Step 1. Take the infinite hyperplane arrangement $\{x_i = j \mid i = 1, \ldots, n,\ j \in \mathbb{Z}\}$.

Step 2. Let $\mathcal{H}_L$ be its intersection with $L \otimes \mathbb{R}$.

Step 3. Form the quotient $\mathcal{H}_L/L$.

In particular, the i-faces of $\mathcal{H}_L/L$ are in 1-1 correspondence with the minimal i-th syzygies of $I_{\Lambda(L)}$. A particular example of the resolution described here is the Eagon-Northcott complex for the 2×2 minors of a generic $2 \times n$ matrix [3]. $\square$

8 Lecture VIII: Local cohomology

Among the first homological objects to be calculated explicitly for squarefree monomial ideals I were the local cohomology modules of S/I with support on $\mathfrak{m}$. Their Hilbert series were described in an unpublished theorem of M. Hochster (which finally found its way into [32]), and their module structure was later described explicitly by H. G. Gräbe [17]. On the other hand, local cohomology with support on the irrelevant monomial ideal in the Cox homogeneous coordinate ring can be used to compute sheaf cohomology on toric varieties, in analogy to the case of projective space [13, 28]. This application has prompted research into local cohomology with support on monomial ideals in general. N. Terai [38] made the calculation of the Hilbert series of the local cohomology modules of S with support on I, inspired by the techniques of Hochster; at the same time, M. Mustaţă [29] found the module structure. Then, using generalizations of Alexander duality for modules, E. Miller [24] showed that the answers provided by Terai and Mustaţă are actually equivalent to those provided by Hochster and Gräbe. Our aim in this final lecture is to present these results, along with the necessary background, in the spirit of geometry from previous lectures.

8.1 Preliminaries

The Čech complex. Local cohomology has many equivalent definitions, each with its own role in the theory. Hochster's local cohomology formula is proved using the most accessible of these, the *Čech complex* (more accurately called the *stable Koszul complex*),

$$\mathbb{C}^{\bullet} : 0 \to S \to \bigoplus_{i=1}^{n} S[x_i^{-1}] \to \cdots \to \bigoplus_{|\sigma|=r} S[x^{-\sigma}] \to \cdots \to S[x_1^{-1}, \ldots, x_n^{-1}] \to 0$$

(see Section 0.1 for notation). This is to be considered as a cochain complex (upper indices increasing from the copy of S sitting in cohomological degree 0), with the map between the summands $S[\mathbf{x}^{-\sigma}] \to S[\mathbf{x}^{-(\sigma \cup i)}]$ in $\mathbb{C}^{\bullet}$ being $\operatorname{sign}(i, \sigma \cup i)$ times the canonical localization homomorphism.

One way to think about the Čech complex in a $\mathbb{Z}^n$-graded way is to replace each summand $S[-\sigma]$ in the Koszul complex $\mathbb{K}^{\bullet}$ by the localization $S[\mathbf{x}^{-\sigma}]$. That this replacement is possible can be seen from monomial matrices, whose scalar entries can be used just as easily as maps between localizations as between free modules; in fact this was the original motivation for defining monomial matrices in [24]. To write one down, we employ the symbol $*$ in an exponent vector to indicate that the variable with that index has been inverted. Thus, if $S = k[x, y, z]$, the "shift" $S[(0, *, 0)]$ represents the localization $S[y^{-1}]$, while $S[(0, 0, 0)]$ retains its usual meaning (no variables are to be inverted).

Example 8.1 The Čech complex in 3 variables looks like

$$0 \longrightarrow S \xrightarrow{\begin{array}{cc} & \begin{matrix} {*}00 & 0{*}0 & 00{*} \end{matrix} \\ 000 & \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \end{array}} \begin{matrix} S[x^{-1}] \\ \oplus \\ S[y^{-1}] \\ \oplus \\ S[z^{-1}] \end{matrix} \xrightarrow{\begin{array}{cc} & \begin{matrix} 0{*}{*} & {*}0{*} & {*}{*}0 \end{matrix} \\ \begin{matrix} {*}00 \\ 0{*}0 \\ 00{*} \end{matrix} & \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ -1 & -1 & 0 \end{pmatrix} \end{array}} \begin{matrix} S[(yz)^{-1}] \\ \oplus \\ S[(xz)^{-1}] \\ \oplus \\ S[(xy)^{-1}] \end{matrix} \xrightarrow{\begin{array}{cc} & {*}{*}{*} \\ \begin{matrix} 0{*}{*} \\ {*}0{*} \\ {*}{*}0 \end{matrix} & \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \end{array}} S[(xyz)^{-1}] \longrightarrow 0\,,$$

