Remark 1.10 Since we are working over a field k, the reader who wishes may substitute reduced homology for cohomology when calculating Betti numbers, since these have the same dimension. $\square$

Further reading The standard reference for squarefree monomial ideals is Stanley [32], but Chapter 5 of the excellent book of Bruns and Herzog [7] is also recommended.

2 Lecture II: Borel-fixed monomial ideals

Squarefree monomial ideals occur mostly in a combinatorial context. Our next monomial ideals, the *Borel-fixed* monomial ideals, have more direct connection to algebraic geometry, where they arise as fixed points of an algebraic group action on the Hilbert scheme. But don't worry: one need not know what the Hilbert scheme is to understand both the group action and the fixed points.

2.1 Group actions

Let's begin by putting this group action in perspective. *Throughout this lecture, the field k has characteristic 0, and all ideals of S that we consider are $\mathbb{Z}$-graded.* We have the following inclusions of matrix groups:

$$\begin{array}{ccll} \mathrm{GL}_n(k) & = & \{\text{invertible } n\times n \text{ matrices}\} & \text{general linear group} \\ \cup & & & \\ B_n(k) & = & \{\text{upper triangular matrices}\} & \text{Borel group} \\ \cup & & & \\ T_n(k) & = & \{\text{diagonal matrices}\} & \text{torus group} \end{array}$$

The general linear group (and hence its subgroups, as well) acts on the polynomial ring as follows. For $g = (g_{ij}) \in \mathrm{GL}_n(k)$ and $f = f(x_1, \ldots, x_n) \in S$, let g act on f by

$$g \cdot f = f(gx_1, \ldots, gx_n) \quad \text{where} \quad gx_j := \sum_{i=1}^{n} g_{ij} x_i \,.$$

Given an ideal $I \subset S$, we can then set

$$g \cdot I = \{g \cdot f \mid f \in I\}\,,$$

and this defines the action on the Hilbert scheme, whose points correspond to homogeneous ideals of S up to primary components at the irrelevant ideal $\mathfrak{m} = \langle x_1, \ldots, x_n\rangle$.

Being fixed by all of GL_n is awfully difficult for an ideal I; the only way this can happen is if $I = 0$ or I is some power $\mathfrak{m}^d$. At the other extreme, an ideal I is torus fixed if and only if I is a monomial ideal. This characterization of monomial ideals was one of the original motivations for studying *toric varieties*, examples of which come from the orbit-closures under T_n of ideals I. In any case, it explains why we are interested in the action of B_n, which will therefore pick out some special kinds of monomial ideals (but not so special as to be powers of $\mathfrak{m}$!). The extra property enjoyed by a Borel-fixed ideal is that smaller-indexed variables can be swapped in for larger ones without leaving the ideal (see, for instance, [12, Theorem 15.23]):

Proposition 2.1 *The following are equivalent for a monomial ideal I:*

1. *I is Borel-fixed;*
2. *If $m \in I$ is any monomial divisible by x_j, then $mx_i/x_j \in I$ for $i < j$.*

To test your understanding, try answering the following questions:

1. An irreducible ideal $I = \langle x_{i_1}^{a_1}, \ldots, x_{i_r}^{a_r} \rangle$ is Borel fixed if and only if ______?
2. Are the primary components of a Borel-fixed ideal Borel-fixed?

2.2 Generic initial ideals

Again let $I \subset S$ be a $\mathbb{Z}$-graded ideal, and let $<$ be any term order (for an introduction to term orders and the other material in this Section, see [12, Chapter 15]). Every $g \in \mathrm{GL}_n$ determines a monomial ideal $\mathrm{in}_<(g \cdot I)$, the *initial monomial ideal* of $g \cdot I$ for the term order $<$. It is a theorem that as a function of g, $\mathrm{in}_<(g \cdot I)$ is constant on a Zariski open subset of GL_n. The constant value on that open set is called the the *generic initial ideal* of I for the term order $<$, and is denoted

$$\mathrm{gin}_<(I) := \mathrm{in}_<(g \cdot I)\,.$$

The point of all this is:

Theorem 2.2 $\mathrm{gin}_<(I)$ *is Borel-fixed.*

See [12, Chapter 15] for a proof.

