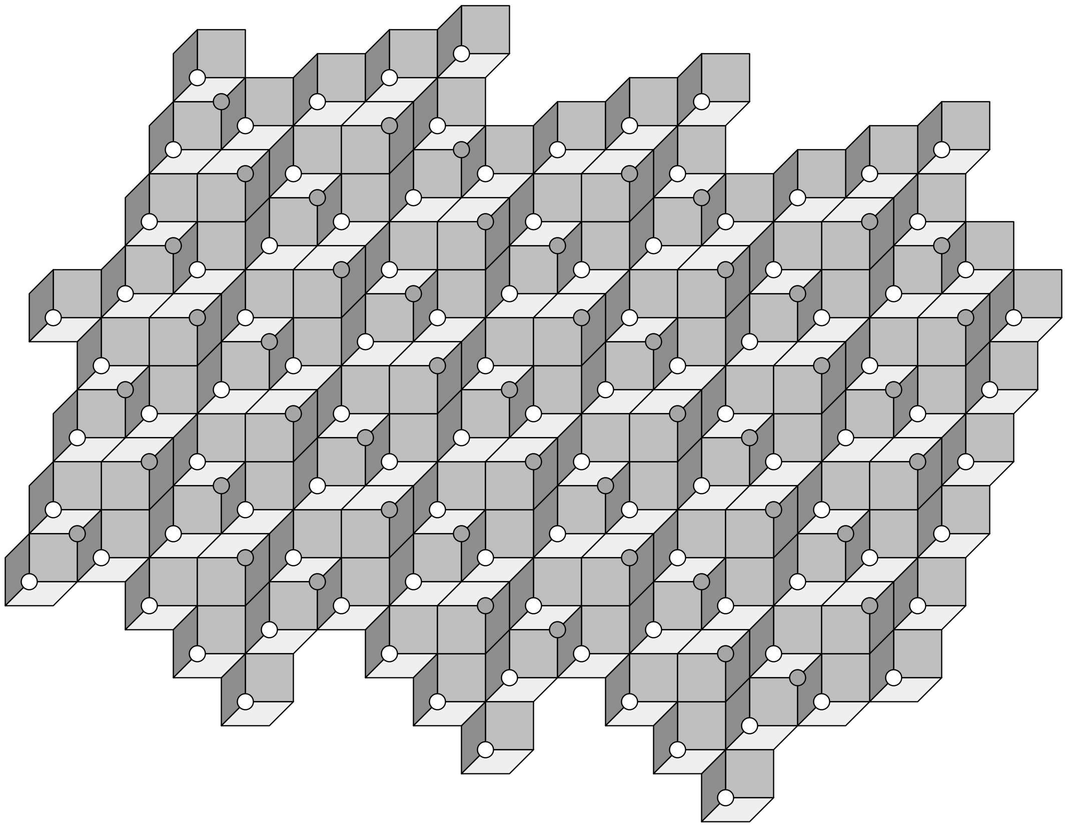
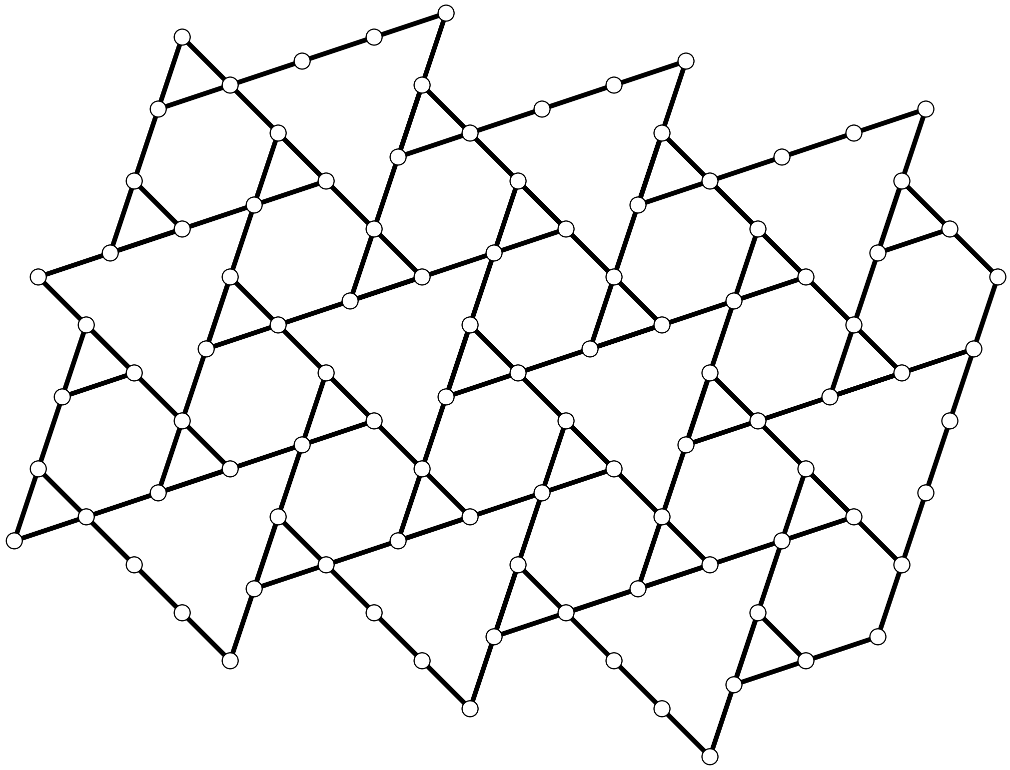
B.4 Solutions for Day 4

