we will see why in the proof of the next corollary. Relying on the second description of the Čech hull in Definition 8.15, this yields

$$H(\check{C}\mathrm{Ext}^r_S(S/I_\Delta, \omega_S); \mathbf{x}) = \sum_{\sigma \in \Delta} \dim_k \widetilde{H}_{n-r-|\sigma|-1}(\mathrm{link}_\sigma \Delta; k) \prod_{j \in \sigma} \frac{x_j}{1 - x_j} \prod_{i \in \overline{\sigma}} \frac{1}{1 - x_i^{-1}}.$$

Shifting the input module by $[\mathbf{1}]$ multiplies this whole expression by $x_1^{-1} \cdots x_n^{-1} = \mathbf{x}^{-\sigma}\mathbf{x}^{-\overline{\sigma}}$, and gives the Hilbert series of $H^r_I(S)$ by Corollary 8.19. All of these steps are reversible, so Hochster's formula can similarly be derived from Terai's. □

Corollary 8.21 *The theorem of Mustaţă (8.12) is equivalent to that of Gräbe (8.9).*

Proof: The proof is the same as the previous corollary, except that we need to keep track of the multiplication maps between the $\mathbb{Z}^n$-graded degrees of the modules in question. The crux is that the homomorphisms

$$\widetilde{H}^{n-r-|\sigma\cup i|-1}(\mathrm{link}_{\sigma\cup i} \Delta; k) \longrightarrow \widetilde{H}^{n-r-|\sigma|-1}(\mathrm{link}_\sigma \Delta; k)$$

coming from Gräbe's theorem become the transpose homomorphisms

$$\widetilde{H}_{n-r-|\sigma|-1}(\mathrm{link}_\sigma \Delta; k) \longrightarrow \widetilde{H}_{n-r-|\sigma\cup i|-1}(\mathrm{link}_{\sigma\cup i} \Delta; k)$$

in the Matlis dual, by definition. Then, by Alexander duality, these agree with the maps

$$\widetilde{H}^{r-2}(\Delta^\vee|_{\overline{\sigma}}; k) \longrightarrow \widetilde{H}^{r-2}(\Delta^\vee|_{\overline{\sigma}\setminus i}; k)$$

in Mustaţă's theorem. □

The Čech hull had been defined before Theorems 8.11 and 8.12 were known, although its relation to local cohomology had not been found. The motivation for Theorem 8.18 was to prove *local duality with monomial support* [24], which generalizes Theorem 8.7—this is where one really needs the fact that the canonical Čech complex has minimal length.

A Appendix: Exercises

The following exercises are derived from those given at the June 1999 COCOA Summer School in Turin, Italy.

A.1 Exercises from Day 1

Exercise A.1.1 Let $n = 6$ and let Δ be the boundary of an octahedron.

(a) Determine I_Δ and $I_{\Delta^\vee}$.
(b) Compute their respective Hilbert series.
(c) Compute their minimal free resolutions.
(d) Interpret the Betti numbers obtained in part (c) in terms of simplicial homology.

Exercise A.1.2 Explain how CoCoA can be used to calculate the homology of a simplicial complex.

Exercise A.1.3 Consider a 5×5-matrix of indeterminates (x_{ij}) and let I_Δ be the ideal in $k[x_{ij}]$ generated by all 100 squarefree monomials of the form $x_{ij}x_{il}$ or $x_{ij}x_{lj}$. The simplicial complex Δ is called the *chessboard complex*.

(a) Using Hilbert series in CoCoA, find the number of 2-dimensional faces of Δ.
(b) Compute the homology groups of Δ, using your algorithm in Exercise A.1.2.

Exercise A.1.4 Let I be the ideal of the cubic Veronese surface in projective 9-space. Compute the generic initial ideal of I for reverse lexicographic order and for purely lexicographic order. How do their minimal free resolutions compare to that of I itself?

Exercise A.1.5 Give an example of a Borel-fixed ideal which is not the initial monomial ideal of any homogeneous prime ideal in $k[x_1, \ldots, x_n]$. Are such examples rare or abundant?

Exercise A.1.6 Let $I \subset \mathbb{C}[x, y, z]$ be the homogeneous radical ideal of seven generic points in $\mathbb{P}^2_{\mathbb{C}}$, where $\mathbb{C}$ is the field of complex numbers. List **all** initial monomial ideals of I, with respect to all term orders.

Exercise A.1.7 Let M be an arbitrary monomial ideal in $\mathbb{C}[x_1, \ldots, x_n]$, and let $\mathcal{B} \subset \mathbb{N}^n$ be the set of all vectors $\mathbf{b}$ such that $\mathbf{x}^{\mathbf{b}}$ is not in M. The *distraction* of M is the radical ideal D_M of all polynomials in $\mathbb{C}[x_1, \ldots, x_n]$ which vanish on the set $\mathcal{B}$.

(a) Determine a finite generating set of D_M.
(b) Show that M is the initial monomial ideal of D_M with respect to any term order.
(c) Determine the prime decomposition of D_M.
(d) The number of prime components of D_M is called the *arithmetic degree* of M. Write a CoCoA program for computing the arithmetic degree.

A.2 Exercises from Day 2

Exercise A.2.1 Let Δ be the simplicial complex on the set $\{x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4, z_1, z_2, z_3, z_4\}$ obtained by **polarization** of the monomial ideal

$$M = \langle x^4, y^4, z^4, x^3y^2z, xy^3z^2, x^2yz^3 \rangle.$$

Determine the number of i-dimensional faces of Δ for $i = 2, 3, 4, 5, 6, 7, 8$.

