Math 403 Homework #5, Spring 2024

Instructor: Ezra Miller

Solutions by: ...your name...

Collaborators: ...list those with whom you worked on this assignment...
(1 point for each of up to 3 collaborators who also list you) /3

Due: 11:59pm Tuesday 9 April 2024

/36

Reading assignments

20. for Tue. 26 March
    - [Treil, §8.5] multilinear algebra, tensor product
    - [Wikipedia, *Tensor product*]

21–22. for Thu. 28 March and Tue. 2 April
    - [Wikipedia, *Exterior algebra*]

Exercises

1. For $P \in \mathbb{R}^{n\times n}$ with $P > 0$, set $\Gamma(P) = \{\lambda \in \mathbb{R}_{\geq 0} \mid P\mathbf{x} \leq \lambda\mathbf{x}$ for some nonzero $\mathbf{x} \geq \mathbf{0}\}$. Show that the dominant eigenvalue $\lambda(P)$ satisfies $\lambda(P) = \min_{\lambda\in\Gamma(P)} \lambda$. /3

2. In class, we stated the theorem that for $P > 0$ a stochastic $n\times n$ matrix with dominant eigenvector $\mathbf{v}$, and $\mathbf{x} \geq 0$ any nonzero vector, $P^k\mathbf{x} \to \alpha\mathbf{v}$ as $k \to \infty$ for some real $\alpha > 0$. But we only proved it when P is diagonalizable. Complete the proof. /3

3. Prove that P has a dominant positive eigenvalue if $P \geq 0$ and $P^k > 0$ for some $k > 0$. /3

4. Prove or disprove: the set of stochastic $n \times n$ matrices is compact and convex. /3

5. Use the universal property of tensor products to prove commutativity: there is a unique isomorphism $V \otimes W \to W \otimes V$ such that $v \otimes w \mapsto w \otimes v$ for all $v \in V$ and $w \in W$. /3

6. Use the universal property of tensor products to prove associativity: there is a unique isomorphism $(U \otimes V) \otimes W \to U \otimes (V \otimes W)$ such that $(u \otimes v) \otimes w \mapsto u \otimes (v \otimes w)$ for all $u \in U$, $v \in V$, and $w \in W$. Hint: You can either use the universal property to produce the map or check that the two parenthesizations have the same universal property regarding bilinear maps on $(U \times V) \times W = U \times (V \times W)$ and appeal to "abstract nonsense": universal constructions are unique up to unique isomorphism. /3

7. Prove that homomorphisms $\varphi : V \to V'$ and $\psi : W \to W'$ result in a canonical homomorphism $\varphi\otimes\psi : V \otimes W \to V' \otimes W'$. Given matrices for φ and ψ, write down a matrix for $\varphi\otimes\psi$. Note: your answer will depend on how you order the basis of $V \otimes W$. /3

8. Prove that a homomorphism $\varphi : V \to W$ results in a canonical homomorphism $\bigwedge^r \varphi : \bigwedge^r V \to \bigwedge^r W$. Given a matrix for φ, write down a matrix for $\bigwedge^r \varphi$. Note: make no attempt to draw a matrix; just describe its entries as labeled by pairs of basis vectors. /3

9. Prove that tensor products commute with direct sums: if I is any (finite or infinite) index set and $V = \bigoplus_{i \in I} V_i$, then there is a natural isomorphism $V \otimes W \to \bigoplus_{i \in I} V_i \otimes W$. /3

10. Construct a natural map $V^* \otimes W^* \to (V \otimes W)^*$. Show that it is injective. If one of V and W has finite dimension, show that the map is an isomorphism. /3

11. Prove the existence of a bilinear map $\bigwedge^r V \times \bigwedge^s V \to \bigwedge^{r+s} V$ taking /3

$$(v_1 \wedge \cdots \wedge v_r, v_1' \wedge \cdots \wedge v_s') \mapsto v_1 \wedge \cdots \wedge v_r \wedge v_1' \wedge \cdots \wedge v_s'.$$

Write $\omega = v_1 \wedge \cdots \wedge v_r$ and $\omega' = v_1' \wedge \cdots \wedge v_r'$, so $\omega \wedge \omega' = v_1 \wedge \cdots \wedge v_r \wedge v_1' \wedge \cdots \wedge v_s'$. Show that $\omega' \wedge \omega = (-1)^{rs} \omega \wedge \omega'$.

12. Show how to recover the atlas for $G_k(F^n)$ in Lecture 9 (lecture notes p.19) from the Plücker coordinates in Lecture 22 (lecture notes, p.46). /3