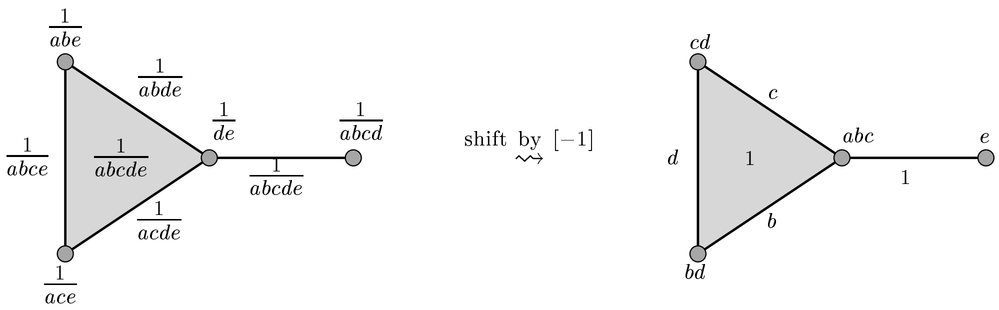
where the vectors are row vectors and the matrices act by multiplication on the right (as is natural for the cochain complex—see Example 0.6). □

Definition 8.2 Given a $\mathbb{Z}^n$-graded module M, the r^{th} *local cohomology module of M with support on* $\mathfrak{m}$ is the r^{th} cohomology $H^r_{\mathfrak{m}}(M) = H^r(M \otimes_S \check{C}^{\bullet})$.

The local cohomology of a finitely generated module M with support on $\mathfrak{m}$ is related to other homological invariants of M, via *local duality*. Before stating this fundamental theorem, we need to say something about Ext and Matlis duality.

Ext modules. Let $0 \leftarrow M \leftarrow \mathbb{F}_{\bullet}$ be a free resolution of M. Then $\operatorname{Hom}_S(\mathbb{F}_{\bullet}, S)$ is a cochain complex of free S-modules whose r^{th} cohomology is $\operatorname{Ext}^r_S(M, S)$, by definition. Recall that $\operatorname{Hom}_S(S[-\mathbf{a}], S) = S[\mathbf{a}]$; thus, in terms of monomial matrices, $\operatorname{Hom}_S(\mathbb{F}_{\bullet}, S)$ is obtained from $\mathbb{F}_{\bullet}$ by taking the negatives of the vector labels (inverses of monomial labels) on the columns and rows while taking the transposes of the scalar matrices.

Example 8.3 If $M = S/I$ has a cellular resolution supported on the labeled cell complex X, then the complex $\operatorname{Hom}_S(\mathbb{F}_X, S)$ is obtained by using as scalars the cochain complex of X, with the inverses of the monomial labels. For instance, if $I = I_\Gamma$ from Example 1.4 which has a cellular resolution $\mathbb{F}_X$ given in Example 1.5, then $\mathbb{F}^X := \operatorname{Hom}_S(\mathbb{F}_X, S)$ is obtained from surrounding the cochain complex of X in a monomial matrix with the labels at left below.



For various reasons, it is common to work with the $\mathbb{Z}^n$-graded shift $\mathrm{Hom}_S(\mathbb{F}_{\boldsymbol{\cdot}}, S)[-\mathbf{1}] = \mathrm{Hom}_S(\mathbb{F}_{\boldsymbol{\cdot}}, S[-\mathbf{1}])$. It is especially convenient for the case of $M = S/I$ for squarefree I, because the vector labels in monomial matrices for this shift are again in $\mathbb{N}^n$ (i.e. the monomials are again in S), as in the right picture above. The module $S[-\mathbf{1}]$ is sometimes called the *canonical module of* S, and denoted by ω_S. $\square$

Matlis duality. A $\mathbb{Z}^n$-graded module M is really just a collection of vector spaces $M_{\mathbf{a}}$ indexed by $\mathbf{a} \in \mathbb{Z}^n$, along with a given set of maps $\lambda_{\mathbf{a}\to\mathbf{b}} := \cdot\mathbf{x}^{\mathbf{b}-\mathbf{a}} : M_{\mathbf{a}} \to M_{\mathbf{b}}$ for each pair $\mathbf{a}, \mathbf{b} \in \mathbb{Z}^n$ with $\mathbf{a} \preceq \mathbf{b}$. Given M we can make another such collection of vector spaces and homomorphisms from the k-duals $\mathrm{Hom}_k(M_{\mathbf{a}}, k)$ and the transposes $\lambda^{\mathbf{a}\leftarrow\mathbf{b}} : \mathrm{Hom}_k(M_{\mathbf{b}}, k) \to \mathrm{Hom}_k(M_{\mathbf{a}}, k)$, as follows.