Solution B.4.1

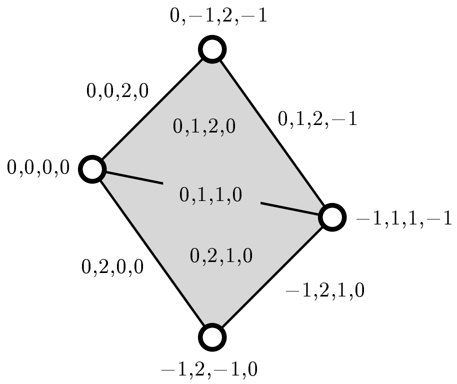


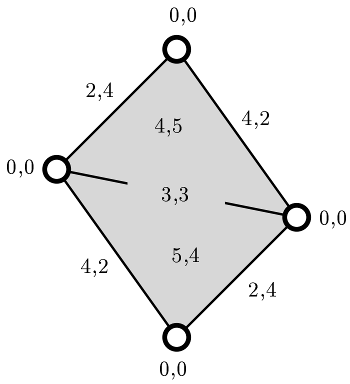


The hull resolution is not minimal: hull(M_L) is obtained from the cell complex above by introducing three short edges into each large "triangle", leaving only hexagons and small triangles. The hull resolution can be made minimal by removing any one of the three edges from each resulting small "down" triangle, but we have chosen the method in the diagram for aesthetic reasons.

Solution B.4.2

The hull complex of M_L consists of shifts of the up and down triangles below.





Since the labels are all distinct, the resolution is minimal. Notice how the opposing edge labels naturally become identified when the matrix defining L is applied to them.

Solution B.4.3

The canonical module ω is spanned as a k-vector subspace of $k(t, t^{-1})$ by the monomials $t^{-2}, t^{-1}, t, t^2, t^3, \ldots$ (with $t^0 = 1$ missing). These powers of t are the inverses of the powers of t that aren't in $k[t^3, t^4, t^5]$; see [7, Corollary 4.3.8]. Therefore, ω is the image under the functor π of the monomial module M_ω which is k-spanned by the *inverses* of the Laurent monomials *not* in M_L, where $L = \ker([4\ 3\ 5])$. What does this have to do with Alexander duality? A picture of the monomial module M_ω is obtained from the picture of M_L in Example 7.10 by turning the page over. This is the same instruction you were supposed to follow to get the Alexander dual ideal from the original in Example 6.12. And remember the definition of Alexander dual simplicial complex: $\Delta^\vee$ consists of the *complements* of faces that are *not* in Δ. For more on these connections, see [23].

Solution B.4.4 *By Bahman Engheta, Leah Gold*, and Ed Mosteig*

Finding the Hilbert series of $k[t^{20}, t^{24}, t^{25}, t^{31}]$ is equivalent to finding the Hilbert series of the isomorphic ring $Q[a, b, c, d]/I_L$, where L is the lattice generated by the kernel of the matrix [20 24 25 31] and the variables have degrees 20, 24, 25, and 31. The following CoCoA routine calculates the Hilbert series of $Q[a, b, c, d]/I_L$.

```
Use S ::= Q[t,x,y,z], Weights(20,24,25,31);
Poincare(S/Toric([[20,24,25,31]]);
```

