Endpoints for multiparameter persistence

Ezra Miller

Duke University, Department of Mathematics
and Department of Statistical Science

ezra@math.duke.edu

Institute for Mathematical and Statistical Innovation (IMSI)

Topological Data Analysis

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Outline

1. Persistent homology
2. Fruit fly wing veins
3. Probability distributions
4. Intervals
5. Functorial endpoints over $\mathbb{R}$
6. Socles in multiple parameters
7. Birth and death posets
8. QR codes
9. Applications

Collaborators

- Faculty


Paul Bendich
Math
Duke


David Houle
Biology
Florida State


Steve Marron
Stat/OR
UNC


Sean Skwerer
AV Robotics
Cruise

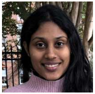
- Postdocs


Woojin Kim
Math
Duke


Maggie Regan
Math
Duke


Ashleigh Thomas
Math Bio
Georgia Tech

- Graduate students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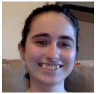
Math
Duke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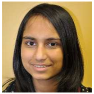
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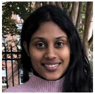
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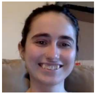
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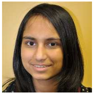
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$\Rightarrow Q$ is a partially ordered set: $X_q \subseteq X_{q'} \Leftrightarrow q \preceq q'$

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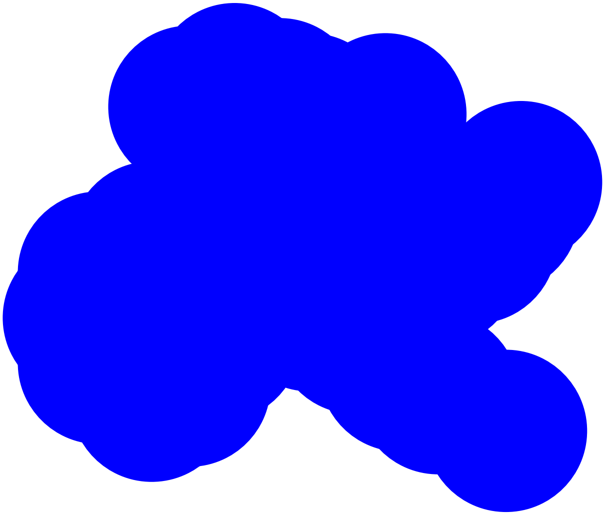
Example: expanding balls



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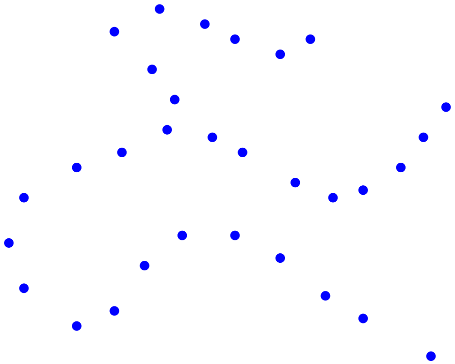

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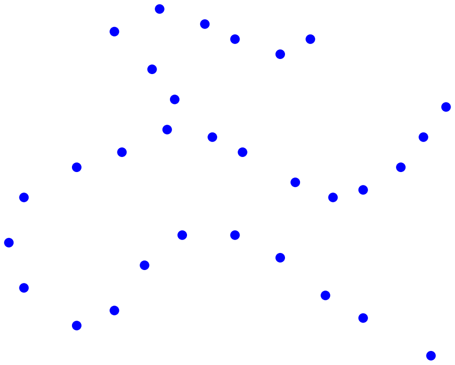
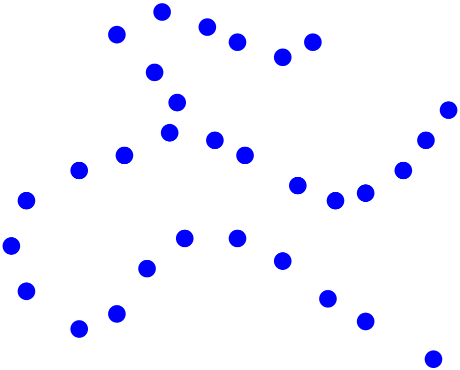


$\dim(H_0) = 31$

Example: expanding balls



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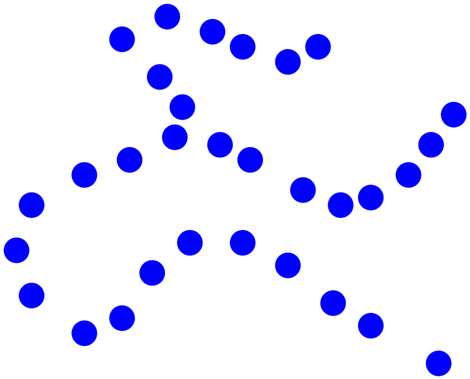