Example 2.3 Let $f, g \in k[x_1, x_2, x_3, x_4]$ be generic forms of degrees d, e, respectively. Here is the ideal $M = \mathrm{gin}_{\mathrm{lex}}(\langle f, g \rangle)$ for a few cases:

$$\begin{array}{lcl} (d,e) = (2,2) & M = & \langle x_2^4, x_1x_3^2, x_1x_2, x_1^2 \rangle \\ & = & \langle x_1, x_2^4 \rangle \cap \langle x_1^2, x_2, x_3^2 \rangle \\ (d,e) = (2,3) & M = & \langle x_2^6, x_1x_3^6, x_1x_2x_4^4, x_1x_2x_3x_4^2, x_1x_2x_3^2, x_1x_2^2, x_1^2 \rangle \\ & = & \langle x_1, x_2^6 \rangle \cap \langle x_1^2, x_2, x_3^6 \rangle \cap \langle x_1^2, x_2^2, x_3, x_4^4 \rangle \cap \langle x_1^2, x_2^2, x_3^2, x_4^2 \rangle \\ (d,e) = (3,3) & M = & \langle x_2^9, x_1x_3^{18}, x_1x_2x_4^{16}, x_1x_2x_3x_4^{14}, \ldots, x_1^3 \rangle \ \ (26 \text{ generators}) \end{array}$$

2.3 The Eliahou-Kervaire resolution

Throughout this Section, let the monomials $m_1, \ldots, m_r$ minimally generate a Borel-fixed ideal, and let u_i be the largest index of a variable dividing m_i.

Lemma 2.4 *Any monomial $m \in \langle m_1, \ldots, m_r \rangle$ can be written uniquely as a product $m = m_i m'$ such that $u_i \leq$ the smallest index of an indeterminate dividing m'.*

Proof: Uniqueness: Suppose $m = m_i m_i' = m_j m_j'$ both satisfy the condition, with $u_i \le u_j$. Then m_i and m_j agree in every variable whose index is $< u_i$. Now if x_{u_i} divides m_j' then $u_i = u_j$ by the assumed condition, whence one of m_i and m_j divides the other, so $i = j$. Otherwise, x_{u_i} does not divide m_j'. In this case the degree of x_{u_i} in m_i is $\le$ the degree of x_{u_i} in m_j, which equals the degree of x_{u_i} in m, so that again m_i divides m_j and $i = j$.

Existence: Suppose that $m = m_j m'$ for some j, but that $u_j > u =$: the smallest index of a variable dividing m'. Then Proposition 2.1 says that we can replace m_j by any minimal generator m_i dividing $m_j x_u / x_{u_j}$. By construction, $u_i \le u_j$, so either $u_i < u_j$, or $u_i = u_j$ and the degree of x_{u_i} in m_i is $\le$ the degree of x_{u_i} in m_j. This shows that we can't keep going on making such replacements forever. $\square$

Now apply this lemma to each $m = m_i x_u$ where $u < u_i$. If $m = m_i x_u = m_j m'$ as in the Lemma, then $x_u \mathbf{e}_i - m' \mathbf{e}_j$ is a syzygy of the ideal. In this way, we get a set of minimal first syzygies for a Borel-fixed ideal.

Example 2.5 A Borel-fixed ideal and its minimal first syzygies:

$$
\begin{array}{rrrrrrr}
\langle x_1x_2x_4^4, & x_1x_2x_3x_4^2, & x_1x_3^6, & x_1x_2x_3^2, & x_2^6, & x_1x_2^2, & x_1^2\rangle \\
x_3\,\mathbf{e}_1 & -x_4^2\,\mathbf{e}_2 & & & & & \\
x_2\,\mathbf{e}_1 & & & & & -x_4^4\,\mathbf{e}_6 & \\
x_1\,\mathbf{e}_1 & & & & & & -x_2x_4^4\,\mathbf{e}_7 \\
 & x_3\,\mathbf{e}_2 & & -x_4^2\,\mathbf{e}_4 & & & \\
 & x_2\,\mathbf{e}_2 & & & & -x_3x_4^2\,\mathbf{e}_6 & \\
 & x_1\,\mathbf{e}_2 & & & & & -x_2x_3x_4^2\,\mathbf{e}_7 \\
 & & x_2\,\mathbf{e}_3 & -x_3^4\,\mathbf{e}_4 & & & \\
 & & x_1\,\mathbf{e}_3 & & & & -x_3^6\,\mathbf{e}_7 \\
 & & & x_2\,\mathbf{e}_4 & & -x_3^2\,\mathbf{e}_6 & \\
 & & & x_1\,\mathbf{e}_4 & & & -x_2x_3^2\,\mathbf{e}_7 \\
 & & & & x_1\,\mathbf{e}_5 & -x_2^4\,\mathbf{e}_6 & \\
 & & & & & x_1\,\mathbf{e}_6 & -x_2^2\,\mathbf{e}_7
\end{array}
$$