Definition 8.4 The *Matlis dual of* M is the module M^* whose $\mathbb{Z}^n$-graded piece in degree $-\mathbf{a}$ is $M^*_{-\mathbf{a}} = \mathrm{Hom}_k(M_{\mathbf{a}}, k)$. For $\mathbf{a} \preceq \mathbf{b} \in \mathbb{Z}^n$, the action of multiplication $\cdot\mathbf{x}^{\mathbf{b}-\mathbf{a}} : M^*_{-\mathbf{b}} \to M^*_{-\mathbf{a}}$ is by $\lambda^{\mathbf{a}\leftarrow\mathbf{b}}$.

From the point of view of Hilbert series, Matlis duality simply reverses the grading.

Lemma 8.5 $H(M^*; x_1, \ldots, x_n) = H(M; x_1^{-1}, \ldots, x_n^{-1})$.

If $M \to N$ is a homomorphism of $\mathbb{Z}^n$-graded modules, then Matlis duality induces a dual map $M^* \leftarrow N^*$. That is, Matlis duality is a contravariant functor. In addition, Matils duality is *exact*, since in each degree it is the functor $\mathrm{Hom}_k(-, k)$, which is exact. We are justified in calling this functor a duality, since a k-vector space is canonically isomorphic to its double-dual, at least when it is finite-dimensional.

Remark 8.6 For those that like the language of categories, a $\mathbb{Z}^n$-graded module M whose $\mathbb{Z}^n$-graded degrees are finite-dimensional is a subcategory of the category $\mathcal{K}$ of finite-dimensional k-vector spaces, with the morphisms in M being determined by monomials in S. Matlis duality is an anti-isomorphism of M with another subcategory M^* of $\mathcal{K}$ which is induced by the usual anti-isomorphism $\mathcal{K} \to \mathcal{K}$ given by the transpose $\mathrm{Hom}_k(-, k)$. $\square$

Now we are ready to state the local duality theorem, equating local cohomology with support on $\mathfrak{m}$ with the Matlis dual of an Ext module. For references and a general proof, see [7]; for a special treatment of the $\mathbb{Z}^n$-graded case, see [16] or [24].

Theorem 8.7 $H^r_{\mathfrak{m}}(M) \cong \mathrm{Ext}^{n-r}_S(M, \omega_S)^*$ *if* M *is finitely generated and* $\mathbb{Z}^n$*-graded.*

8.2 Maximal support

The theorem of Hochster [32, Theorem II.4.1] gives the $\mathbb{Z}^n$-graded Hilbert series $H(H^r_{\mathfrak{m}}(S/I_\Delta); \mathbf{x})$ for the local cohomology of S/I_Δ with support on $\mathfrak{m}$ in terms of the homology of the links in the Stanley-Reisner complex Δ.

Theorem 8.8 $H(H^r_{\mathfrak{m}}(S/I_\Delta); \mathbf{x}) = \displaystyle\sum_{\sigma\in\Delta} \dim_k \widetilde{H}^{r-|\sigma|-1}(\mathrm{link}_\sigma \Delta; k) \prod_{j\in\sigma} \frac{x_j^{-1}}{1 - x_j^{-1}}$.

Let us parse the statement. The product $\prod_{j\in\sigma}\frac{x_j^{-1}}{1-x_j^{-1}}$ is the sum of all Laurent monomials whose exponent vectors are nonpositive and have support exactly σ. Therefore, the formula for the Hilbert series of $H^r_{\mathfrak{m}}(S/I_\Delta)$ is just like the one for S/I_Δ in the third line of the displayed equation in Section 1.2, except that we are considering monomials with negative exponents, and we have added in nonnegative constants $\dim_k \widetilde{H}^{r-|\sigma|-1}(\mathrm{link}_\sigma\,\Delta; k)$ depending on r and σ.

As one might expect from the relation between $\mathbb{K}^\bullet$ and $\mathbb{C}^\bullet$, the proof of Theorem 8.8 is quite similar to that of Theorem 1.8, being accomplished (as usual) by checking which simplicial complex has its cochain complex in each $\mathbb{Z}^n$-graded degree. The main complication is in determining what relation the module $S/I_\Delta \otimes S[\mathbf{x}^{-\sigma}]$ has to the face ring of something (it turns out that the "something" is a link in Δ). The reader wishing details should consult [7] or [32].

From Hochster's Hilbert series formula, we find that whenever $b_i < -1$, the vector spaces $H^r_{\mathfrak{m}}(S/I_\Delta)_{\mathbf{b}}$ and $H^r_{\mathfrak{m}}(S/I_\Delta)_{\mathbf{b}+\mathbf{e}_i}$ have the same dimension. It is natural, therefore, to think that multiplication by x_i is an isomorphism between these graded components. This is indeed the case, as one can check even for the complex $S/I_\Delta\otimes\mathbb{C}^\bullet$ whose homology is $H^r_{\mathfrak{m}}(S/I_\Delta)$. But what about when $b_i = -1$? The answer is provided by an observation of H. G. Gräbe [17].

