Math 627 Homework #1, Fall 2022

Instructor: Ezra Miller

Solutions by: ...your name...

Collaborators: ...list those with whom you worked on this assignment...

Due: Tuesday 13 September 2022

Reading assignments in [Vakil]

- by Tuesday 6 September: §2.1, §2.2, §2.3, §2.4
- by Thursday 8 September: §2.6
- by Tuesday 13 September: §13.1, §13.2.2, §13.3.3, §13.3.D
- by Thursday 15 September: §2.7, §2.5

Exercises: In [Vakil], exercises have labels C.S.N, for "Chapter C, Section S, Exercise N", where $\mathrm{C}, \mathrm{S} \in \mathbb{Z}_+$ and $\mathrm{N} \in \mathrm{A}, \ldots, \mathrm{Z}$. Exercises marked "[essential]" are essential.

2.2.C

2.2.G (a) [essential]

(b)

(c) Demonstrate that 2.2.F is a special case of part (a) by considering the projection $Y \times X \to X$.

2.2.I

2.3.C [essential] (Note: Most commonly, sheaf-hom is denoted using some form of calligraphy or math italics, such as $\mathcal{H}om(\mathcal{F}, \mathcal{G})$, since $\mathrm{Hom}(\mathcal{F}, \mathcal{G})$ is most often interpreted as the group of homomorphisms $\mathcal{F} \to \mathcal{G}$ between objects $\mathcal{F}$ and $\mathcal{G}$ in the category of sheaves.)

2.3.J [essential]

2.4.E [essential]

2.4.M

2.4.P [essential]

2.6.A

2.6.B

2.6.C

2.6.F [The part about the global section functor not being exact is required]

2.6.G [essential]

Non-book Exercises: These exercises give an inkling of the flexibility of sheaf theory.

1. Fix a real vector space Q of finite dimension and a cone $C \subseteq Q$, meaning an additive submonoid of Q (so $\mathbf{0} \in C$ and $C + C \subseteq C$). Assume C has only the trivial unit $\mathbf{0}$. Prove that the relation on Q that sets $q \preceq q'$ if $q' \in q + C$ constitutes a partial order. Prove that every partial order on Q such that $p \preceq q \Rightarrow p + r \preceq q + r$ for all $r \in Q$ arises this way.

Definition. Fix a partially ordered real vector space Q whose positive cone C is closed in the usual topology and contains all positive real rescalings of itself. An *upset* in Q is a subset U closed under addition by C, so $U + C \subseteq U$.

- The *conic topology* on Q consists of the upsets that are open in the ordinary topology.
- The *Alexandrov topology* consists of all of the upsets in Q.

To avoid confusion when it might occur, write

- Q^{con} for the set Q with the conic topology,
- Q^{ale} for the set Q with the Alexandrov topology, and
- Q^{ord} for the set Q with its ordinary topology.

In the situation of this definition, prove the following.

2. The identity on Q yields continuous maps of topological spaces

$$\iota : Q^{\mathrm{ord}} \to Q^{\mathrm{con}} \qquad \text{and} \qquad \jmath : Q^{\mathrm{ale}} \to Q^{\mathrm{con}}.$$

3. Any sheaf $\mathscr{F}$ on Q^{ord} pulled back from Q^{con} has natural maps on stalks

$$\mathscr{F}_q \to \mathscr{F}_{q'} \text{ for } q \preceq q' \text{ in } Q.$$

4. Similarly, any sheaf $\mathscr{G}$ on Q^{ale} has natural maps on stalks

$$\mathscr{G}_q \to \mathscr{G}_{q'} \text{ for } q \preceq q' \text{ in } Q.$$

5. If sheaves $\mathscr{F}$ on Q^{ord} and $\mathscr{G}$ on Q^{ale} are both pulled back from the same sheaf $\mathscr{E}$ on Q^{con}, then the diagrams of vector spaces indexed by Q in items 3 and 4 are the same.
6. The pushforward functor $\jmath_*$ is exact, and $\jmath_* \jmath^{-1} \mathscr{E} \cong \mathscr{E}$.

Note: the diagrams in items 3 and 4 are called *Q-modules*. The functor in item 4 from sheaves on Q^{ale} to Q-modules is an equivalence of categories.

References

[Vakil] Ravi Vakil, *The Rising Sea: Foundations of Algebraic Geometry*, November 18, 2017