Exercises

1. Prove that set of invertible elements in any monoid is a group.

2. Solve for $y$ given that $xyz^{-1}w = 1$ in a group.

3. Assume that the equation $xyz = 1$ holds in a group $G$. Does it follow that $yzx = 1$? How about $yxx = 1$?

4. Fix elements $a$ and $b$ in a group $G$. Show that the equation $ax = b$ has a unique solution in $G$.

5. Determine the elements of the cyclic group generated by the matrix $egin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$ explicitly.

6. Let $a$ and $b$ be elements of a group $G$. Assume that $a$ has order 5 and that $a^3b = ba^3$. Prove that $ab = ba$.

7. Prove that a nonempty subset $H$ of a group $G$ is a subgroup if for all $x, y \in H$ the element $xy^{-1}$ lies in $H$.

8. An $n^{\text{th}}$ root of unity is a complex number $z$ such that $z^n = 1$. Prove that the $n^{\text{th}}$ roots of unity form a cyclic subgroup of $\mathbb{C}^\times$ of order $n$. More generally, show that every finite subgroup of the multiplicative group of any field is cyclic.

9. Let $H$ be the subgroup generated by two elements $a$ and $b$ of a group $G$. Prove that if $ab = ba$ then $H$ is abelian.

10. Describe all groups that contain no proper subgroup. Describe all groups that contain no proper nontrivial subgroup.

11. Let $G$ be a cyclic group of order $n$, and let $r$ be an integer dividing $n$. Prove that $G$ contains exactly one subgroup of order $r$.

12. Let $G$ be a cyclic group of order 6. How many of its elements generate $G$? How about if $G$ has order 5, 8, or 10? And the general case of order $n$?

13. Prove that a group in which every element except the identity has order 2 is abelian.

14. How many elements of order 2 does the symmetric group $S_4$ have? What about $S_5$ or $S_6$?

15. Prove that the set of elements of finite order in an abelian group is a subgroup. Find a group whose elements of finite order do not constitute a subgroup.

16. Let $M$ be a finite monoid that satisfies the cancellation law. Prove that $M$ is a group.