Glossary of notation

We use the standard arithmetic, algebraic, and logical symbols, including: "$=$" and "$\cong$" for equality and isomorphism; "$\varnothing$" and "$\{\ldots\}$" for the empty set and the set consisting of "$\ldots$"; "$\cap$" and "$\cup$" for intersection and union; "$\bigoplus$" and "$\prod$" for direct sum and product; $\otimes$ for tensor product; "$\in$" and "$\subseteq$" for set membership and containment (allowing equality; we use "$\subset$" if strict containment is intended); "$\wedge$" and "$\vee$" for meet and join; "M/N" for the quotient of M by N; and "$\langle\ldots\rangle$" for the ideal generated by "$\ldots$".

We use square brackets [...] to delimit matrices appearing "as is", whereas we use parentheses (...) to delimit column vectors written horizontally in the text. Thus, column vectors represented vertically in displayed equations or figures are delimited by square brackets.

Our common symbols beyond the very standard ones above are defined in the following table. The notations listed are those that span more than one chapter. If the notation has a specific definition, we have given the page number for it; otherwise, we simply list the page number of a typical (often not the first) usage.

symbol	typical usage or definition	page
$\succeq$	partial order on $\mathbb{N}^n$	11
$\mathbf{0}$	the zero vector	63, 133
$\mathbf{1}$	$(1,\ldots,1) \in \mathbb{N}^n$	76
A	abelian group with distinguished elements $\mathbf{a}_1,\ldots,\mathbf{a}_n$	150
$\mathbf{A}$	integer matrix whose columns $\mathbf{a}_1,\ldots,\mathbf{a}_n$ generate A	133
$\mathbf{a}$	vector $(a_1,\ldots,a_n)$ in $\mathbb{N}^n$	3
	element in A (often, a vector $(a_1,\ldots,a_d)$ in $\mathbb{Z}^d$)	133
$\mathbf{a}_F$	vector label on face F of labeled cell complex	62
$\mathbf{a}_i$	$\deg(x_i)$, one of the distinguished elements $\mathbf{a}_1,\ldots,\mathbf{a}_n \in A$	149
$\mathbf{a}_\sigma$	$\deg(m_\sigma) = \bigvee_{i\in\sigma} \mathbf{a}_i$	107
$\mathbf{a} \smallsetminus \mathbf{b}$	complementation of $\mathbf{b}$ in $\mathbf{a}$, for Alexander duality	88
$\langle \mathbf{a}, \mathbf{t}\rangle$	linear form $a_1t_1 + \cdots + a_dt_d$	166
$\mathbf{b}$	analogous to $\mathbf{a}$	4, 129
$\lvert\mathbf{b}\rvert$	$b_1 + \cdots + b_n$	30
$\beta_{i,\mathbf{a}}(M)$	The i^{th} Betti number of M in degree $\mathbf{a}$	157
$\mathrm{Buch}(I)$	Buchberger graph of I	48
C	a real polyhedral cone (usually a rational polyhedral cone in $\mathbb{R}^d$)	134

