$. (You may use the previous exercise.) /3

14. Let $\gamma : (-\varepsilon, \varepsilon) \to GL_n(F)$ be a differentiable path, where $F \in \{\mathbb{R}, \mathbb{C}\}$. Use Cramer's rule (or some other method) to show that the inverse path $t \mapsto \gamma(t)^{-1}$ is differentiable. /3