1 Lecture I: Squarefree monomial ideals

1.1 Equivalent descriptions

A monomial $\mathbf{x}^{\mathbf{a}}$ is called *squarefree* if every coordinate of $\mathbf{a}$ is 0 or 1. A monomial ideal is called *squarefree* if it is generated by squarefree monomials. The information carried by squarefree monomial ideals can be characterized in many ways. Some of the most important are given in the following

Theorem 1.1 *The following are equivalent:*

1. *Squarefree monomial ideals in* $k[x_1, \ldots, x_n]$
2. *Unions of coordinate subspaces in* k^n
3. *Unions of coordinate subspaces in* $\mathbf{P}^{n-1}$
4. *Simplicial complexes on* $\{1, \ldots, n\} := \{1, 2, \ldots, n\}$

The ideal $I = I_\Delta$ *in (1) is called the* face ideal *or* Stanley-Reisner ideal *of the simplicial complex* Δ *in (4), and* S/I_Δ *is called the* face ring *or* Stanley-Reisner ring *of* Δ.

Proof: The general idea is

$$\text{ideal } I \leftrightsquigarrow \text{affine variety of } I \leftrightsquigarrow \text{projective variety of } I$$
$$\text{and}$$
$$\text{coordinate subspace} \leftrightsquigarrow \text{simplex}.$$

Precisely:

1⇝2: Given a squarefree monomial ideal I, let $V(I) \subset k^n$ be the algebraic subset on which it vanishes. From the algorithm in Section 0.6 it follows that $I = \mathfrak{m}^{\sigma_1} \cap \cdots \cap \mathfrak{m}^{\sigma_r}$ is an intersection of monomial prime ideals. Thus $V(I) = V(\mathfrak{m}^{\sigma_1}) \cup \cdots \cup V(\mathfrak{m}^{\sigma_r})$ is a union of coordinate subspaces because $V(\mathfrak{m}^\sigma)$ is the vector subspace of k^n spanned by the standard basis vectors $\{\mathbf{e}_j \mid j \notin \sigma\}$.

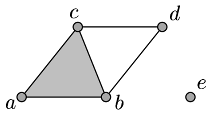
2⇝3: Recall that $\mathbb{P}^{n-1} = (k^n \setminus \{\mathbf{0}\})/k^*$ is a quotient of k^n. In particular, the quotient of any nonzero coordinate subspace of k^n is a coordinate subspace of $\mathbb{P}^{n-1}$. This gives a 1-1 correspondence between affine and projective coordinate subspaces if we agree that the empty set is a projective coordinate subspace, spanned by the empty set of coordinate points, corresponding to the affine linear subspace $\{\mathbf{0}\}$.

3⇝4: The simplicial complex on $\{1, \ldots, n\}$ corresponding to a union of coordinate subspaces is the one whose faces consist of those sets $\sigma \subseteq \{1, \ldots, n\}$ such that $\operatorname{span}(\mathbf{e}_i \mid i \in \sigma)$ is contained in the union. It is a simplicial complex because if a subspace V is contained in our union, then so is every subspace of V.

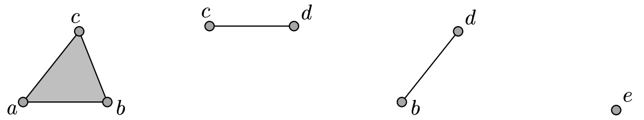
4⇝3⇝2⇝1: The algebraic sets in (3) and (2) are the unions of the subspaces $\operatorname{span}(\mathbf{e}_i \mid i \in \sigma)$ for which σ is in our simplicial complex. Each coordinate subspace $\operatorname{span}(\mathbf{e}_i \mid i \in \sigma)$ is equal to $V(\mathfrak{m}^{\overline{\sigma}})$, where $\overline{\sigma} := \{1, \ldots, n\} \setminus \sigma$, and the ideal in (1) is the intersection of these $\mathfrak{m}^{\overline{\sigma}}$. □