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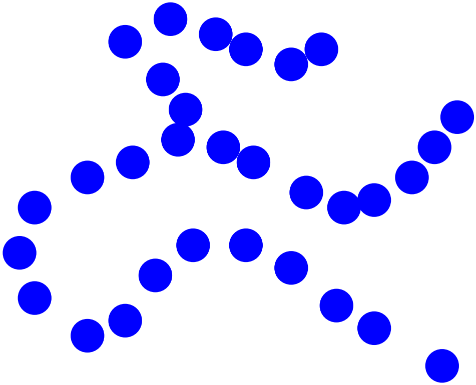
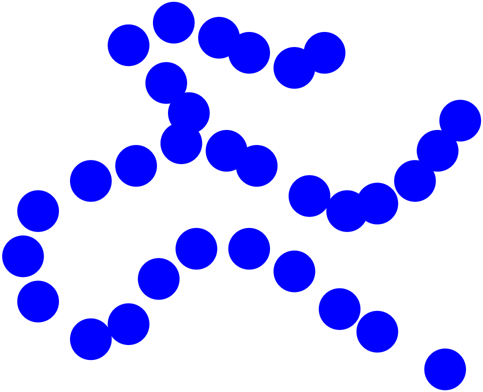


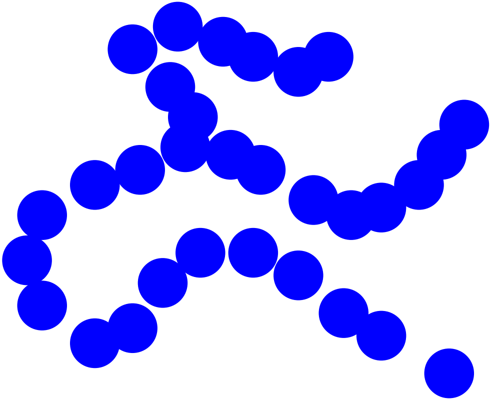
$\dim(H_0) = 26$

Example: expanding balls



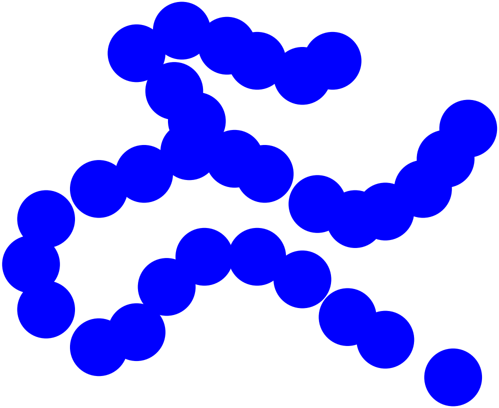
$\dim(H_0) = 21$

Example: expanding balls



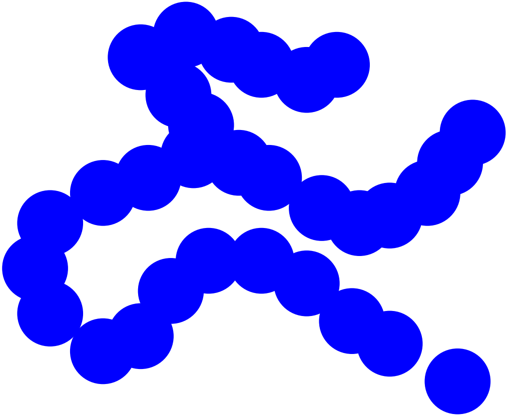
$\dim(H_0) = 12$

Example: expanding balls



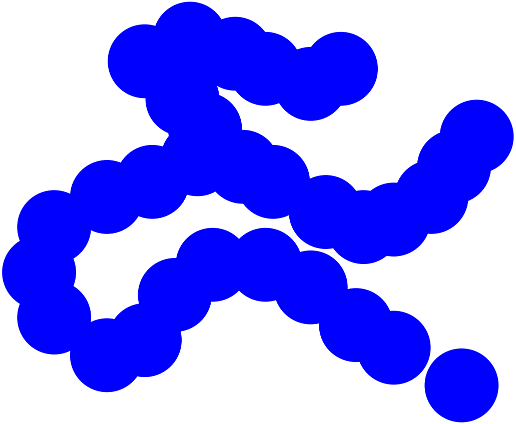
$\dim(H_0) = 6$

Example: expanding balls



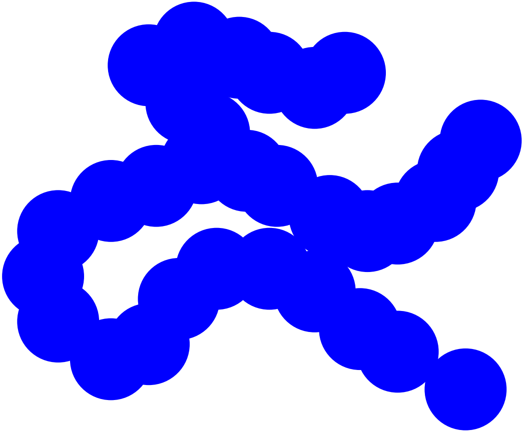
$\dim(H_0) = 2$

Example: expanding balls



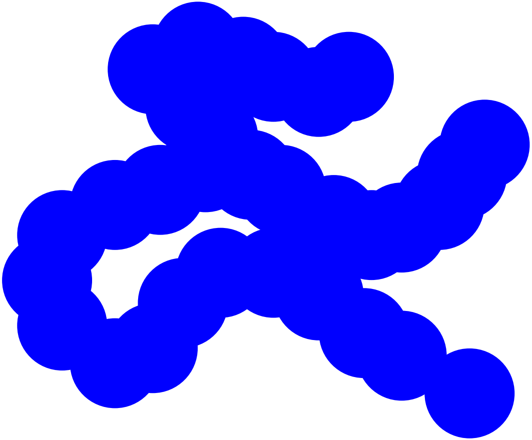
$\dim(H_0) = 2$