Theorem 8.9 *Identify each graded piece $H^r_{\mathfrak{m}}(S/I_\Delta)_{\mathbf{b}}$ with $\widetilde{H}^{r-|\sigma|-1}(\mathrm{link}_\sigma\,\Delta; k)$ whenever $\mathbf{b} \preceq \mathbf{0}$ and $\mathrm{supp}(\mathbf{b}) = \sigma$. Given $\sigma \in \Delta$ and $i \notin \sigma$ there are maps*

$$\widetilde{H}^{r-|\sigma\cup i|-1}(\mathrm{link}_{\sigma\cup i}\,\Delta; k) \longrightarrow \widetilde{H}^{r-|\sigma|-1}(\mathrm{link}_\sigma\,\Delta; k)$$

which agree with multiplication by x_i on the graded piece $H^r_{\mathfrak{m}}(S/I_\Delta)_{\mathbf{b}}$ as above whenever $b_i = -1$.

The choice of cohomology over homology in the Theorem may seem to be an error, given that the inclusion $\mathrm{link}_{\sigma\cup i}\,\Delta \to \mathrm{link}_\sigma\,\Delta$ of simplicial complexes induces maps on cohomology the other way. However, Gräbe's maps are not the usual ones induced by the inclusion; instead, they arise because the *co*chain complex $\widetilde{\mathcal{C}}^\bullet(\mathrm{link}_{\sigma\cup i}\,\Delta; k)$ can be made into a *sub*complex of $\widetilde{\mathcal{C}}^\bullet(\mathrm{link}_\sigma\,\Delta; k)$ by sending $e^*_\tau \mapsto \mathrm{sign}(i, \sigma\cup i)e^*_{\tau\cup i}$. The geometric content of such maps is not so easy to describe directly, but see Corollary 8.21, where they are related to geometry in $\Delta^\vee$ via Theorem 8.12, below.

8.3 Monomial support

An element m in a module M is said to have *support on an ideal I* if m is annihilated by some power of I. The reason why $H^r_{\mathfrak{m}}(-)$ is said to have *support on* $\mathfrak{m}$ is because $x_1, \ldots, x_n$ generate $\mathfrak{m}$, so the kernel of the map $M \otimes S \to \bigoplus_i M \otimes S[x_i^{-1}]$ is the set of elements which are annihilated by some power of each x_i—that is, those elements with support on $\mathfrak{m}$. If, in the Čech complex on s variables, we replace each variable x_i by a monomial m_i, then we get a complex $\mathbb{C}^\bullet(m_1, \ldots, m_s)$ whose first map is $S \to \bigoplus_i S[m_i^{-1}]$. Tensoring $\mathbb{C}^\bullet(m_1, \ldots, m_s)$ with M, the kernel of this first map becomes the set of elements in M which have support on $I = \langle m_1, \ldots, m_s\rangle$.

Definition 8.10 The r^{th} *local cohomology of a module M with support on* $I = \langle m_1, \ldots, m_s \rangle$ is the r^{th} cohomology $H_I^r(M)$ of the complex $M \otimes_S \mathbb{C}^{\bullet}(m_1, \ldots, m_s)$.

Again, there are many equivalent ways to define local cohomology with support on I, with the Čech complex being perhaps the most straightforward. One of the others characterizes $H_I^{\bullet}(M)$ as the right derived functors of $H_I^0(M) =$ "taking the set of elements with support on I". The standard way to calculate such derived functors is to take an injective resolution of M, apply $H_I^0(-)$ to it, and set $H_I^r(M)$ to be the r^{th} cohomology of the resulting complex. (For $\mathbb{Z}^n$-graded M and I, a $\mathbb{Z}^n$-graded injective resolution of M suffices.) But because I is finitely generated, $H_I^0(M)$ depends only on the radical of I, so we can assume I is a radical ideal whenever we write $H_I^{\bullet}(-)$. In this lecture, that means I is a squarefree monomial ideal. We will mention more of the equivalent characterizations of $H_I^{\bullet}(-)$ below; in particular, one of the purposes of Section 8.4 following this one is to show which other complexes can replace $\mathbb{C}^{\bullet}(m_1, \ldots, m_s)$ in the definition.

Inspired by Theorem 8.8 and its proof, N. Terai calculated the following [38].

