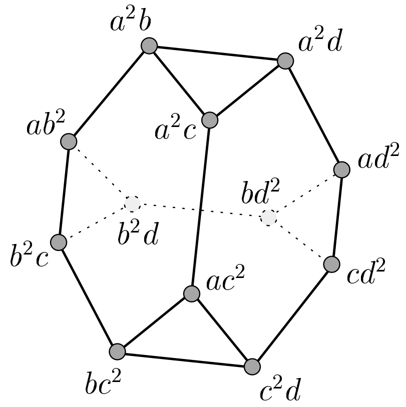
5 Lecture V: Cellular Resolutions

A number of times in the lectures up until now, we found that the constants in monomial matrices in free resolutions could be described in terms of geometric objects. Sometimes the geometric object seemed to come from nowhere (like Example 1.5), and sometimes from combinatorial data hidden in the generators and their least common multiples. Our aim in this lecture is to show how all monomial ideals "resolve themselves" via geometric resolutions, as suggested by the following picture. Here, the 12 vertices, 18 edges, and 8 faces in the polytope correspond to the Betti numbers 12, 18, and 8.



$$0 \longleftarrow S \longleftarrow S^{12} \longleftarrow S^{18} \longleftarrow S^{8} \longleftarrow S^{1} \longleftarrow 0$$

5.1 The basic construction

Let X be a *polyhedral cell complex* with r vertices (see [7, Section 6.2]). For instance, the cells of X may be the faces of a polytope, or a simplicial complex. Label each of the vertices of X by the generators $\mathbf{x}^{\mathbf{a}_1}, \ldots, \mathbf{x}^{\mathbf{a}_r}$ of a monomial ideal, and then label each face F of X by the least common multiple of the labels of its vertices:

$$\mathbf{x}^{\mathbf{a}_F} := \operatorname{lcm}(\mathbf{x}^{\mathbf{a}_i}, i \in F).$$

The exponent, $\mathbf{a}_F$ is called the *label* of the face F.

The polyhedral complex X is equipped with a *reduced chain complex*, which specializes to the usual reduced chain complex when X is simplicial.

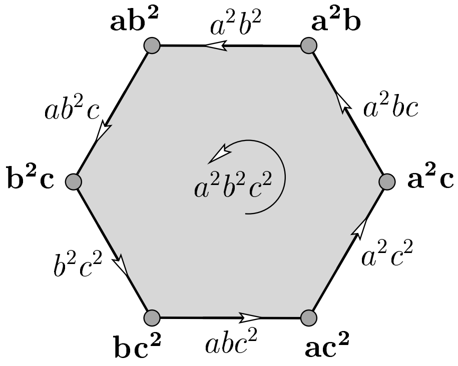
Definition 5.1 The *cellular free complex* $\mathbb{F}_X$ *supported on* X is the complex of $\mathbb{Z}^n$-graded free S-modules (with basis) given by monomial matrices as follows: use the boundary complex for X as the scalar entries, labeling the rows and columns by the face labels on X.

Without using monomial matrices, $\mathbb{F}_X$ and its boundary ∂ can be described as

$$\mathbb{F}_X = \bigoplus_{F \in X} S[-\mathbf{a}_F], \qquad \partial(F) = \sum_{\text{facets } G \text{ of } F} \operatorname{sign}(G, F) \, \frac{\mathbf{x}^{\mathbf{a}_F}}{\mathbf{x}^{\mathbf{a}_G}} \, G\,,$$

where F and G are thought of both as faces of X and as basis vectors in degrees $\mathbf{a}_F$ and $\mathbf{a}_G$. The sign for (G, F) is ± 1, and is part of the data in the boundary of the chain complex of X.

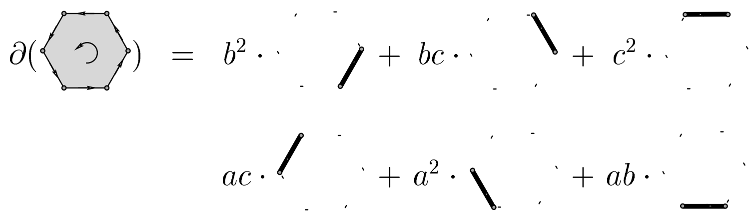
Example 5.2 The labeled hexagon



represents the complex $\mathbb{F}_X$, given by monomial matrices as:

$$0 \leftarrow S \xleftarrow{\begin{array}{c} \begin{array}{cccccc} a^2c & a^2b & ab^2 & b^2c & bc^2 & ac^2 \end{array} \\ 1\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} \end{array}} S^6 \xleftarrow{\begin{array}{c|cccccc} & a^2c^2 & a^2bc & a^2b^2 & ab^2c & b^2c^2 & abc^2 \\ \hline a^2c & 1 & -1 & 0 & 0 & 0 & 0 \\ a^2b & 0 & 1 & -1 & 0 & 0 & 0 \\ ab^2 & 0 & 0 & 1 & -1 & 0 & 0 \\ b^2c & 0 & 0 & 0 & 1 & -1 & 0 \\ bc^2 & 0 & 0 & 0 & 0 & 1 & -1 \\ ac^2 & -1 & 0 & 0 & 0 & 0 & 1 \end{array}} S^6 \xleftarrow{\begin{array}{c|c} & a^2b^2c^2 \\ \hline a^2c^2 & 1 \\ a^2bc & 1 \\ a^2b^2 & 1 \\ ab^2c & 1 \\ b^2c^2 & 1 \\ abc^2 & 1 \end{array}} S \leftarrow 0$$

The arrows denote the *orientations* of the faces, which determine the values of $\operatorname{sign}(G, F)$. For example, in the alternate way of writing cellular free complexes,



5.2 Exactness of cellular complexes

A subset $Q \subseteq \mathbb{Z}^n$ is an *order ideal* if $\mathbf{b} - \mathbf{a} \in Q$ whenever $\mathbf{b} \in Q$ and $\mathbf{a} \in \mathbb{N}^n$. Loosely, Q is "closed under going down" in the partial order on $\mathbb{Z}^n$. For an order ideal Q, we define a subcomplex of a labeled polyhedral complex X,

$$X_Q := \{F \in X \mid \mathbf{a}_F \in Q\}.$$

As a special case, for each $\mathbf{b} \in \mathbb{Z}^n$ define $X_{\preceq \mathbf{b}}$ to be the subcomplex of X consisting of faces whose degrees are coordinate-wise at most $\mathbf{b}$. For another example, say that $\mathbf{b}' \prec \mathbf{b}$ if $\mathbf{b}' \preceq \mathbf{b}$ and $\mathbf{b}' \neq \mathbf{b}$, and let $X_{\prec \mathbf{b}}$ be the subcomplex of X consisting of faces whose degrees are $\prec \mathbf{b}$.

Theorem 5.3 *$\mathbb{F}_X$ is exact if and only if $X_{\preceq \mathbf{b}}$ is acyclic over k (has no reduced homology) for all $\mathbf{b} \in \mathbb{N}^n$. In this case, $\mathbb{F}_X$ is called a* cellular resolution *of the monomial ideal M generated by its vertex labels, and the Betti numbers of M are*

$$\beta_{i,\mathbf{b}}(M) = \dim_k \widetilde{H}_{i-1}(X_{\prec \mathbf{b}}; k)\,.$$

Proof sketch: The free modules which contribute to the $\mathbb{Z}^n$-graded degree $\mathbf{b}$ part of the cellular free complex $\mathbb{F}_X$ are precisely those whose generators are in degrees $\preceq \mathbf{b}$; this proves the statement about acyclicity. The statement about Betti numbers comes from tensoring $\mathbb{F}_X$ with k. The resulting complex in degree $\mathbf{b}$ is the relative chain complex $\widetilde{\mathcal{C}}_{\bullet}(X_{\preceq \mathbf{b}}, X_{\prec \mathbf{b}}; k)$, whose homology is as claimed by the long exact sequence for relative homology, *because $X_{\preceq \mathbf{b}}$ is acyclic*—that is, because $\mathbb{F}_X$ is a resolution. □

Here is a nice example of how one can use the acyclicity criterion in a specific case. Recall that a polytope of dimension d is *simple* if every vertex meets d edges.

Proposition 5.4 *Let X be a simple polytope with facets $F_1, \ldots, F_n$ and vertices $v_1, \ldots, v_r$. Label each vertex v_i of X by the squarefree monomial $\prod_{v_i \notin F_j} x_j$. Then the labeled cell complex X supports a cellular resolution $\mathbb{F}_X$ of the monomial ideal I generated by the labels on its vertices. The resolution $\mathbb{F}_X$ is minimal and linear.*

Proof: A face G of X is contained in a facet F_j if and only if all of its vertices are contained in F_j. Hence, the label on G is

$$m_G = \frac{x_1 \cdots x_n}{\prod_{F_j \supseteq G} x_j} = \prod_{F_j \not\supseteq G} x_j\,.$$