symbol	typical usage or definition	page
$\mathbb{C}$	field of complex numbers	191
$\mathbb{C}^*$	group of nonzero complex numbers	192
$\mathcal{C}(M;\mathbf{t})$	multidegree of module M in variables $\mathbf{t}$	167
$\mathcal{C}(X;\mathbf{t})$	multidegree of variety (or scheme) X in variables $\mathbf{t}$	167
$\widetilde{\mathcal{C}}_\bullet(X;\Bbbk)$	reduced chain complex of cell complex X with coefficients in $\Bbbk$	9
$\widetilde{\mathcal{C}}^\bullet(X;\Bbbk)$	reduced cochain complex of cell complex X with coefficients in $\Bbbk$	10
conv	convex hull	71
$\mathbf{c}$	analogous to $\mathbf{a}$ and $\mathbf{b}$ or else to $\mathbf{u}$ and $\mathbf{v}$	15, 144
D	a (reduced) pipe dream	312
$D(w)$	diagram of partial permutation w	294
d	rank of A, when A is torsion-free	133
deg	degree map $\mathbb{Z}^n \to A$	149
det	determinant of a square matrix	274
dim	dimension	4, 301
Δ	simplicial complex	4
$\Delta^\star$	Alexander dual simplicial complex	16
Δ_I	Scarf complex of I	110
∂	boundary map	9
	differential	62
	topological boundary	124
∂_i	i^{th} divided difference operator	304
$\mathbf{e}$	basis vector of free S-module	107
	basis vector of $\mathbb{Z}^d$ or $\mathbb{R}^d$	129
$\mathcal{E}ss(w)$	essential set of partial permutation w	294
F	face of cell complex	62
	face of semigroup	133
$\mathcal{F}$	free module or resolution	156
$\mathcal{F}_X$	cellular free complex supported on labeled cell complex X	63
f	a polynomial	142
GL_n	general linear group	21
$H(M;\mathbf{t})$	Hilbert series of M in variables $\mathbf{t}$	153
$\widetilde{H}_\bullet(X;\Bbbk)$	reduced homology of X with coefficients in $\Bbbk$	65
$\widetilde{H}^\bullet(X;\Bbbk)$	reduced cohomology of X with coefficients in $\Bbbk$	10
$\underline{\mathrm{Hom}}$	module of graded homomorphisms	215
$\mathcal{H}_Q$	minimal generating set of pointed semigroup Q	137
	Hilbert basis of saturated semigroup Q or cone $\mathbb{R}_{\geq 0}Q$	138
hull(I)	hull complex of I	73
I	an ideal	3
$I^\star$	Alexander dual of I	16, 68
$I^{[\mathbf{a}]}$	Alexander dual of I with respect to $\mathbf{a}$	88
I_Δ	Stanley–Reisner ideal for simplicial complex Δ	5
I_ϵ	deformation of I	115
I_L	lattice ideal for sublattice $L \subseteq \mathbb{Z}^n$	130
I_w	Schubert determinantal ideal for partial permutation w	292
$(I:J)$	*colon ideal* $\{x \mid Jx \subseteq I\}$	90
$(I:J^\infty)$	saturation $\bigcup_m (I:J^m)$ of I with respect to J	132

symbol	typical usage or definition	page
$\mathsf{in}(f)$	initial term of f	24
$\mathsf{in}(I)$	initial ideal of I	24
$\mathsf{in}(M)$	initial submodule of M	27
J	an ideal	44
$\mathbb{K}_{\bullet}$	Koszul complex	13
$K^{\mathbf{b}}(I)$	upper Koszul simplicial complex	16
$\mathcal{K}(M;\mathbf{t})$	K-polynomial of M in variables $\mathbf{t}$	157
$\Bbbk$	field (sometimes with chapter-wide hypotheses)	3
$\Bbbk[\mathbf{x}]$	polynomial ring in variables $\mathbf{x}$	3
$\Bbbk[Q]$	semigroup ring for semigroup Q over $\Bbbk$ (sometimes $\Bbbk = \mathbb{Z}$)	129
$\Bbbk\{T\}$	vector space $\bigoplus_{\mathbf{a}\in T} \Bbbk \cdot \mathbf{t}^{\mathbf{a}}$, usually as $\Bbbk[Q]$-module	133
L	lattice in $\mathbb{Z}^n$ (often the kernel of $\mathbb{Z}^n \to A$)	130
$L_{\mathbb{R}}^{\perp}$	orthogonal complement in $\mathbb{R}^n$ of the real span of L	144
$\mathbf{L}$	integer matrix with cokernel A (so the rows generate L)	131
lcm	least common multiple	42
$\mathrm{link}_{\Delta}(\sigma)$	link of σ in Δ	17
$l(w)$	length of partial permutation w	294
λ	a real number	177
	a partition	285
λ_{qp}	scalar entries in monomial matrix	12, 217
M	a module	11
$M^{\vee}$	Matlis dual of module M	216
$M_{\mathbf{a}}$	graded component of M in degree $\mathbf{a}$	153
$M(\mathbf{a})$	graded translate of M satisfying $M(\mathbf{a})_{\mathbf{b}} = M_{\mathbf{a}+\mathbf{b}}$	153
$M_{k\ell}$	matrices with k rows and ℓ columns over the field $\Bbbk$	290
m_i	minimal generator of monomial ideal $\langle m_1, \ldots, m_r \rangle$	28
m_{σ}	least common multiple of $\{m_i \mid i \in \sigma\}$	107
$\mathfrak{m}$	graded maximal ideal	257
$\mathfrak{m}^{\mathbf{b}}$	irreducible monomial ideal $\langle x_i^{b_i} \mid b_i \geq 1 \rangle$	87
$\mathbb{N}$	the natural numbers $\{0, 1, 2, \ldots\}$	3
n	number of variables in polynomial ring S	3
$n!$	n factorial $= n(n-1)\cdots 3 \cdot 2 \cdot 1$	356
$[n]$	the set $\{1, \ldots, n\}$	81, 274
$\binom{n}{k}$	binomial coefficent $\frac{n!}{k!(n-k)!}$	48
ν	a normal vector	77, 199
$\Omega_Q^{\bullet}$	dualizing complex for affine semigroup Q	233
ω_Q	canonical module for semigroup ring $\Bbbk[Q]$	233
P_F	monomial prime ideal of semigroup ring	134
$\mathbb{P}^r$	projective space of dimension r	198
$\mathcal{P}$	a polytope or polyhedron	62, 197
$\mathcal{P}_{\lambda}$	hull polyhedron for real number $\lambda \gg 0$	177
$\mathfrak{p}$	a prime ideal	165
Q	subsemigroup of A generated by $\mathbf{a}_1, \ldots, \mathbf{a}_n$	150
Q_{sat}	saturation of semigroup Q	140
$\mathcal{Q}$	a polytope	62
R	a ring	159