Theorem 8.11 *The $\mathbb{Z}^n$-graded Hilbert series of $H_{I_\Delta}^r(S)$ is*

$$H(H_{I_\Delta}^r(S); \mathbf{x}) = \sum_{\sigma \in \Delta} \dim_k \widetilde{H}_{n-r-|\sigma|-1}(\mathrm{link}_\sigma \Delta; k) \prod_{i \in \overline{\sigma}} \frac{x_i^{-1}}{1 - x_i^{-1}} \prod_{j \in \sigma} \frac{1}{1 - x_j}.$$

This series is interpreted in a manner similar to Hochster's: $\prod_{i \in \overline{\sigma}} \frac{x_i^{-1}}{1-x_i^{-1}} \prod_{j \in \sigma} \frac{1}{1-x_j}$ is the sum of all Laurent monomials $\mathbf{x}^{\mathbf{b}}$ whose negative parts $\mathbf{b}_- := \sum_{b_i < 0} -b_i \mathbf{e}_i$ have support precisely $\overline{\sigma}$. Terai's proof follows the outline of Hochster's, using $\mathbb{C}^{\bullet}(m_1, \ldots, m_s)$ in place of $\mathbb{C}^{\bullet}$. Terai's argument is somewhat more complicated, though, since the simplicial complexes in his formula have n vertices, while those obtained from the Čech complex have as many vertices as there are generators of I_Δ.

Again, it is clear from Theorem 8.11 that many of the multiplication maps $\cdot x_i$ are between vector spaces of the same dimension, and as before, one can verify using the Čech complex of I_Δ that these maps are isomorphisms. While Terai made his calculation, M. Mustaţă simultaneously and independently observed these isomorphisms and went further, giving the module structure in terms of simplicial cohomology [29].

Theorem 8.12 *Each graded piece $H_{I_\Delta}^r(S)_{\mathbf{b}}$ is isomorphic to $\widetilde{H}^{r-2}(\Delta^\vee|_{\overline{\sigma}}; k)$, where $\mathbf{b} \in \mathbb{Z}^n$ and* $\mathrm{supp}(\mathbf{b}_-) = \overline{\sigma}$. *For $i \in \overline{\sigma}$, the maps on simplicial cohomology*

$$\widetilde{H}^{r-2}(\Delta^\vee|_{\overline{\sigma}}; k) \longrightarrow \widetilde{H}^{r-2}(\Delta^\vee|_{\overline{\sigma} \setminus i}; k)$$

induced by the inclusions $\Delta^\vee|_{\overline{\sigma} \setminus i} \to \Delta^\vee|_{\overline{\sigma}}$ agree with multiplication by x_i on the graded piece $H_{\mathfrak{m}}^r(S/I_\Delta)_{\mathbf{b}}$ whenever $b_i = -1$. If $b_i \neq -1$, then multiplication by x_i is an isomorphism on $H_{\mathfrak{m}}^r(S/I_\Delta)_{\mathbf{b}}$.

Upon comparison of Mustaţă's theorem with Terai's, it shouldn't be too surprising that the graded pieces of $H_{I_\Delta}^r(S)$ can be identified with $\widetilde{H}^{r-2}(\Delta^\vee|_{\overline{\sigma}}; k)$; after all,

$$\widetilde{H}_{n-r-|\sigma|-1}(\mathrm{link}_\sigma \Delta; k) \cong \widetilde{H}^{r-2}(\Delta^\vee|_{\overline{\sigma}}; k)$$

by Alexander duality in the form of Corollary 6.6. We will see in the next section why the theorems in this section look so similar to those in the previous section, and why duality enters the picture.

There is another proof of the theorems in this section which exploits *cellular injective resolutions*, particularly that of the canonical module ω_S, whose minimal $\mathbb{Z}^n$-graded injective resolution is a simplex (and Matlis dual to the usual Čech complex $\mathbb{C}^{\bullet}$). The point is that applying $H^0_{I_\Delta}(-)$ directly to the minimal $\mathbb{Z}^n$-graded injective resolution of $S = \omega_S[\mathbf{1}]$ yields a *cellular complex of injectives* whose homology is automatically given in each degree as the simplicial cohomology of a link in Δ. In fact, this proof is almost the same as that of Theorem 1.8; see [27] for details.

8.4 The Čech hull

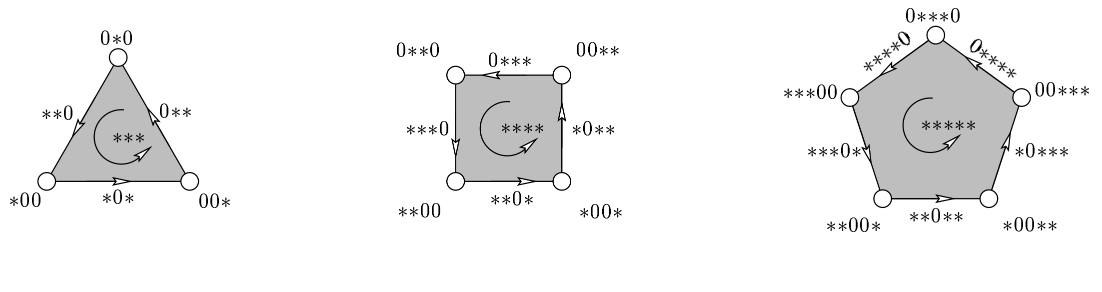
Mustaţă's proof of Theorem 8.12 used cellular resolutions in a particularly nice way, in combination with the characterization of $H^{\bullet}_I(M)$ as a limit over $t \in \mathbb{N}$ of modules $\mathrm{Ext}^{\bullet}_S(S/I^t, M)$. He was able to use the Taylor resolution (Section 5.3) to get very nice finite approximations to the limit. Although we will bypass these limits here, let's see what the Taylor resolution of S/I has to do with local cohomology supported on I.