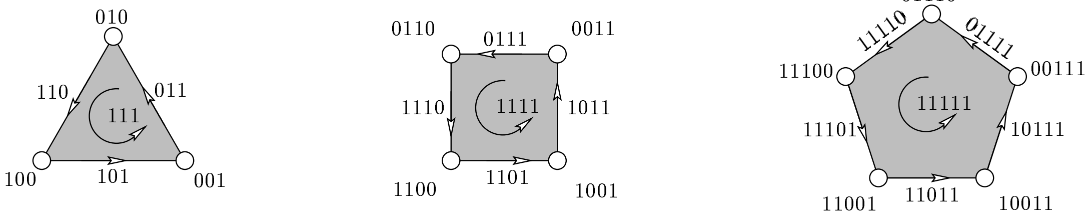
Acyclicity is easy to check: $X_{\preceq \mathbf{b}}$ is either empty, or it is the face of X obtained by intersecting all facets F_j for which x_j does not divide $x^{\mathbf{b}}$. In particular, $X_{\preceq \mathbf{b}}$ is contractible for all $\mathbf{b}$.

Minimality and linearity of $\mathbb{F}_X$ are both consequences of the fact that X is a simple polytope. Indeed, every face of a simple polytope is expressed uniquely as an intersection of facets. It follows from the equation for m_G that the labels on distinct faces G are unequal, which implies minimality. The formula for m_G also shows that m_G is a monomial of degree

$$\deg(m_G) = n - \dim(X) + \dim(G)$$

whence $\mathbb{F}_X$ is linear by the definition of the boundary map in $\mathbb{F}_X$. (Minimality also follows from linearity.) □

Example 5.5 When Q is two-dimensional, these resolutions follow the pattern



Note how the number of variables increases with the number of facets of Q. □

5.3 Examples of cellular resolutions

Taylor's resolution [36]. Let X be a simplex whose vertices are labeled by a set of monomials $\langle m_1, \ldots, m_r\rangle$. The resulting cellular free complex is a cellular resolution of $M = \langle m_1, \ldots, m_r\rangle$ called the *Taylor resolution*. Of course, the Taylor resolution tends not to be minimal: its length is r, and it has rank 2^r, and these should be compared with the bounds obtained using Scarf complexes in Lecture IV.

Scarf complex. If M is generic, then the minimal resolution of S/M is cellular, supported on the Scarf complex Δ_M.

CoScarf complex. There are ideals which are opposite to generic ideals, in a sense to be made precise in Lecture VI. They are called *cogeneric* monomial ideals, and their minimal resolutions are also cellular, supported on polyhedral complexes called *coScarf complexes*.

Permutohedron ideals. Let $\mathbf{u} = (u_1, u_2, \ldots, u_n) \in \mathbb{Z}^n$ with $u_1 < u_2 < \ldots < u_n$. By permuting the coordinates of $\mathbf{u}$, we obtain $n!$ points in $\mathbb{Z}^n \subset \mathbb{R}^n$ which are the vertices of an $(n-1)$-dimensional polytope called a *permutohedron* $P(\mathbf{u})$. The *permutohedron ideal* is the ideal $M(\mathbf{u})$ minimally generated by the monomials obtained by permuting the exponents of the monomial $\mathbf{x}^{\mathbf{u}} = x_1^{u_1} x_2^{u_2} \cdots x_n^{u_n}$. Labeling the vertices of the permutohedron with the generators of the permutohedron ideal in the natural way, we get a cellular resolution which is the minimal resolution of $M(\mathbf{u})$.

We now describe the degrees associated to each face of $P(\mathbf{u})$. Let $[n] := \{1, \ldots, n\}$ and $\mathbf{v} \in \mathbb{R}^n$. For each $\sigma \subseteq [n]$, define $v_\sigma := \sum_{i\in\sigma} v_i$ and $\alpha_\sigma := \sum_{i=1}^{|\sigma|} u_i$. The permutohedron has the inequality description

$$P(\mathbf{u}) := \left\{\mathbf{v} \in \mathbb{R}^n \mid v_{[n]} = \alpha_{[n]} \text{ and } v_\sigma \geq \alpha_\sigma \text{ for all } \sigma \subset [n]\right\}.$$

Each i-dimensional face is determined by a chain of distinct proper subsets of $[n]$

$$\sigma_1 \subset \sigma_2 \subset \cdots \subset \sigma_{n-i-1}$$

by setting $v_{\sigma_i} = \alpha_{\sigma_i}$ in the inequality description for $P(\mathbf{u})$. Given any such chain, define $\sigma_0 = \varnothing$ and $\sigma_{n-i} = [n]$. The label for the corresponding face F is then