We find that the numerator $p(t)$ is

$$\begin{aligned}
&-t^{183} - t^{178} - t^{177} - t^{167} - t^{166} - t^{161} \\
&+t^{158} + t^{153} + t^{152} + t^{147} + t^{146} + t^{143} + t^{142} + t^{141} + t^{137} + t^{136} + t^{135} + t^{130} \\
&-t^{122} - t^{112} - t^{110} - t^{96} - t^{93} - t^{80} - t^{75} + 1\,.
\end{aligned}$$

Each term t^d appearing in the above polynomial corresponds to a face of hull(I_L). The exponent d gives the degree of the face in $\mathbb{Z}^4/L \cong \mathbb{Z}$. Hence there are 7 edges with degrees 75, 80, 93, 96, 110, 112, and 122; 12 faces of dimension 2 with degrees 130, 135, 136, 137, 141, 142, 143, 146, 147, 152, 153, and 158; and 6 faces of dimension 3 with degrees 161, 166, 167, 177, 178, and 183.

Solution B.4.5 *By Bahman Engheta, Leah Gold*, and Ed Mosteig*

Let $S = Q[t^{20}, t^{24}, t^{25}, t^{31}]$. Then to find the largest amount of money which cannot be expressed by the 4 coins, we simply need to find the largest d for which t^d is not in S. That is, find the largest d such that $\dim_k(S_d) = 0$. The Hilbert series for S, which we found in the previous Solution can be written as

$$\begin{aligned}
1 &+ t^{20} + t^{24} + t^{25} + t^{31} + t^{40} + t^{44} + t^{45} + t^{48} + t^{49} + t^{50} + t^{51} + t^{55} + t^{56} + t^{60} + t^{62} \\
&+ t^{64} + t^{65} + t^{68} + t^{69} + t^{70} + t^{71} + t^{72} + t^{73} + t^{74} + t^{75} + t^{76} + t^{79} + t^{80} + t^{81} + t^{82} \\
&+ \textstyle\sum_{d=84}^{\infty} t^d\,.
\end{aligned}$$

As we know, the coefficient of t^d in the Hilbert series is $\dim_k(S_d)$. Hence, the largest amount of money which cannot be expressed by the 4 coins is 83 units. This number is most simply calculated as the degree of the Hilbert series as a rational function—that is, the degree of the numerator ($= 183$) minus the sum of the degrees of the variables ($= 100$).

Solution B.4.6

This exercise is analogous to, but more complicated than, Exercise A.2.5. Since L is generic, the hull complex is the Scarf complex, which makes things easier. Here's the problem: although the terms in the numerator of the $\mathbb{Z}^n/L$-graded Hilbert series of S/I_L are in bijection with the faces of hull(I_L) for generic L, it's harder to get the partial order information on the faces from the same source. Therefore, we prefer to get the partial order on the face labels before quotienting hull(M_L) by L. We will accomplish this by choosing a `WeightsMatrix` which records the finer $\mathbb{Z}^n$-grading in a coherent way.

First things first: let's find generators for I_L. We need to have a positive functional which vanishes on L, since CoCoA requires homogeneity with respect to the first row of the `WeightsMatrix` in order to apply the operator `Toric`.

```
M := Mat([[-7,-5,3,8],[4,-7,9,-1]]);
K := LinKer(M);
K;
[[-3, 0, 1, -3], [-5, 25, 22, 3]]
```

```
-------------------------------
K[2] - 2K[1];
[1, 25, 20, 9]
-------------------------------
```

As in Solution B.2.5, we now define our ring with one dummy variable `t` to make things homogeneous. The rest of the `WeightsMatrix` keeps track of the $\mathbb{Z}^n$-grading. For now, the ideal `J` is defined by the binomials coming directly from the two row vectors of `M`.

```
Use S ::= Q[t,a,b,c,d], Weights(Mat([[1,1,25,20,9],
                                     [0,1,0,0,0],
                                     [0,0,1,0,0],
                                     [0,0,0,1,0],
                                     [0,0,0,0,1]]));
Pos(V) :=  1/2(V + [Abs(X) | X In V]);
Neg(V) := -1/2(V - [Abs(X) | X In V]);
J := Ideal([LogToTerm(Concat([0],Pos(X)))
         - LogToTerm(Concat([0],Neg(X))) | X In M]);
J;
Ideal(-a^7b^5 + c^3d^8, a^4c^9 - b^7d)
-------------------------------
```

The lattice ideal $\mathtt{I} = I_L$ contains `J`, and is obtained from `J` by saturation with respect to the product of all the variables. CoCoA has a new built-in command to do this:

```
I := Toric(Gens(J));
I;
Ideal(-a^7b^5 + c^3d^8, a^4c^9 - b^7d, a^11c^6 - b^2d^9, -a^18b^3c^3 +
d^17, -a^3b^12 + c^12d^7, -b^19 + ac^21d^6)
-------------------------------
```

The faces of $\mathrm{hull}(I_L) = \mathrm{hull}(M_L)/L$ are represented once each in the $\mathbb{Z}^n/L$-graded Hilbert series of S/I_L, and our `Weights` are set up to give us $\mathbb{Z}^n$-graded labels for them, listed in `Scarf` (along with the corresponding ± 1 coefficient in the Hilbert series).