Exercise A.2.2 Let $<$ be the purely lexicographic term order. Using CoCoA, compute the generic initial ideal $\mathrm{gin}_<(M)$ and its minimal free resolution, for the ideal M in the previous Exercise.

Exercise A.2.3 Pick 100 monomials in x, y, z at random with exponents between 0 and 1000. Compute the minimal free resolution and the Hilbert series of the ideal they generate. Repeat the experiment ten times. Explain your data. Try again with more monomials...

Exercise A.2.4 Draw the *second barycentric subdivision of a triangle*. Construct a monomial ideal in $k[x, y, z]$ which has that resolution. Such ideals exist by Schnyder's Theorem.

Exercise A.2.5 Explain how the Hilbert function command in CoCoA can be used to compute the Scarf complex of a generic monomial ideal. Apply your method to compute Δ_M for

$$M = \langle a^5, b^5, c^5, d^5, ab^2c^3d^4, a^2b^3c^4d, a^3b^4cd^2, a^4bc^2d^3 \rangle.$$

The Scarf complex Δ_M is a triangulation of the tetrahedron; draw it.

Exercise A.2.6 Compute the irreducible decomposition of the monomial ideal M in the previous Exercise.

Exercise A.2.7 What is the maximum number of irreducible components of an artinian ideal generated by 10 monomials in 4 variables? Can you find an example that attains the bound?

Exercise A.2.8 Consider the non-generic monomial ideal $M = \langle x, y, z \rangle^3$. Construct at least three different free resolutions of M by *deformation of exponents*.

A.3 Exercises from Day 3

Exercise A.3.1 Find a monomial ideal I such that $(I^\vee)^\vee \neq I$. Characterize such I.

Exercise A.3.2 Alexander duality commutes with taking radicals: $\sqrt{I^\vee} = (\sqrt{I})^\vee$.

Exercise A.3.3 Let I be a monomial ideal given in terms of its minimal generators, and $\mathbf{a} \in \mathbb{N}^n$ coordinatewise bigger than the exponent vectors of the generators. What is the easiest (or fastest) method in CoCoA for calculating the Alexander dual $I^{[\mathbf{a}]}$?

Exercise A.3.4 The irrelevant ideal of the product of projective spaces $\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^1$ is a squarefree monomial ideal M in $k[x_0, x_1, x_2, y_0, y_1, y_2, z_0, z_1]$.

(a) Find the minimal generators of M.
(b) Calculate the minimal free resolution of M in CoCoA.
(c) Interpret the Betti numbers of your resolution in (b) in terms of polytopes.
(d) Show that the minimal free resolution of M is a cellular resolution.
(e) Why is the Alexander dual of M Cohen-Macaulay?

Exercise A.3.5 Draw the minimal free resolution of the cogeneric ideal

$$\langle x^1, y^4, z^6\rangle \cap \langle x^2, y^6, z^1\rangle \cap \langle x^3, y^3, z^3\rangle \cap \langle x^4, y^5, z^2\rangle \cap \langle x^5, y^1, z^5\rangle \cap \langle x^6, y^2, z^4\rangle .$$

Check your result using CoCoA.

Exercise A.3.6 What is the maximal number of minimal generators of an intersection of 12 irreducible monomial ideals in $k[x_1, x_2, x_3, x_4]$?

Exercise A.3.7 Show that the hull resolution of $\langle x^4, y^4, x^3z, y^3z, x^2z^2, y^2z^2, xz^3, yz^3\rangle$ is minimal. What is its irreducible decomposition? Where is this information hiding in the hull complex?

Exercise A.3.8 Compute the hull resolution of the ideal

$$I = \langle x_1x_2, x_1x_3, x_1x_4, x_1x_5, x_2x_3, x_2x_4, x_2x_5, x_3x_4, x_3x_5, x_4x_5\rangle .$$

Can you state a general result for squarefree monomial ideals?

A.4 Exercises from Day 4

Exercise A.4.1 Let M denote the monomial module generated by all Laurent monomials $x^iy^jz^k$ with the properties that $i + j + k = 0$ and not all three coordinates of (i, j, k) are even. Draw a picture of this module. Determine the minimal free resolution of M over $k[x, y, z]$.

Exercise A.4.2 Let L be the kernel of the matrix $\begin{pmatrix} 3 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 \end{pmatrix}$. Show that the hull resolution of the monomial module M_L is minimal. What happens modulo the action by the lattice L?

Exercise A.4.3 Describe the canonical module of the ring $k[t^3, t^4, t^5]$ as the quotient of a lattice module in $k[x^{\pm 1}, y^{\pm 1}, z^{\pm 1}]$ by a lattice action. Is there a relation to Alexander duality?

Exercise A.4.4 Using CoCoA, compute the $\mathbb{Z}$-graded Hilbert series of the algebra $k[t^{20}, t^{24}, t^{25}, t^{31}]$ in the form

$$\frac{p(t)}{(1 - t^{20})(1 - t^{24})(1 - t^{25})(1 - t^{31})} .$$

Give a polyhedral explanation for each term appearing in the polynomial $p(t)$.

Exercise A.4.5 Suppose you travel to a country whose currency has four coins valued 20, 24, 25, and 31. What is the largest amount of money which cannot be expressed by these coins?

Exercise A.4.6 Explain how the hull complex of a **generic** lattice ideal can be computed in CoCoA. Apply your procedure to compute the hull complex for the ideal I_L of the 2-dimensional sublattice L of $\mathbb{Z}^4$ spanned by the vectors $(-7, -5, 3, 8)$ and $(4, -7, 9, -1)$.

Exercise A.4.7 Compute the hull resolution for the ideal of 2×2-minors of a generic 2×4-matrix.