Recall that the Taylor complex of the squarefree monomial ideal $I = \langle \mathbf{x}^{\sigma_1}, \ldots, \mathbf{x}^{\sigma_s} \rangle$ is supported on a simplex X whose s vertices are labeled by the minimal generators. The label on a face $G \in X$ is the monomial $m_G = \mathbf{x}^{\sigma_G} := \mathrm{lcm}\,(\mathbf{x}^{\sigma_j}, j \in G)$, and the monomial matrices for the Taylor resolution are filled with the boundary maps of X. On the other hand, the monomial matrices for the Čech complex $\mathbb{C}^{\bullet}(\mathbf{x}^{\sigma_1}, \ldots, \mathbf{x}^{\sigma_s})$ are filled with the coboundary maps of the *same* simplex X, while the monomial labels m_G are replaced by the vector labels $*\sigma_G$ that have a $*$ in the i^{th} slot for $i \in \sigma_G$ and 0 otherwise (cf. Example 8.1). The key point is that inverting $\mathbf{x}^{\sigma_i}$ for $i \in G$ is the same thing as inverting m_G; i.e. $S[\mathbf{x}^{-\sigma_i} \mid i \in G] = S[\mathbf{x}^{-\sigma_G}]$.

E. Miller showed how this transition from Taylor complex to Čech complex works for all free resolutions of S/I [24]. When the shifts are squarefree, one can use:

Definition 8.13 Suppose that $\mathbb{F}.$ is a free resolution of S/I_Δ that has monomial matrices $\{\Lambda_r\}$ with squarefree row and column labels. The *generalized Čech complex* $\mathbb{C}^{\bullet}_{\mathbb{F}}$ is the complex of localizations of S whose monomial matrix Λ^r is obtained from Λ_r by taking the transpose of the scalar entries and replacing each label σ by $*\sigma$. If $\mathbb{F}.$ is minimal, $\mathbb{C}^{\bullet}_{\mathbb{F}}$ is called the *canonical Čech complex of I* and denoted $\mathbb{C}^{\bullet}_I$.

Example 8.14 The canonical Čech complexes for polygons look like



using the cellular minimal resolutions of Example 5.5. In terms of the applications of local cohomology to sheaf cohomology on toric varieties, these diagrams illustrate how the canonical Čech complex reflects the geometry particularly well when the toric variety is projective and smooth (or merely simplicial). □

We have what seems to be a magical way of taking a complex $\mathbb{F}_{\bullet}$ of free modules and producing a complex $\check{\mathbb{C}}^{\bullet}_{\mathbb{F}}$ of localizations of S. But as with other kinds of magic, the secrets of this procedure become more transparent when it is broken down into smaller steps. We start by getting the transposes of the scalar matrices to show up, which they do in $\mathbb{F}^{\bullet} := \mathrm{Hom}_S(\mathbb{F}_{\bullet}, \omega_S)$ as in Example 8.3. Observe that when $S/I \leftarrow \mathbb{F}_{\bullet}$ is a free resolution, the complex $\mathbb{F}^{\bullet}$ of free modules has cohomology $\mathrm{Ext}^{\bullet}_S(S/I, \omega_S)$. Now how do we create localizations?

Definition 8.15 The *Čech hull* of an $\mathbb{N}^n$-graded module M is the $\mathbb{Z}^n$-graded module $\check{C}M$ whose degree $\mathbf{b}$ piece is

$$(\check{C}M)_{\mathbf{b}} = M_{\mathbf{b}_+} \quad \text{where} \quad \mathbf{b}_+ = \sum_{b_i \geq 0} b_i \mathbf{e}_i \,.$$

Equivalently,

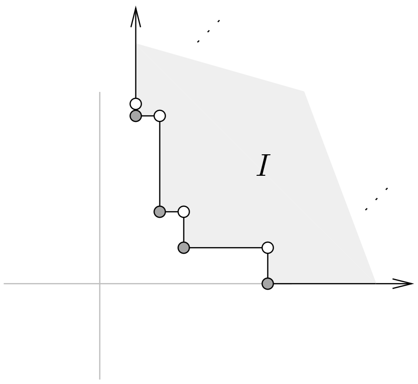
$$\check{C}M = \bigoplus_{\mathbf{b} \in \mathbb{N}^n} M_{\mathbf{b}} \otimes_k k[x_i^{-1} \mid b_i = 0]\,.$$

