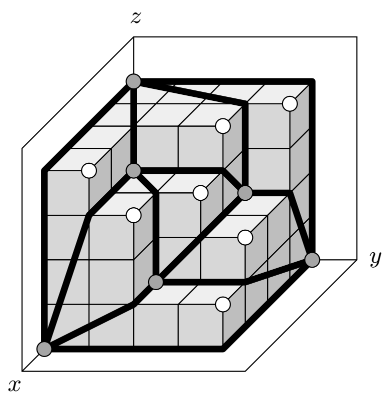
# Combinatorial Commutative Algebra

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*To Elen and Hyungsook*

# Preface

The last decade has seen a number of exciting developments at the intersection of commutative algebra with combinatorics. New methods have evolved out of an influx of ideas from such diverse areas as polyhedral geometry, theoretical physics, representation theory, homological algebra, symplectic geometry, graph theory, integer programming, symbolic computation, and statistics. The purpose of this volume is to provide a self-contained introduction to some of the resulting combinatorial techniques for dealing with polynomial rings, semigroup rings, and determinantal rings. Our exposition mainly concerns combinatorially defined ideals and their quotients, with a focus on numerical invariants and resolutions, especially under gradings more refined than the standard integer grading.

This project started at the COCOA summer school in Torino, Italy, in June 1999. The eight lectures on monomial ideals given there by Bernd Sturmfels were later written up by Ezra Miller and David Perkinson and published in [MP01]. We felt it would be nice to add more material and turn the COCOA notes into a real book. What you hold in your hand is the result, with Part I being a direct outgrowth of the COCOA notes.

Combinatorial commutative algebra is a broad area of mathematics, and one can cover but a small selection of the possible topics in a single book. Our choices were motivated by our research interests and by our desire to reach a wide audience of students and researchers in neighboring fields. Numerous references, mostly confined to the Notes ending each chapter, point the reader to closely related topics that we were unable to cover.

A milestone in the development of combinatorial commutative algebra was the 1983 book by Richard Stanley [Sta96]. That book, now in its second edition, is still an excellent source. We have made an attempt to complement and build on the material covered by Stanley. Another boon to the subject came with the arrival in 1995 of the book by Bruns and Herzog [BH98], also now in its second edition. The middle part of that book, on "Classes of Cohen–Macaulay rings", follows a progression of three chapters on combinatorially defined algebras, from Stanley–Reisner rings through semigroup rings to determinantal rings. Our treatment elaborates on these three themes. The influence of [BH98] can seen in the subdivision of our book into three parts, following the same organizational principle.

We frequently refer to two other textbooks in the same Springer series as ours, namely Eisenbud's book on commutative algebra [Eis95] and Ziegler's book on convex polytopes [Zie95]. Students will find it useful to place these two books next to ours on their shelves. Other books in the GTM series that contain useful material related to combinatorial commutative algebra are [BB04], [Eis04], [EH00], [Ewa96], [Grü03], [Har77], [MacL98], and [Rot88].

There are two other fine books that offer an introduction to combinatorial commutative algebra from a perspective different than ours, namely the ones by Hibi [Hib92] and Villarreal [Vil01]. Many readers of our book will enjoy learning more about computational commutative algebra as they go along; for this we recommend the books by Cox, Little, and O'Shea [CLO98], Greuel and Pfister [GP02], Kreuzer and Robbiano [KR00], Schenck [Sch03], Sturmfels [Stu96], and Vasconcelos [Vas98]. Additional material can be found in the proceedings volumes [EGM98] and [AGHSS04].