Remark 1.2 A little bit of caution is warranted: in 4⇝3⇝2⇝1, it is not always true that the ideal of polynomials vanishing on a collection of coordinate subspaces is a monomial ideal! This means that the correspondence in the Theorem is *not* the

Zariski correspondence: there is a problem if k is finite. On the other hand, when k is infinite, the Zariski correspondence between ideals and algebraic sets does induce the 1-1 correspondence between squarefree monomial ideals and unions of coordinate subspaces. At any rate, the Theorem does not depend on k being infinite. □

Example 1.3 The simplicial complex $\Delta =$ 

from Example 0.3, replacing the variables x_1, x_2, x_3, x_4, x_5 by a, b, c, d, e, has Stanley-Reisner ideal



$$\begin{aligned} I_\Delta &= \langle d, e\rangle \cap \langle a, b, e\rangle \cap \langle a, c, e\rangle \cap \langle a, b, c, d\rangle \\ &= \langle ad, ae, bcd, be, ce, de\rangle. \end{aligned}$$

We have expressed I_Δ via its irreducible decomposition and its minimal generators. Above each irreducible component is drawn the corresponding facet of Δ. □

Given a simplicial complex Δ, it should be clear by now what the irreducible decomposition of its Stanley-Reisner ideal I_Δ means: each irreducible component $\mathfrak{m}^\sigma$ corresponds to a facet $\overline{\sigma} := \{1, \ldots, n\} \setminus \sigma$ of Δ. But what about the minimal generators? In order for $\mathbf{x}^\tau$ to be in the intersection $I_\Delta = \bigcap_{i=1}^r \mathfrak{m}^{\sigma_i}$, it is necessary and sufficient that τ share at least one element with σ_i for each i between 1 and r. In terms of Δ, this means that for every facet $\overline{\sigma} \in \Delta$, τ has at least one vertex not in $\overline{\sigma}$. In other words, τ is not contained in any facet (and therefore in any face) of Δ; we say that τ is a *nonface* of Δ. Thus the squarefree monomials in I_Δ correspond precisely to the nonfaces of Δ:

$$I_\Delta = \langle \mathbf{x}^\tau \mid \tau \notin \Delta \rangle \, .$$

Since being a nonface is preserved under taking supersets, the minimal generators of I_Δ are therefore $\{\mathbf{x}^\tau \mid \tau \text{ is a minimal nonface of } \Delta\}$. Alternatively, the nonzero squarefree monomials of S/I_Δ correspond to the faces of Δ. Example 1.3 can be used as a test case.

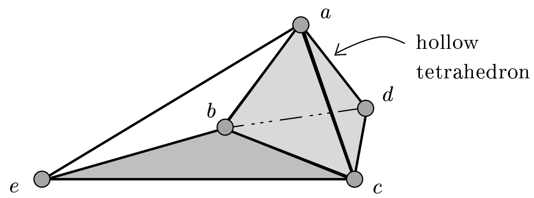
1.2 Hilbert series

Now we want to write down the Hilbert series of S/I_Δ for the Stanley-Reisner ring of a simplicial complex Δ. We know already from the previous section which *squarefree* monomials are not in I_Δ. But because the generators of I_Δ are themselves squarefree, a monomial $\mathbf{x}^\mathbf{a}$ is not in I_Δ if and only if $\mathbf{x}^{\mathrm{supp}(\mathbf{a})}$ is not in I_Δ, where $\mathrm{supp}(\mathbf{a}) = \{i \in \{1, \ldots, n\} \mid a_i \neq 0\}$ is the *support* of $\mathbf{a}$. Therefore,

$$H(S/I_\Delta; x_1, \ldots, x_n) \;=\; \sum \{\mathbf{x}^\mathbf{a} \mid \mathbf{a} \in \mathbb{N}^n \text{ and } \mathrm{supp}(\mathbf{a}) \in \Delta\}$$