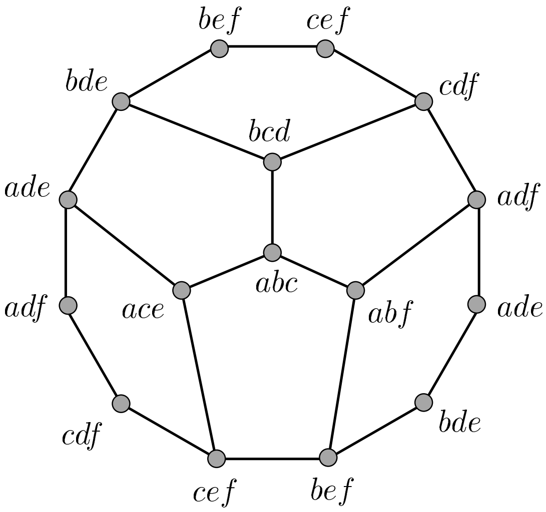
$$\mathbf{x}^{\mathbf{a}_F} = \prod_{j=1}^{n-i} \prod_{\ell\in\sigma_j\setminus\sigma_{j-1}} x_\ell^{\max\{\sigma_j\setminus\sigma_{j-1}\}}.$$

The hexagon in Example 5.2 is the minimal resolution of a permutohedron ideal $M(0, 1, 2)$. For more on the structure of permutohedra see [6], or check [23, Section 5] for more details on the connections with cellular resolutions.

Tree ideals. Let $M = \langle (\prod_{s\in I} x_s)^{n-|I|+1} \mid \varnothing \neq I \subseteq \{1,\ldots,n\}\rangle$ be the *tree ideal* in n variables (the name comes from the fact that M has the same number of standard monomials as there are labeled trees on $n+1$ vertices). Its minimal resolution is cellular, supported on the barycentric subdivision of an $(n-1)$-simplex. These will be investigated in Lecture VI, where we will find a relation to permutohedron ideals.

Irrelevant ideal of a toric variety. For a smooth (or just simplicial) projective toric variety there is a simple polytope (the moment polytope) whose vertices are labeled by the minimal generators of the so-called *irrelevant ideal of the Cox homogeneous coordinate ring* [9] as in Proposition 5.4. The corresponding cellular free complex is acyclic and minimal. We will say more about these ideals in Lecture VI.

The minimal triangulation of $\mathbb{RP}^2$. As in [7, Chapter 5], consider the Stanley-Reisner ideal of the minimal triangulation of the real projective plane. The cellular dual to the triangulation is a cell complex X consisting of six pentagons. We label the ten vertices of X with the minimal generators of the ideal:



If $\operatorname{char} k \neq 2$, then X is acyclic, and the cellular complex $\mathbb{F}_X$ is the minimal free resolution

$$0 \longleftarrow S \longleftarrow S^{10} \longleftarrow S^{15} \longleftarrow S^{6} \longleftarrow 0\,.$$

If $\operatorname{char} k = 2$, then X is not acyclic, and $\mathbb{F}_X$ is not a resolution of the ideal.

5.4 The hull resolution

Finally, our road leads to geometry: this section outlines a construction of Bayer and Sturmfels [4] which demonstrates how a canonical free resolution of a monomial ideal can be extracted from the way in which the exponents of the monomials sit inside of $\mathbb{R}^n$. Given $t \in \mathbb{R}$ and $\mathbf{a} \in \mathbb{Z}^n$ set $t^{\mathbf{a}} := (t^{a_1},\ldots,t^{a_n}) \in \mathbb{R}^n$. For a monomial ideal M and $t \in \mathbb{R}$ consider the polyhedron $P_t := \operatorname{conv}\{t^{\mathbf{a}} \mid \mathbf{x}^{\mathbf{a}} \in M\}$.

Lemma 5.6 *The face poset of P_t is independent of t for $t \gg 0$. The vertices of P_t correspond precisely to the minimal generators of M.*

The polyhedron P_t is never bounded, since there are monomials of arbitrarily high degree in any (nonzero) monomial ideal.

Example 5.7 If $M = S$, so $S/M = 0$, then P_t is the shift by $(1, \ldots, 1)$ of the real nonnegative orthant $\mathbb{R}^n_+$. If $M = \mathfrak{m} = \langle x_1, \ldots, x_n \rangle$, then P_t is obtained from this shifted positive orthant by chopping off the vertex. □

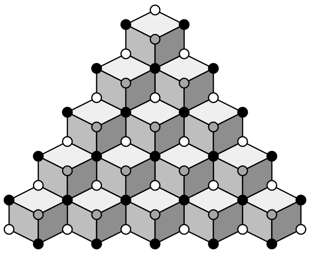
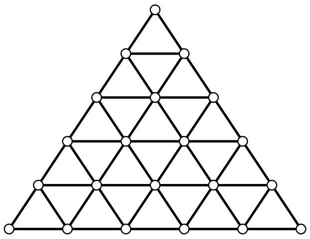
Definition 5.8 The *hull complex*, $\mathrm{hull}(M)$, is the polyhedral complex of bounded faces of P_t for $t \gg 0$. This complex is naturally labeled, with each vertex corresponding to a minimal generator of M.