symbol	typical usage or definition	page
$\mathbb{R}$	field of real numbers	41
$\mathbb{R}^n_{\geq 0}$	orthant of all nonnegative real vectors	72
$\mathbb{R}_{\geq 0}Q$	real cone generated by affine semigroup Q	134
$\mathcal{RP}(w)$	set of reduced pipe dreams for partial permutation w	312
$r_{pq}(w)$	rank of submatrix $w_{p\times q}$ of partial permutation w	290
S	polynomial ring $\Bbbk[\mathbf{x}]$	3
S^G	ring of invariants in S under action of group G	193, 364
S_n	symmetric group of permutations of $\{1,\ldots,n\}$	291
$\operatorname{supp}(\mathbf{a})$	support $\{i \in \{1,\ldots,n\} \mid a_i \neq 0\}$	7
$\mathbf{s}$	auxiliary symbol/variables analogous to $\mathbf{t}$	164
σ	squarefree vector or face of simplicial complex	4–5
$\overline{\sigma}$	complement $\{1,\ldots,n\} \smallsetminus \sigma$	5
σ_i	transposition switching i and $i+1$	298
$\mathfrak{S}_w(\mathbf{t})$	Schubert polynomial	304
$\mathfrak{S}_w(\mathbf{t}-\mathbf{s})$	double Schubert polynomial	304
Tor_i^S	i^{th} Tor module	15
$\mathbf{t}$	dummy variable for monomials in semigroup rings	129
	dummy variable for Hilbert series and K-polynomials	154
	variables $t_1,\ldots,t_d$ for K-polynomials and multidegrees	166
τ	analogous to σ	4
$\mathbf{u}$	vector $(u_1,\ldots,u_n)$ in $\mathbb{Z}^n$	130
$v \leq w$	Bruhat and weak orders on partial permutations	295, 299
$\mathbf{v}$	vector $(v_1,\ldots,v_n)$ in $\mathbb{Z}^n$	130
w	weight vector in $\mathbb{R}^n_{\geq 0}$	142
	partial permutation (matrix)	290
w_0	long word (permutation), reversing the order of $1,\ldots,n$	291
$\mathbf{w}$	vector $(w_1,\ldots,w_n)$ in $\mathbb{Z}^n$	179
X	cell complex, often labeled	62
$\underline{X}$	underlying unlabeled cell complex	92
$X_{\prec\mathbf{b}}$	subcomplex of X on face with labels $\prec \mathbf{b}$	64
$X_{\preceq\mathbf{b}}$	subcomplex of X on face with labels $\preceq \mathbf{b}$	64
$\overline{X}_w$	matrix Schubert variety for partial permutation w	290
$\mathbf{x}$	variables $x_1, x_2, \ldots$ in polynomials rings	3
	coordinates $x_1, x_2, \ldots$ on affine space	192
	variables $x_{\alpha\beta}$ in a square or rectangular array	290
$\mathbf{x^a}$	monomial $x_1^{a_1}\ldots x_n^{a_n}$	3
$\mathbf{x^a} < \mathbf{x^b}$	comparison of monomials under term order $<$	24
$\mathbf{x}_{p\times q}$	upper-left $p \times q$ submatrix of matrix $\mathbf{x}$	290
$\mathbf{y}$	auxiliary variables analogous to $\mathbf{x}$	25, 139
$\mathbb{Z}$	ring of integers	6
$\mathbb{Z}F$	group generated by face F of affine semigroup	134
$Z_{p\times q}$	upper-left $p \times q$ submatrix of matrix Z	290
$\mathbf{z}$	Laurent variables $z_1,\ldots,z_n$; coordinates on $(\mathbb{C}^*)^n$	192