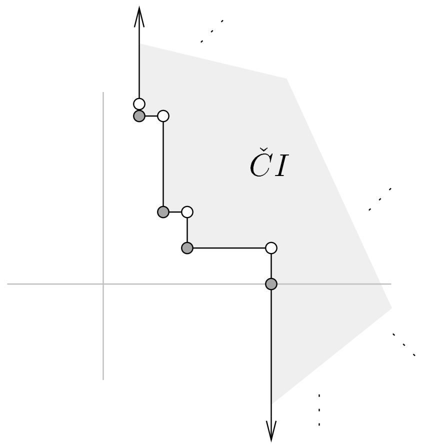
The action of multiplication by x_i is

$$\cdot x_i : (\check{C}M)_{\mathbf{b}} \to (\check{C}M)_{\mathbf{e}_i + \mathbf{b}} = \begin{cases} \text{identity} & \text{if } b_i < 0 \\ \cdot x_i : M_{\mathbf{b}_+} \to M_{\mathbf{e}_i + \mathbf{b}_+} & \text{if } b_i \geq 0 \end{cases}.$$

Note that $\mathbf{e}_i + \mathbf{b}_+ = (\mathbf{e}_i + \mathbf{b})_+$ whenever $b_i \geq 0$.

This definition was made in [23] for monomial ideals (as part of Alexander duality) where $\check{C}I$ is characterized as the largest monomial module inside of $T = S[x_1^{-1}, \ldots, x_n^{-1}]$ whose intersection with S is precisely I. The staircase diagram of $\check{C}I$ is obtained by pushing to negative infinity any point on the staircase diagram for I which touches the boundary of the positive orthant:





Heuristically, the first description of $\check{C}M$ in the definition says that if you want to know what $\check{C}M$ looks like in degree $\mathbf{b} \in \mathbb{Z}^n$, then check what M looks like in the nonnegative degree closest to $\mathbf{b}$; the second description says that the vector space $M_{\mathbf{a}}$ for $\mathbf{a} \in \mathbb{N}^n$ is copied into all degrees $\mathbf{b}$ such that $\mathbf{b}_+ = \mathbf{a}$.

Getting the desired localizations to appear is easy, using the Čech hull:

Lemma 8.16 $\check{C}(S[-\overline{\sigma}]) = S[\mathbf{x}^{-\sigma}][-\mathbf{1}]$.

If $\lambda : M \to N$ is a map of $\mathbb{N}^n$-graded modules, then there is a map $\check{C}\lambda : \check{C}M \to \check{C}N$ obtained by setting $(\check{C}\lambda)_{\mathbf{b}} = \lambda_{\mathbf{b}_+}$ for $\mathbf{b} \in \mathbb{Z}^n$. In particular, if $\mathbb{F}$ is a complex of free modules generated in squarefree degrees, then $\check{C}\mathbb{F}$ is a complex of $\mathbb{Z}^n$-graded localizations of S, by the Lemma. Recall from Example 8.3 that $\omega_S = S[-\mathbf{1}]$.

Proposition 8.17 *Suppose that $\mathbb{F}_{\bullet}$ is a free resolution of S/I_Δ whose shifts are all squarefree, and let $\mathbb{F}^{\bullet} = \mathrm{Hom}_S(\mathbb{F}_{\bullet}, \omega_S)$. Then $(\check{C}\mathbb{F}^{\bullet})[\mathbf{1}] = \mathbb{C}^{\bullet}_{\mathbb{F}}$.*

Proof: Since taking $\mathrm{Hom}_S(-, \omega_S)$ reverses the arrows (i.e. takes the transposes), the result follows from the Lemma because $\mathrm{Hom}_S(S[-\sigma], \omega_S) = S[-\overline{\sigma}]$. □

The complex $\check{C}\mathbb{F}^{\bullet}$ depends only on $\mathbb{F}_{\bullet}$ and not on the monomial matrix used to represent it; thus the same is true of $\mathbb{C}^{\bullet}_{\mathbb{F}}$ by the Proposition. In particular, the isomorphism class of the canonical Čech complex $\mathbb{C}^{\bullet}_I$, as a complex of $\mathbb{Z}^n$-graded S-modules, depends only on I. Even better, the Proposition can be used to define $\mathbb{C}^{\bullet}_{\mathbb{F}}$ for any free resolution of S/I. Doing so, Miller proved that there are lots of complexes of localizations of S which can take the place of $\mathbb{C}^{\bullet}(\mathbf{x}^{\sigma_1}, \ldots, \mathbf{x}^{\sigma_s})$ in Definition 8.10.

