Free modules/PI

**Def:** The rank of a free module $F$ over a nonzero commutative ring $R$ is $\text{rank } F = \text{basis of } F$.

**Lemma:** does not depend on basis.

**Pf:** Suppose $F \cong \bigoplus_{s \in S} R$. Let $\mathfrak{p} \subseteq R$ be maximal.

$$F/\mathfrak{p}F \cong \bigoplus_{s \in S} R/\mathfrak{p}$$

is a vector space over $R/\mathfrak{p}$ of dim $|S|$. □

**Thm:** Fix $F$ free/PI over $\text{PID } R$ and a submodule $M \subseteq F$. Then $M$ is free of rank $\leq \text{rank } F$.

**Pf:** $F \cong \bigoplus_{x \in \Lambda} Rx$, for a basis $\{x_{\lambda}\}_{\lambda \in \Lambda}$.

$J \subseteq \Lambda \Rightarrow M_J \overset{\text{def}}{=} M \cap \bigoplus_{i \in J} Rx_i$ has the form

$\bullet$ and $M_J \cong \bigoplus_{i \in J} Ry_i$ for some $y_i \in M_J$.

Warning: some of the $y_i$ might be 0

or not. Order the set $Y$ of

families $\{y_i\}_{i \in J}$ for which $\exists$ basis $\{x_{\lambda}\}_{\lambda \in \Lambda}$ satisfying $(\ast)$

by inclusion: $\{y_i\}_{i \in J} \subseteq \{y_i'\}_{i \in J'}$ if $J \subseteq J'$ and $y_{i'} = y_i \forall i \in J$.

If $C$ is a chain in $Y$ then $\bigcup_{c \in C} c \subseteq Y$ since any dependence relation involves only finitely many $y_i$.

Hence $\exists$ family $\{y_i\}_{i \in J}$ maximal in $Y$. Want $J = \Lambda$.

Suffices: $k \in \Lambda \setminus J \Rightarrow \ast$. Let $K = J \cup \{k\}$ and $M \hookrightarrow F \twoheadrightarrow Rx_k$.

Then $\pi_k(M_K) = \langle a \rangle x_k \subseteq Rx_k$ since $R$ is a PID.

But $\ker \pi_k|_{M_K} = M_J$, so

$$0 \to M_J \to M_K \to \pi_k(M_K) \to 0$$

is exact and splits because $\langle a \rangle x_k$ is free!

Thus $\{y_i\}_{i \in K} \in Y$ if $y_k = ax_k$. □

So $J = \Lambda$. □

**Cor:** $M$ finitely generated/PI over $\text{PID } R$ and $N \subseteq M$ submodule $\Rightarrow N$ f.g.

**Pf:** $f: R^n \to M$ $\Rightarrow f^{-1}(N)$ free of rank $\leq n$

$\Rightarrow N$ f.g. since $f^{-1}(N) \to N$. □