Math 403 Homework #4, Spring 2024
Instructor: Ezra Miller

Solutions by: ...your name...
Collaborators: ...list those with whom you worked on this assignment...
(1 point for each of up to 3 collaborators who also list you)

Due: 11:59pm Saturday 23 March 2024

Reading assignments

16–17. for Tue. 5 March and Thu. 7 March
• [Tapp, Chap.5] Lie algebras as tangent spaces to the identity
• [Lax, Chap.9: Matrix-valued Functions, through §Matrix Exponential]
• [Wikipedia, Exponential map (Lie theory)]

18. for Tue. 19 March
• [Lax, Chapter 16, p.237–240] entrywise positive matrices, Perron’s theorem

19. for Thu. 21 March
• [Lax, Chapter 16, p.240–245] stochastic and nonnegative matrices, Frobenius thm

20. for Tue. 26 March
• [Treil, §8.5] multilinear algebra, tensor product
• [Wikipedia, Tensor product]

21–22. for Thu. 28 March and Tue. 2 April
• [Wikipedia, Exterior algebra]

Exercises

1. Show that \( \|AB\|_2 \leq \|A\|_2 \|B\|_2 \) whenever the product \( AB \) is defined. /3

2. Show that \( \lim_{k \to \infty} A^k = 0 \) if and only if its spectral radius satisfies \( \rho(A) < 1 \). /3

3. Show that if \( \nu \) is a consistent norm on \( \mathbb{C}^{n \times n} \), then \( \lim_{k \to \infty} \nu(A^k)^\frac{1}{k} = \rho(A) \). /3

4. Prove or disprove, and salvage the statement as best you can in case you find it’s false: /3

   If \( \| (\tilde{\lambda} I - A)^{-1} \| \geq \eta \) then there is an eigenvalue \( \lambda \) of \( A \in \mathbb{C}^{n \times n} \) satisfying

   \[
   |\tilde{\lambda} - \lambda| \leq 2(\|A\| + \eta^{-1})\eta^{-\frac{1}{n}}.
   \]

5. For \( A \in \mathbb{C}^{m \times n} \), set \( \|A\|_{\infty} = \max_{\|x\|_{\infty} = 1} \|Ax\|_{\infty} \). Prove that \( \|A\|_{\infty} = \max \sum_{j} |a_{ij}| \). /3

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6. Assume that the union of \( m \) out of the \( n \) Gerschgorin disks \( G_i \) is disjoint from the other \( n - m \) disks. Prove that the union of \( m \) disks contains precisely \( m \) of the eigenvalues. Hint: how do the eigenvalues move as \( A \) proceeds along a straight line to \( A' \)?

7. Let \( G(A) \) be the union of the Gerschgorin disks of \( A \). Show that the intersection \( \bigcap_S G(S^{-1}AS) \) over all nonsingular matrices \( S \) equals the spectrum of \( A \).

8. Prove that the spectrum of \( A \) is contained in \( G(A) \cap G(A^\top) \). Illustrate with the \( 3 \times 3 \) matrix with entries \( a_{ij} = i/j \).

9. Fix the matrix

\[
A = \begin{bmatrix}
7 & -16 & 8 \\
-16 & 7 & -8 \\
8 & -8 & -5
\end{bmatrix}.
\]

(a) Use Gerschgorin’s Theorem to say as much as you can about the locations of the eigenvalues of and the spectral radius of \( A \).

(b) Consider \( D^{-1}AD \) for a diagonal matrix \( D \) to see if you can improve your estimates for the eigenvalue locations.

(c) Compute the actual eigenvalues and comment on the quality of your estimates in (a) and (b).

10. Find the Lie algebra \( \mathfrak{so}_n \) of the special orthogonal group \( SO_n(\mathbb{R}) \).

11. Fix a matrix group \( G \subseteq M_n(F) \) over the field \( F \in \{\mathbb{R}, \mathbb{C}\} \) and a matrix \( A \in G \). Show that the tangent space \( T_A(G) \) of \( G \) at \( A \) is \( A\mathfrak{g} \), and show that this equals \( \mathfrak{g}A \).

12. Prove that \( \det(e^A) = e^{\text{tr}(A)} \).

13. Explain why \( \mathfrak{su}_n = \mathfrak{u}_n \cap \mathfrak{sl}_n(\mathbb{C}) \). (You may use the previous exercise.)

14. Let \( \gamma : (-\varepsilon, \varepsilon) \to GL_n(F) \) be a differentiable path, where \( F \in \{\mathbb{R}, \mathbb{C}\} \). Use Cramer’s rule (or some other method) to show that the inverse path \( t \mapsto \gamma(t)^{-1} \) is differentiable.