Theorem 8.18 $H^r_{I_\Delta}(M) = H^r(M \otimes \mathbb{C}^{\bullet}_{\mathbb{F}})$ *for any free resolution $\mathbb{F}_{\bullet}$ of S/I_Δ.*

If $\mathbb{F}_{\bullet}$ is the Taylor resolution, this Theorem agrees with Definition 8.13. It holds even for $\mathbb{Z}$-graded or ungraded modules M. Taking $M = S$, we see immediately how to calculate $H^r_{I_\Delta}(S)$.

Corollary 8.19 $H^r_{I_\Delta}(S) = (\check{C}\mathrm{Ext}^r_S(S/I_\Delta, \omega_S))[\mathbf{1}]$.

Proof: The Čech hull is *exact*—that is, applying $\check{C}$ to an exact sequence yields an exact sequence—because it copies the degree $\mathbf{b}_+$ part of any complex into degree $\mathbf{b}$. In particular, $\check{C}$ commutes with taking cohomology, as does shifting by $[\mathbf{1}]$. □

Corollary 8.20 *Terai's formula in Theorem 8.11 is equivalent to Hochster's formula in Theorem 8.8.*

Proof: $\mathrm{Ext}^r_S(S/I_\Delta, \omega_S) \cong H^{n-r}_{\mathfrak{m}}(S/I_\Delta)^*$ by Theorem 8.7, so

$$H(\mathrm{Ext}^r_S(S/I_\Delta, \omega_S); \mathbf{x}) = \sum_{\sigma \in \Delta} \dim_k \widetilde{H}_{n-r-|\sigma|-1}(\mathrm{link}_\sigma \Delta; k) \prod_{j \in \sigma} \frac{x_j}{1 - x_j}$$

by Lemma 8.5 applied to Theorem 8.8 (r and x_j^{-1} have been replaced by $n-r$ and x_j). Note that we replace the superscripts on $\widetilde{H}$ by subscripts when we take Matlis duals;

we will see why in the proof of the next corollary. Relying on the second description of the Čech hull in Definition 8.15, this yields

$$H(\check{C}\mathrm{Ext}^r_S(S/I_\Delta, \omega_S); \mathbf{x}) = \sum_{\sigma\in\Delta} \dim_k \widetilde{H}_{n-r-|\sigma|-1}(\mathrm{link}_\sigma \Delta; k) \prod_{j\in\sigma} \frac{x_j}{1-x_j} \prod_{i\in\overline{\sigma}} \frac{1}{1-x_i^{-1}}.$$

Shifting the input module by $[\mathbf{1}]$ multiplies this whole expression by $x_1^{-1}\cdots x_n^{-1} = \mathbf{x}^{-\sigma}\mathbf{x}^{-\overline{\sigma}}$, and gives the Hilbert series of $H^r_I(S)$ by Corollary 8.19. All of these steps are reversible, so Hochster's formula can similarly be derived from Terai's. □

Corollary 8.21 *The theorem of Mustaţă (8.12) is equivalent to that of Gräbe (8.9).*

Proof: The proof is the same as the previous corollary, except that we need to keep track of the multiplication maps between the $\mathbb{Z}^n$-graded degrees of the modules in question. The crux is that the homomorphisms

$$\widetilde{H}^{n-r-|\sigma\cup i|-1}(\mathrm{link}_{\sigma\cup i}\Delta; k) \longrightarrow \widetilde{H}^{n-r-|\sigma|-1}(\mathrm{link}_\sigma \Delta; k)$$

coming from Gräbe's theorem become the transpose homomorphisms

$$\widetilde{H}_{n-r-|\sigma|-1}(\mathrm{link}_\sigma \Delta; k) \longrightarrow \widetilde{H}_{n-r-|\sigma\cup i|-1}(\mathrm{link}_{\sigma\cup i}\Delta; k)$$

in the Matlis dual, by definition. Then, by Alexander duality, these agree with the maps

$$\widetilde{H}^{r-2}(\Delta^\vee|_{\overline{\sigma}}; k) \longrightarrow \widetilde{H}^{r-2}(\Delta^\vee|_{\overline{\sigma}\setminus i}; k)$$

in Mustaţă's theorem. □

The Čech hull had been defined before Theorems 8.11 and 8.12 were known, although its relation to local cohomology had not been found. The motivation for Theorem 8.18 was to prove *local duality with monomial support* [24], which generalizes Theorem 8.7—this is where one really needs the fact that the canonical Čech complex has minimal length.

A Appendix: Exercises

The following exercises are derived from those given at the June 1999 COCOA Summer School in Turin, Italy.

A.1 Exercises from Day 1

Exercise A.1.1 Let $n = 6$ and let Δ be the boundary of an octahedron.

(a) Determine I_Δ and $I_{\Delta^\vee}$.
(b) Compute their respective Hilbert series.
(c) Compute their minimal free resolutions.
(d) Interpret the Betti numbers obtained in part (c) in terms of simplicial homology.