Drafts of this book have been used for graduate courses taught by Victor Reiner at the University of Minnesota and by the authors at UC Berkeley. In our experience, covering all 18 chapters would require a full-year course, either two semesters or three quarters (one for each of Part I, Part II, and Part III). For a first introduction, we view Chapter 1 and Chapters 3–8 as being essential. However, we recommend that this material be supplemented with a choice of one or two of the remaining chapters, to get a feel for a specific application of the theory presented in Chapters 7 and 8. Topics that stand alone well for this purpose are Chapter 2 (which could, of course, be presented earlier), Chapter 9, Chapter 10, Chapter 11, Chapter 14, and Chapter 18. We have also observed success in covering Chapter 12 with only the barest introduction to injective modules from Chapter 11, although Chapters 11 and 12 work even more coherently as a pair. Other two-chapter sequences include Chapters 11 and 13 or Chapters 15 and 16. Although the latter pair forms a satisfying end, it becomes even more so as a triplet with Chapter 17. Advanced courses could begin with Chapters 7 and 8 and continue with the rest of Part II, or instead continue with Part III.

In general, we assume knowledge of commutative algebra (graded rings, free resolutions, Gröbner bases, and so on) at a level on par with the undergraduate textbook of Cox, Little, and O'Shea [CLO97], supplemented with a little bit of simplicial topology and polyhedral geometry. Although these prerequisites are fairly modest, the mix of topics calls for considerable mathematical maturity. Also, more will be gained from some of the later chapters with additional background in homological algebra or algebraic geometry. For the former, this is particularly true of Chapters 11 and 13, whereas for the latter, we are referring to Chapter 10 and Chapters 15–18. Often we work with algebraic groups, which we describe explicitly by saying what form the matrices have (such as "block lower-triangular"). All of our arguments that use algebraic groups are grounded firmly in the transparent linear algebra that they represent. Typical conclusions reached using algebraic geometry are the smoothness and irreducibility of orbits. Typical uses of homological algebra include statements that certain operations (on resolutions, for example) are well-defined independent of the choices made.

Each chapter begins with an overview and ends with Notes on references and pointers to the literature. Theorems are, for the most part, attributed only in the Notes. When an exercise is based on a specific source, that source is credited in the Notes. For the few exercises used in the proofs of theorems in the main body of the text, solutions to the nonroutine ones are referenced in the Notes. The References list the pages on which each source is cited. The mathematical notation throughout the book is kept as consistent as possible, making the glossary of notation particularly handy, although some of our standard symbols occasionally moonlight for brief periods in nonstandard ways, when we run out of letters. Cross-references have the form "Item aa.bb" if the item is number bb in Chapter aa. Finally, despite our best efforts, errors are sure to have kept themselves safely hidden from our view. Please do let us know about all the bugs you may discover.

In August 2003, a group of students and postdocs ran a seminar at Berkeley covering topics from all 18 chapters. They read the manuscript carefully and provided numerous comments and improvements. We wish to express our sincere gratitude to the following participants for their help: Matthias Beck, Carlos D'Andrea, Mike Develin, Nicholas Eriksson, Daniel Giaimo, Martin Guest, Christopher Hillar, Serkan Hoşten, Lionel Levine, Edwin O'Shea, Julian Pfeifle, Bobby Poon, Nicholas Proudfoot, Brian Rothbach, Nirit Sandman, David Speyer, Seth Sullivant, Lauren Williams, Alexander Woo, and Alexander Yong. Additional comments and help were provided by David Cox, Alicia Dickenstein, Jesus De Loera, Joseph Gubeladze, Mikhail Kapranov, Diane Maclagan, Raymond Hemmecke, Bjarke Roune, Olivier Ruatta, and Günter Ziegler. Special thanks are due to Victor Reiner, for the many improvements he contributed, including a number of exercises and corrections of proofs. We also thank our coauthors Dave Bayer, Mark Haiman, David Helm, Allen Knutson, Misha Kogan, Laura Matusevich, Isabella Novik, Irena Peeva, David Perkinson, Sorin Popescu, Alexander Postnikov, Mark Shimozono, Uli Walther, and Kohji Yanagawa, from whom we have learned so much about combinatorial commutative algebra, and whose contributions form substantial parts of this book.

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