$$
\begin{aligned}
&= \sum_{\sigma\in\Delta} \sum \{\mathbf{x}^{\mathbf{a}} \mid \mathbf{a}\in\mathbb{N}^n \text{ and } \operatorname{supp}(\mathbf{a})=\sigma\} \\
&= \sum_{\sigma\in\Delta} \prod_{i\in\sigma} \frac{x_i}{1-x_i} \\
&= \frac{1}{(1-x_1)\cdots(1-x_n)} \cdot \underbrace{\left\{ \sum_{\sigma\in\Delta} \prod_{i\in\sigma} x_i \cdot \prod_{j\notin\sigma}(1-x_j) \right\}}_{\text{numerator of the Hilbert series}} .
\end{aligned}
$$

Example 1.4 Consider the simplicial complex Γ depicted below. (The reason for not calling it Δ is because Γ is the Alexander dual of the simplicial complex Δ of Examples 0.3 and 1.3, and calling them both Δ would be confusing in Lecture VI.)



The simplicial complex Γ

The Stanley-Reisner ideal of Γ is

$$
\begin{aligned}
I_\Gamma &= \langle de, abe, ace, abcd\rangle \\
&= \langle a,d\rangle \cap \langle a,e\rangle \cap \langle b,c,d\rangle \cap \langle b,e\rangle \cap \langle c,e\rangle \cap \langle d,e\rangle ,
\end{aligned}
$$

and its Hilbert series is

$$
\begin{aligned}
&1 + \tfrac{a}{1-a} + \tfrac{b}{1-b} + \tfrac{c}{1-c} + \tfrac{d}{1-d} + \tfrac{e}{1-e} + \tfrac{ab}{(1-a)(1-b)} + \tfrac{ac}{(1-a)(1-c)} + \tfrac{ad}{(1-a)(1-d)} \\
&\quad + \tfrac{ae}{(1-a)(1-e)} + \tfrac{bc}{(1-b)(1-c)} + \tfrac{bd}{(1-b)(1-d)} + \tfrac{be}{(1-b)(1-e)} + \tfrac{cd}{(1-c)(1-d)} + \tfrac{ce}{(1-c)(1-e)} \\
&\quad + \tfrac{abc}{(1-a)(1-b)(1-c)} + \tfrac{abd}{(1-a)(1-b)(1-d)} + \tfrac{acd}{(1-a)(1-c)(1-d)} + \tfrac{bcd}{(1-b)(1-c)(1-d)} + \tfrac{bce}{(1-b)(1-c)(1-e)} \\
&= \frac{1 - abcd - abe - ace - de + abce + abde + acde}{(1-a)(1-b)(1-c)(1-d)(1-e)} .
\end{aligned}
$$

See Example 1.5, below, for a quick way to get this last equality. □

The formula for the Hilbert series of S/I_Δ perhaps becomes a little neater when we coarsen to the $\mathbb{Z}$-grading. Letting $f_i := |F_i(\Delta)|$ = the number of i-faces of Δ (Section 0.5), we get

$$
\begin{aligned}
H(S/I_\Delta; t,\ldots,t) &= \frac{1}{(1-t)^n} \sum_{i=0}^{d} f_{i-1}\, t^i (1-t)^{n-i} \\
&= \frac{h_0 + h_1 t + h_2 t^2 + \cdots + h_d t^d}{(1-t)^d}
\end{aligned}
$$

where $d = \dim(\Delta)+1$ and the vector $(h_0, h_1, \ldots, h_d)$, which is defined by this equation, is the *h-vector* of Δ. We will provide no more generalities about the h-vector or the f-vector $(f_{-1}, f_0, \ldots, f_d)$ since they are, to some approximation, the subjects of a whole chapter of Stanley's book [32].

1.3 Free resolutions

There is no general formula for the maps in a minimal free resolution of an arbitrary squarefree monomial ideal I_Δ. However, we can figure out what the Betti numbers are in terms of the simplicial cohomology of the Stanley-Reisner complex Δ. Before doing this, let's first have a look at some free resolutions. In what follows, it is convenient and customary to identify the subset $\sigma \subseteq \{1, \ldots, n\}$ with its characteristic vector in $\{0,1\}^n$, which has a 1 in the i^{th} slot when $i \in \sigma$. This causes the notation $\mathbf{x}^\sigma = \prod_{i \in \sigma} x_i$ to make even more sense than it did in Example 0.1.3.