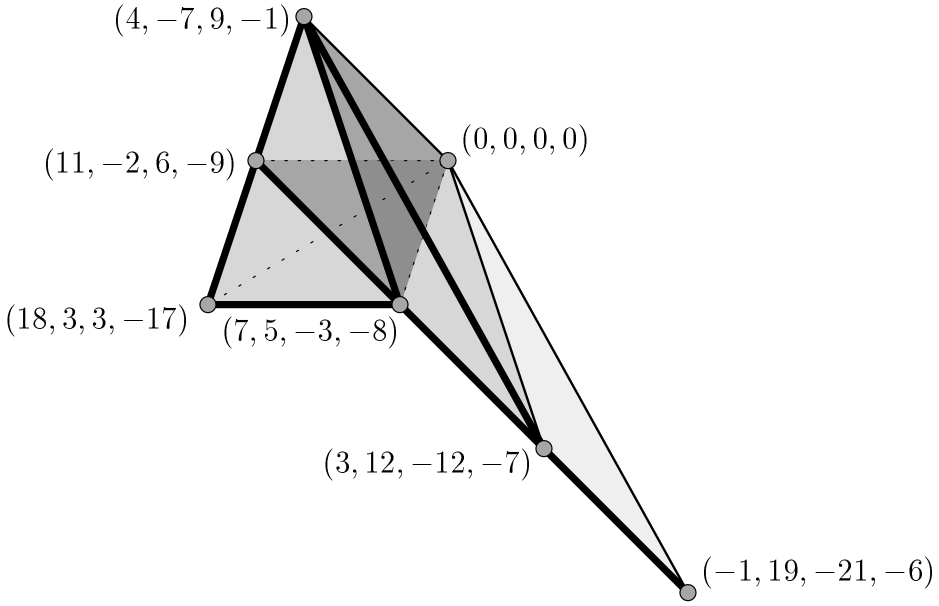
```
P := Poincare(S/I);
P;
---  Non-simplified HilbertPoincare' Series  ---
( - t^487a^7b^12c^9 + t^484a^4b^12c^9 + t^478a^3b^19 - t^475b^19 -
t^316a^11b^5c^9 + t^312a^7b^5c^9 + t^307a^7b^12 - t^303a^3b^12 -
t^263a^18b^5c^6 + t^256a^11b^5c^6 + t^213a^18b^3c^6 +
t^203a^18b^5c^3 + t^191a^11c^9 - t^184a^4c^9 - t^153a^18b^3c^3 -
t^131a^11c^6 - t^132a^7b^5 + 1) / ( (1-t) (1-ta) (1-t^25b) (1-t^20c)
(1-t^9d) )
-------------------------------
```

```
Scarf := [[X[1],Tail(X[2])] | X In P[1]];
Scarf;
[[-1, [7, 12, 9, 0]], [ 1, [4, 12, 9, 0]], [ 1, [3, 19, 0, 0]],
 [-1, [0, 19, 0, 0]], [-1, [11, 5, 9, 0]], [ 1, [7,  5, 9, 0]],
 [ 1, [7, 12, 0, 0]], [-1, [3, 12, 0, 0]], [-1, [18, 5, 6, 0]],
 [ 1, [11, 5, 6, 0]], [ 1, [18, 3, 6, 0]], [ 1, [18, 5, 3, 0]],
 [ 1, [11, 0, 9, 0]], [-1, [4,  0, 9, 0]], [-1, [18, 3, 3, 0]],
 [-1, [11, 0, 6, 0]], [-1, [7,  5, 0, 0]], [ 1, [ 0, 0, 0, 0]]]
-------------------------------
```

