1. Determine the automorphism groups of the integers \( \mathbb{Z} \), the symmetric group \( S_3 \), and the cyclic group \( C_{10} \).

2. Find all subgroups of \( S_3 \) and determine which are normal.

3. Given two homomorphisms \( \varphi \) and \( \psi \) from a group \( G \) to \( G' \), let \( H \subseteq G \) be the subset where \( \varphi \) and \( \psi \) agree: \( H = \{ x \in G \mid \varphi(x) = \psi(x) \} \). Is \( H \) a subgroup of \( G' \)?

4. Prove that the center of any group is a normal subgroup.

5. If \( \varphi : G \to G' \) is a surjective homomorphism and \( N \leq G \) is a normal subgroup, prove that the image \( \varphi(N) \leq G' \) is also a normal subgroup.

6. Is the intersection \( R \cap R' \) of two equivalence relations in \( S \times S \) an equivalence relation on \( S \)? Is the union?

7. Prove that every group whose order is a power of a prime \( p \) contains an element of order \( p \).

8. Let \( F \) be a field and \( W \) the solution set in \( F^n \) of a system of homogeneous linear equations \( Ax = 0 \). Show that the solution set of any inhomogeneous system \( Ax = b \) is a coset of \( W \).

9. Prove that every index 2 subgroup is normal. Exhibit a non-normal index 3 subgroup.

10. Classify all groups of order 6. Hint: is there an element of order 6? Of order 3 but not of order 6? Or no element of order 3?

11. If \( G \) and \( G' \) are finite groups whose orders are relatively prime, prove that there is a unique homomorphism \( G \to G' \).

12. Fix subgroups \( H \) and \( K \) of a group \( G \). Prove that the intersection \( xH \cap yK \) of cosets is either empty or else is a coset of \( H \cap K \). Conclude that if \( H \) and \( K \) have finite index in \( G \) then so does \( H \cap K \).

13. Prove that a group of order 30 can have at most seven subgroups of order 5.

14. Fix a surjective group homomorphism \( \varphi : G \to G' \) with kernel \( K \). Show that the set of subgroups of \( G \) containing \( K \) and the set of all subgroups of \( G' \) are in bijection via the map \( H \mapsto \varphi(H) \). If \( H \leq G \), must it be that \( \varphi(H) \leq G' \)?

15. Is the symmetric group \( S_3 \) a direct product of nontrivial groups?
16. Prove that the product of two infinite cyclic groups is not cyclic. Is the same true without the word “infinite”?

17. Fix a group $G$ whose order is $|G| = ab$. Suppose that $G$ has subgroups $H$ and $K$ with orders $|H| = a$ and $|K| = b$. Assume that $|H \cap K| = 1$. Prove that $HK = G$. Is $G$ isomorphic to the product group $H \times K$?

18. Suppose that a group $G$ has a partition $P$ with the property that for any pair of blocks $A$ and $B$ of the partition, the product $AB$ is contained entirely within a block of $P$. Let $N$ be the block that contains the identity $e$ of $G$. Prove that $N \leq G$ and that $P$ is the partition of $G$ into the set of cosets of $N$.

19. Let $H = \{\pm 1, \pm i\} \subset \mathbb{C}^\times$, the subgroup of fourth roots of unity. Describe the cosets of $H$ in $\mathbb{C}^\times$ explicitly (geometrically), and prove that $\mathbb{C}^\times / H \cong \mathbb{C}^\times$.

20. Fix a group $G$. Let $N = \langle xyx^{-1}y^{-1} | x, y \in G \rangle$ be the subgroup of $G$ generated by the commutators of pairs of elements of $G$. Prove that $N$ is normal and the quotient $G / N$ is abelian. Moreover, show that any homomorphism $G \to G'$ to an abelian group $G'$ contains $N$ in its kernel.

21. Assume that both $H$ and $K$ are normal subgroups of a group $G$ and that $|H \cap K| = 1$. Prove that $xy = yx$ for all $x \in H$ and $y \in K$. Hint: prove that $xyx^{-1}y^{-1} \in H \cap K$.

22. Find a nonabelian group $G$ and a proper normal subgroup $N$ such that $G/N$ is abelian.