Example 1.5 Let Γ be the simplicial complex from Example 1.4. The Stanley-Reisner ring S/I_Γ has minimal free resolution

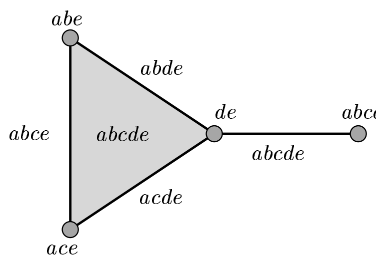
$$0 \longleftarrow S \xleftarrow{\begin{array}{r} \begin{array}{cccc} de & abe & ace & abcd \end{array} \\ 1 \begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix} \end{array}} S^4 \xleftarrow{\begin{array}{r} \begin{array}{cccc} abce & abde & acde & abcde \end{array} \\ \begin{array}{r} de \\ abe \\ ace \\ abcd \end{array} \begin{pmatrix} 0 & -1 & -1 & -1 \\ 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}} S^4 \xleftarrow{\begin{array}{r} abcde \\ \begin{array}{r} abce \\ abde \\ acde \\ abcde \end{array} \begin{pmatrix} -1 \\ 1 \\ -1 \\ 0 \end{pmatrix} \end{array}} S \longleftarrow 0$$

where the maps are denoted by *monomial matrices* as in Section 0.2. For an example of how to recover the usual matrix notation for maps of free S-modules, the middle matrix can be written as

$$\begin{pmatrix} 0 & -ab & -ac & -abc \\ c & d & 0 & 0 \\ -b & 0 & d & 0 \\ 0 & 0 & 0 & e \end{pmatrix}$$

without the border entries.

As a preview to Lecture V, the reader is invited to figure out how the following labeled simplicial complex corresponds to the above free resolution:



Hint: compare the free resolution and the labeled simplicial complex with the numerator of the Hilbert series in Example 1.4. □

Example 1.6 The minimal free resolution of $k = S/\mathfrak{m}$ is the *Koszul complex* $\mathbb{K}_\bullet$, and is easiest to describe using monomial matrices. Recall that in the reduced chain complex of the simplex consisting of all subsets of $\{1, \ldots, n\}$, the basis vectors are called e_σ for $\sigma \subseteq \{1, \ldots, n\}$. The monomial matrices for the maps in $\mathbb{K}_\bullet$ are obtained simply by labeling the column and the row corresponding to e_σ by σ itself (or $\mathbf{x}^\sigma$) and shifting homologically so that the empty set $\varnothing$ is in homological degree 0. If $n = 3$, for instance, we get the following resolution (with x_1, x_2, x_3 replaced by x, y, z).

$$\mathbb{K}_\bullet : \quad 0 \longleftarrow S \xleftarrow{\begin{array}{c|ccc} & x & y & z \\ 1 & 1 & 1 & 1 \end{array}} S^3 \xleftarrow{\begin{array}{c|ccc} & yz & xz & xy \\ x & 0 & 1 & 1 \\ y & 1 & 0 & -1 \\ z & -1 & -1 & 0 \end{array}} S^3 \xleftarrow{\begin{array}{c|c} & xyz \\ yz & 1 \\ xz & -1 \\ xy & 1 \end{array}} S \longleftarrow 0$$

The method of proof for many statements about monomial ideals is to determine what is happening in each $\mathbb{Z}^n$-graded degree of a complex of S-modules. To illustrate, we do this now for $\mathbb{K}_\bullet$ in some detail.