Theorem 5.9 *The cellular free complex $\mathbb{F}_{\mathrm{hull}(M)}$ is a resolution of M.*

This *hull resolution* gives a canonical resolution of length at most n (the number of variables) for any monomial ideal, though it is usually non-minimal. Also, the free summands occurring in the hull resolution all have shifts which are least common multiples of generators, so that, for instance, the Betti numbers of a squarefree ideal are all in squarefree degrees.

Example 5.10

1. When $M = \mathfrak{m}$, the bounded faces are the simplex left by the truncated vertex in Example 5.7. This again yields a cellular description of the Koszul complex, as in Section 1.3.
2. The staircase diagram of $M = \langle x, y, z \rangle^5$ is at left below. For the hull resolution, look at the convex hull of $\{(t^i, t^j, t^k) \mid i + j + k = 5\}$ from the point $(1, \ldots, 1)$, for $t > 1$.

The hull resolution of $\langle x, y, z \rangle^5$, at right above, respects the S_3-symmetry. In general, the hull resolution of $\langle x, y, z \rangle^d$ has two classes of second syzygies: the "up" triangles and the "down" triangles. The three edges of any "down" triangle have the same label (the coordinates of the black dots in the staircase), and are the reason for non-minimality: there should be two edges in each such degree.

It is always possible to remove edges from a cellular resolution of an ideal in three variables to get a minimal cellular resolution [27]. In this case, we have to remove one edge from each "down" triangle. When the power d is congruent to 0 or 1 mod 3, it is even possible to retain the S_3-symmetry [25].

3. The Scarf resolution of Section 5.3 is the hull resolution if M is generic! This is nontrivial (see [4]), and will not be proved here.
4. The minimal resolution of a permutohedron ideal (Section 5.3) is the hull resolution.
5. Not every minimal cellular resolution is a hull resolution. We will see systematic failures in the next lecture, provided by cogeneric monomial ideals. For a specific counterexample, the cellular resolution of the real projective plane in Section 5.3 is not the hull resolution. It is a good exercise to check this directly by writing down the hull complex explicitly, but there is a much easier reason: the hull complex is independent of characteristic, whereas the minimal resolution of the real projective plane is not.
6. In fact, the cellular resolutions of Proposition 5.4 are usually not hull resolutions, either. This is essentially because not every simple polytope has the same combinatorial type as the convex hull of a collection of squarefree vectors (i.e. vectors in $\{0,1\}^n$). □

6 Lecture VI: Alexander duality

The essence of Alexander duality for monomial ideals is the familiar optical illusion in which isometric drawings of cubes look alternately like they're pointing "in" or "out" (see Example 6.12). This duality generalizes the combinatorial notion of Alexander duality for simplicial complexes, where it is manifested quite simply as a switch between generators and irreducible components. More generally, this switch works on resolutions of monomial ideals: the data contained in the common multiples of generators is similar (but dual) to the data contained in the greatest common divisors of the irreducible components. The goal of this lecture is to define Alexander duality for ideals, and give a glimpse of how it is manifested in resolutions.

6.1 Squarefree monomial ideals

The *Alexander dual* $I^\vee$ of a squarefree monomial ideal I is obtained by switching the roles of the minimal generators and the prime components. A minimal generator of the form $\mathbf{x}^\sigma$ becomes the prime component $\mathfrak{m}^\sigma$. If $I = I_\Delta$ is the face ideal of Δ, then the simplicial complex $\Delta^\vee$ is defined by $I_{\Delta^\vee} = I_\Delta^\vee$. More directly, $\Delta^\vee = \{\overline{\tau} \mid \tau \notin \Delta\}$ consists of the complements of the nonfaces of Δ. It is straightforward (but very confusing for the beginner) to check that $(I^\vee)^\vee = I$. Perhaps the easiest way to make this check is by viewing Alexander duality as duality in the boolean lattice of subsets of $\{1,\ldots,n\}$.

Example 6.1 The ideals I_Δ and I_Γ of Examples 1.3 and 1.4 are Alexander dual; their generators and irreducible components are arranged to make this clear. □