Hopefully the lifts to $\mathbb{Z}^n$ we have chosen for the labels are coherent enough that the partial ordering on them is the desired Scarf complex. For each face `X` of $\mathrm{hull}(I_L)$, `Leq` picks out the other faces whose labels divide the label on `X` and whose dimension is of opposite parity (this is just a convenience which avoids listing too many faces).

```
Divides(Y,X) := Mod(LogToTerm(Y),LogToTerm(X)) = 0;
Leq(X,Scarf) := [Y[2] | Y In Scarf And Divides(X[2],Y[2])
                        And X[1] = -Y[1] And Y[2] <> [0,0,0,0]];
PartOrder := [[X[2],Leq(X,Scarf)] | X In Scarf];
PartOrder;
[[[7, 12, 9, 0], [[4, 12, 9, 0], [7, 5,  9, 0], [7, 12, 0, 0]]],
 [[4, 12, 9, 0], [[3, 12, 0, 0], [4, 0,  9, 0]]],
 [[3, 19, 0, 0], [[0, 19, 0, 0], [3, 12, 0, 0]]],
 [[0, 19, 0, 0], [ ]],
 [[11, 5, 9, 0], [[7, 5,  9, 0], [11, 5, 6, 0], [11, 0, 9, 0]]],
 [[7,  5, 9, 0], [[4, 0,  9, 0], [7,  5, 0, 0]]],
 [[7, 12, 0, 0], [[3, 12, 0, 0], [7,  5, 0, 0]]],
 [[3, 12, 0, 0], [ ]],
 [[18, 5, 6, 0], [[11, 5, 6, 0], [18, 3, 6, 0], [18, 5, 3, 0]]],
 [[11, 5, 6, 0], [[11, 0, 6, 0], [7,  5, 0, 0]]],
 [[18, 3, 6, 0], [[18, 3, 3, 0], [11, 0, 6, 0]]],
 [[18, 5, 3, 0], [[18, 3, 3, 0], [7,  5, 0, 0]]],
 [[11, 0, 9, 0], [[4,  0, 9, 0], [11, 0, 6, 0]]],
 [[4,  0, 9, 0], [ ]],
 [[18, 3, 3, 0], [ ]],
 [[11, 0, 6, 0], [ ]],
 [[7,  5, 0, 0], [ ]],
 [[0,  0, 0, 0], [ ]]]
-------------------------------
```

This information is enough to draw a picture, although we will see that there's still some partial ordering left to do.



If B is the simplicial complex outlined in bold and containing $\{3$ triangles plus an edge$\}$, then the diagram is supposed to look like the cone over B. The complex $\mathrm{hull}(I_L)$ is obtained by removing B from the above cone, and consists of 3 tetrahedra, 8 triangles, 6 edges, and 1 vertex; check `Res(S/I)` for verification. Each face of B is identified in $\mathrm{hull}(I_L)$ with a face appearing elsewhere in the diagram. For instance,

$$\big\{(4,-7,9,-1),(7,5,-3,-8),(3,12,-12,-7)\big\} \equiv \big\{(0,0,0,0),(3,12,-12,-7),(-1,19,-21,-6)\big\}\,,$$

is an equivalence modulo L, as is

$$\big\{(11,-2,6,-9),(18,3,3,-17),(7,5,-3,-8)\big\} \equiv \big\{(4,-7,9,-1),(11,-2,6,-9),(0,0,0,0)\big\}\,.$$

The labels in `PartOrder` are the least common multiples of the labels on the vertices.

You may notice something strange about `PartOrder`: it not quite complete. In particular, each tetrahedron seems to have only 3 facets, and each triangle only 2 facets. The $\mathbb{Z}^n$-grading has "forgotten" that the triangle

$$\big\{(0,0,0,0),(3,12,-12,-7),(-1,19,-21,-6)\big\}$$

is a face of the tetrahedron

$$\big\{(0,0,0,0),(4,-7,9,-1),(7,5,-3,-8),(3,12,-12,-7)\big\}$$

after we quotient by L. In general, we have to add to the partial order the relations determined by the fact that every face in $\mathrm{hull}(I_L)$ has one of its boundary faces in B. See [30] for details.

Question for further thought: Were we lucky that the lifting of the $\mathbb{Z}^n/L$-grading gave us a partially ordered set that was already a simplicial complex, without having to fill in more faces, or is this a general phenomenon? What choices did we make (implicitly or otherwise) that made things come out right? To understand the question better, look what happens to the last diagram in Example 7.10 if we choose the $\mathbb{Z}^n$-graded lift $(1, 0, 1)$ for the interior edge instead of $(0, 3, 0)$.

Solution B.4.7 See Example 7.17; we do not give details here.

References

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