Key Fact: These syzygies form the reduced Gröbner basis for the presentation module $\mathcal{M}$ of the Borel-fixed ideal with respect to any position-over-term (POT) order. Indeed, the initial terms of the syzygies above are relatively prime by [10, Proposition 2.9.4]. In the specific example here, the initial module is

$$
\begin{array}{rlll}
\mathrm{in}(\mathcal{M}) = & \langle x_1\,\mathbf{e}_1, & x_2\,\mathbf{e}_1, & x_3\,\mathbf{e}_1, \\
 & x_1\,\mathbf{e}_2, & x_2\,\mathbf{e}_2, & x_3\,\mathbf{e}_2, \\
 & x_1\,\mathbf{e}_3, & x_2\,\mathbf{e}_3, & \\
 & x_1\,\mathbf{e}_4, & x_2\,\mathbf{e}_4, & \\
 & x_1\,\mathbf{e}_5, & & \\
 & x_1\,\mathbf{e}_6\rangle & & \subset k[\mathbf{x}]^7
\end{array}
$$

Its resolution is a direct sum of Koszul complexes:

$$\begin{array}{ccccccccccc}
 & & S\,\mathbf{e}_1 & \longleftarrow & S^3 & \longleftarrow & S^3 & \longleftarrow & S & \longleftarrow & 0 & \\
 & & S\,\mathbf{e}_2 & \longleftarrow & S^3 & \longleftarrow & S^3 & \longleftarrow & S & \longleftarrow & 0 & \oplus \\
 & & S\,\mathbf{e}_3 & \longleftarrow & S^2 & \longleftarrow & S & \longleftarrow & 0 & & & \oplus \\
 & & S\,\mathbf{e}_4 & \longleftarrow & S^2 & \longleftarrow & S & \longleftarrow & 0 & & & \oplus \\
 & & S\,\mathbf{e}_5 & \longleftarrow & S & \longleftarrow & 0 & & & & & \oplus \\
 & & S\,\mathbf{e}_6 & \longleftarrow & S & \longleftarrow & 0 & & & & & \oplus \\
 & & & & & & & & & & & \\
0 & \longleftarrow & \operatorname{in}\mathcal{M} & \longleftarrow & S^{12} & \longleftarrow & S^8 & \longleftarrow & S^2 & \longleftarrow & 0 &
\end{array}$$

The resolution of in($\mathcal{M}$) is linear and lifts (by adding trailing terms) to the minimal *Eliahou-Kervaire* resolution of $\mathcal{M} \subset S^7$ and of $S^7/\mathcal{M}$, the given Borel-fixed ideal [14].

$$0 \longleftarrow S \xleftarrow{\quad (x_1x_2x_4^4 \quad x_1x_2x_3x_4^2 \quad \cdots \quad x_1^2) \quad} S^7 \longleftarrow S^{12} \longleftarrow S^8 \longleftarrow S^2 \longleftarrow 0$$

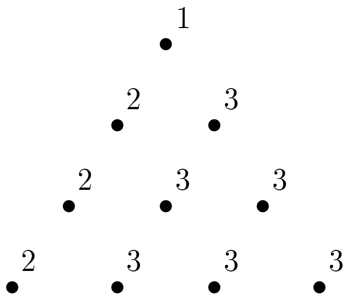
This lifting of linear resolutions of initial modules is a general phenomenon, being a consequence of the upper-semicontinuity of Betti numbers in flat families. And for Borel-fixed ideals, the Koszul behavior exhibited above in the resolution of the initial module is fundamental. Therefore, because we understand the numerics of Koszul complexes, we understand the numerics of Borel-fixed ideals:

Theorem 2.6 (Eliahou-Kervaire [14]) *If $I = \langle m_1, \ldots, m_r \rangle$ is Borel-fixed and u_i is the largest index of a variable dividing m_i, then*

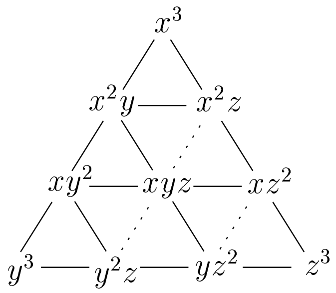
1. *The number of j-th syzygies of I is $\sum_{i=1}^{r} \binom{u_i - 1}{j}$, and*
2. *The numerator of the Hilbert series is I equals*

$$1 - \sum_{i=1}^{r} m_i \prod_{j=1}^{u_i - 1} (1 - x_j)\,.$$

Example 2.7 Powers $\mathfrak{m}^d$ of the maximal ideal are Borel-fixed and hence resolved by Eliahou-Kervaire. In the case $n = d = 3$, we get:



u_i



$\langle x, y, z \rangle^3$

$$
\begin{array}{ccccccccc}
S & \longleftarrow & S^{10} & \longleftarrow & S^{15} & \longleftarrow & S^{6} & \longleftarrow & 0 \\
 & & \begin{array}{c} 1 \\ 1\ 1 \\ 1\ 1\ 1 \\ 1\ 1\ 1\ 1 \end{array} & & \begin{array}{c} 0 \\ 1\ 2 \\ 1\ 2\ 2 \\ 1\ 2\ 2\ 2 \end{array} & & \begin{array}{c} 0 \\ 0\ 1 \\ 0\ 1\ 1 \\ 0\ 1\ 1\ 1 \end{array} & &
\end{array}
$$

The triangles of numbers below the resolution are the $\binom{u_i - 1}{j}$ of Theorem 2.6; they indicate how the initial module decomposes as a direct sum of Koszul complexes. It may seem that any minimal resolution breaks the symmetry under the group S_3 permuting the variables, but this is not necessarily so—see [25]. □

2.4 Lex-segment ideals

Let $<$ be a term order and $H\colon \mathbb{N} \to \mathbb{N}$ the $\mathbb{Z}$-graded Hilbert function of a homogeneous ideal I. The *segment ideal*, $I_{H,<}$, is the ideal k-spanned by the first $H(i)$ monomials with respect to $<$ in each degree i. It is Borel-fixed. If $<$ is the lexicographical order, we get the *lex-segment* ideal, $I_{H,\text{lex}}$. Historically, the reason for studying lex-segment ideals was because of their supremely bad (good?) numerical behavior.

Theorem 2.8 (Macaulay, [22]) *$I_{H,\text{lex}}$ has the highest degree generators among all (monomial) ideals with the same coarse Hilbert function.*

In Macaulay's theorem, it is enough to restrict our attention to monomial ideals, since any initial ideal of a $\mathbb{Z}$-graded ideal I has generators with at least the degrees of the generators of I.

The degrees of the generators, of course, are measuring the zeroth Betti numbers. One can also ask which ideals have the worst behavior with respect to the degrees of the higher Betti numbers. The ultimate statement is that lex-segment ideals take the cake, simultaneously for all Betti numbers.

Theorem 2.9 (Bigatti, Hulett [5, 21]) *$I_{H,\text{lex}}$ has the most minimal i-th syzygies for all i, among all (monomial) ideals with the same Hilbert function.*

Again, the upper semi-continuity of Betti numbers implies that we need only compare the lex-segment ideals with other monomial ideals.

3 Lecture III: Monomial ideals in three variables

Squarefree and Borel-fixed ideals each have their own advantages, the former yielding insight into combinatorics, and the latter into extremal numerical behavior in algebraic geometry. Their utility stems in both cases from our ability to express the appropriate information in terms of the defining properties of these special classes of monomial ideals; and in the Borel-fixed case, we are actually able to write down an explicit minimal free resolution.

However, this is not possible for general monomial ideals, at least not without making arbitrary choices. Even in the Borel-fixed case, the choices have really already