The essential observation is that a free module generated by 1_τ in degree τ is nonzero in degree σ precisely when $\tau \subseteq \sigma$ (equivalently, when $\mathbf{x}^\tau$ divides $\mathbf{x}^\sigma$). It helps to think of $S \cdot 1_\tau$ as the principal ideal $\langle \mathbf{x}^\tau \rangle$, so the statement becomes

$$\langle \mathbf{x}^\tau \rangle_\sigma \neq 0 \iff \mathbf{x}^\sigma \in \langle \mathbf{x}^\tau \rangle \iff \mathbf{x}^\tau \text{ divides } \mathbf{x}^\sigma .$$

The only contribution to the degree $\mathbf{0}$ part of $\mathbb{K}_\bullet$, for example, comes from the free module corresponding to $\varnothing$, which is generated by $1_\varnothing$ in degree $\mathbf{0}$. More generally, the degree σ part of $\mathbb{K}_\bullet$ comes from those rows and columns labeled by faces of σ. In other words, we restrict $\mathbb{K}_\bullet$ to degree σ by ignoring summands $S \cdot 1_\tau$ for which τ is not a face of σ. Therefore, $(\mathbb{K}_\bullet)_\sigma$ is, as a chain complex of k-vector spaces, precisely equal to the reduced chain complex of the simplex σ! This explains why the homology of $\mathbb{K}_\bullet$ is just k in degree $\mathbf{0}$ and zero elsewhere: a simplex σ is contractible, so it has no reduced homology (unless $\sigma = \varnothing$—see Remark 0.5). □

Example 1.7 Instead of using the reduced chain complex of the simplex $\{1, \ldots, n\}$, we could use the reduced *cochain* complex to get another version of the Koszul complex. This time, we denote it by $\mathbb{K}^\bullet$ and put the label $\mathbf{x}^\sigma$ on the column and row corresponding to $e^*_{\overline{\sigma}}$, where $\overline{\sigma} = \{1, \ldots, n\} \setminus \sigma$. Of course, if we want to consider $\mathbb{K}^\bullet$ as a free resolution of k, we must shift the homological degrees so that $e^*_\varnothing$ sits in homological degree n, and more generally $e^*_{\overline{\sigma}}$ sits in homological degree $n - |\overline{\sigma}| = |\sigma|$.

As a complex of k-vector spaces, $(\mathbb{K}^\bullet)_\sigma = \bigoplus_{\tau \subseteq \sigma} S[\tau - \sigma]_\sigma$ is the subcomplex of the cochain complex $\widetilde{\mathcal{C}}^\bullet(\{1, \ldots, n\}; k)$ spanned by the basis elements $\{e^*_{\overline{\tau}} \mid \tau \subseteq \sigma\} = \{e^*_{\tau \cup \overline{\sigma}} \mid \tau \subseteq \sigma\} = \{\mathbf{x}^\tau \cdot 1_{\sigma - \tau} \mid \tau \subseteq \sigma\}$. In fact, though,

$$\begin{array}{rcl} (\mathbb{K}^\bullet)_\sigma & \cong & \widetilde{\mathcal{C}}^\bullet(\sigma) \\ e^*_{\tau \cup \overline{\sigma}} & \mapsto & \operatorname{sign}(\tau, \overline{\sigma}) e^*_\tau \end{array}$$

where $\operatorname{sign}(\tau, \overline{\sigma})$ is the sign of the permutation which puts the list $(\tau, \overline{\sigma})$ into increasing order. This point of view will be essential for the proof of Theorem 1.8. Just as in the last example, this explains why $\mathbb{K}^\bullet$ has homology k in degree $\mathbf{0}$ and 0 otherwise. □

For each $\sigma \subseteq \{1, \ldots, n\}$, define the *restriction of* Δ *to* σ by

$$\Delta|_\sigma := \{\tau \in \Delta \mid \tau \subseteq \sigma\}.$$

Hochster's formula expresses the number of summands generated in degree σ at the i^{th} stage in a minimal free resolution of S/I_Δ as follows.

Theorem 1.8 (Hochster [19]) $\beta_{i-1,\sigma}(I_\Delta) = \beta_{i,\sigma}(S/I_\Delta) = \dim_k \widetilde{H}^{|\sigma|-i-1}(\Delta|_\sigma; k)$.

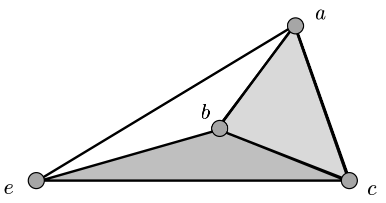
It turns out that all of the syzygies occur in degrees given by some incidence vector σ; we will have a quick and easy proof available as soon as we introduce the *hull resolution* in Lecture V (see the comment after Theorem 5.9). For now, we apply this fact to note that Hochster's formula accounts for all of the nonzero Betti numbers.

Proof: The first equality is obvious, since a minimal free resolution of I_Δ is achieved by snipping off the copy of S occurring in homological degree 0 of the minimal free resolution of S/I_Δ. For the second equality, we use the commutativity of Tor: we can calculate $\mathrm{Tor}_i^S(k, S/I_\Delta)$ by tensoring the Koszul complex $\mathbb{K}^\bullet$ with S/I_Δ as follows.

In each squarefree degree, $(S/I_\Delta \otimes_S \mathbb{K}^\bullet)_\sigma$ is a quotient of $(\mathbb{K}^\bullet)_\sigma$ and, as in Example 1.7, will be the reduced cochain complex of some simplicial complex. For $\tau \subseteq \sigma$, the basis vector $e^*_{\tau\cup\overline{\sigma}} = \mathbf{x}^\tau \otimes 1_{\sigma-\tau} \in (\mathbb{K}^\bullet)_\sigma$ becomes zero in the tensor product $S/I_\Delta \otimes \mathbb{K}^\bullet$ if and only if $\mathbf{x}^\tau = 0$ in S/I_Δ, and this occurs if and only if $\tau \notin \Delta$. Therefore, the k-basis for $(S/I_\Delta \otimes \mathbb{K}^\bullet)_\sigma$ is $\{e^*_{\tau\cup\overline{\sigma}} \mid \tau \in \Delta|_\sigma\}$. Under the isomorphism in Example 1.7, we find that $(S/I_\Delta \otimes \mathbb{K}^\bullet)_\sigma \cong \widetilde{\mathcal{C}}^\bullet(\Delta|_\sigma; k)$, but considered as a homological complex (decreasing indices) with $e^*_\varnothing$ in homological degree $|\sigma|$, and more generally e^*_τ in homological degree $|\sigma| - |\tau| = |\sigma| - \dim\tau - 1$. Taking the i^{th} homology of this complex yields $\widetilde{H}^{|\sigma|-i-1}(\Delta|_\sigma; k)$, as desired. □

Example 1.9 Let Γ be as in Examples 1.4 and 1.5. Taking $\sigma = \{a, b, c, d, e\}$, corresponding to the monomial $abcde$, we have $\Gamma|_\sigma = \Gamma$. Therefore, we can use Hochster's formula to compute the dimensions of the cohomology groups of Γ. From the labelings of the matrices, we see $\beta_{3,\sigma}(S/I_\Gamma) = \beta_{2,\sigma}(S/I_\Gamma) = 1$, and the other Betti numbers in this degree are zero. Thus, $\widetilde{H}^1(\Gamma; k) \cong \widetilde{H}^2(\Gamma; k) \cong k$, and the other reduced cohomology groups of Γ are 0. The nonzero cohomology comes from the "empty" circle $\{a, b, e\}$ and the "empty" sphere $\{a, b, c, d\}$.

For another example, take $\sigma = \{a, b, c, e\}$, corresponding to the monomial $abce$. The restriction $\Gamma|_\sigma$ is the simplicial complex



Hochster's formula gives $\widetilde{H}^{|\sigma|-1-2}(\Gamma|_\sigma; k) = \widetilde{H}^1(\Gamma|_\sigma; k) \cong k$, and the other cohomology groups are trivial. □