Example: expanding balls



$\dim(H_0) = 1 \qquad \dim(H_1) = 2$

Example: expanding balls



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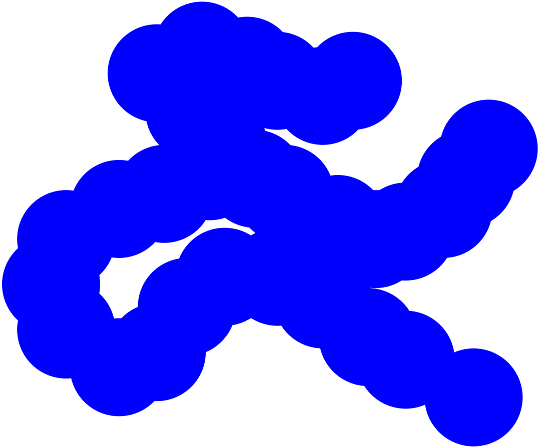

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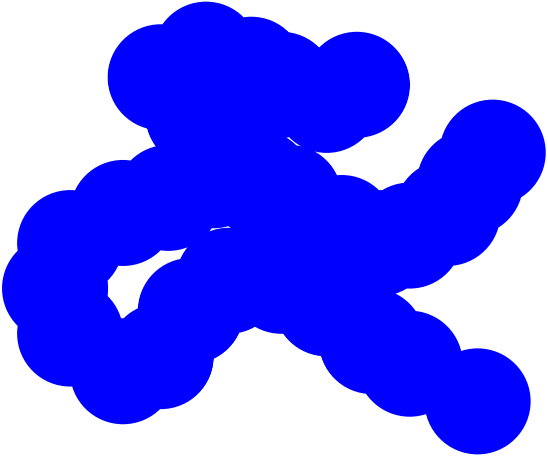
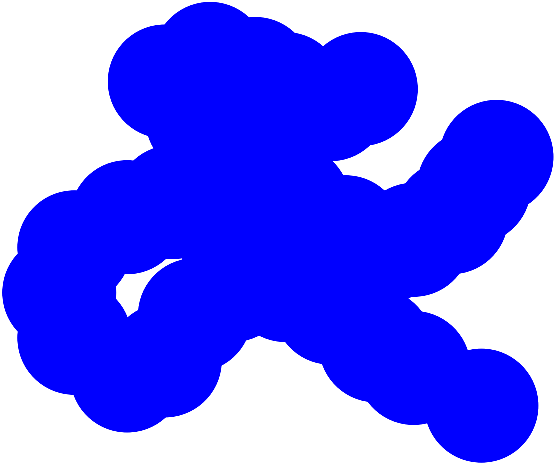


$\dim(H_0) = 1$ $\dim(H_1) = 3$

Example: expanding balls



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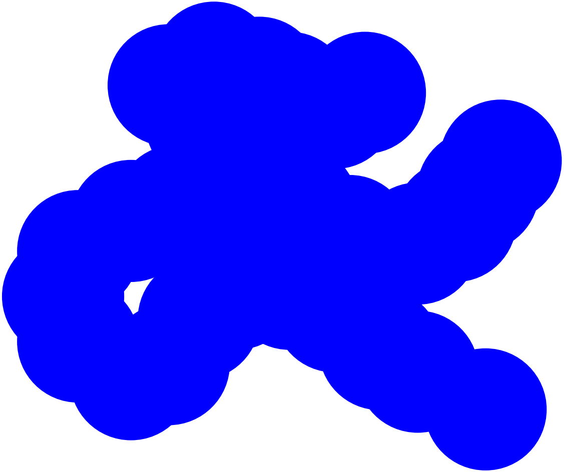

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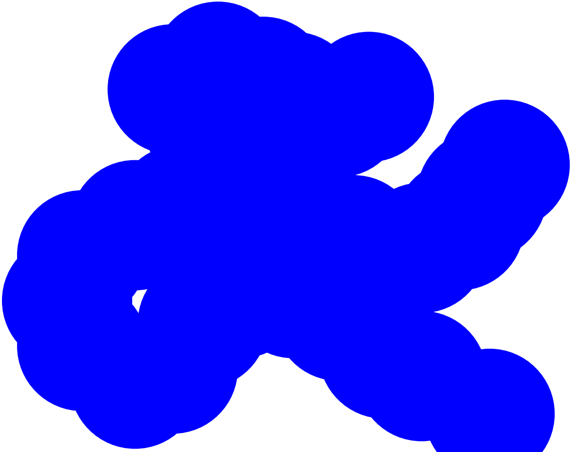

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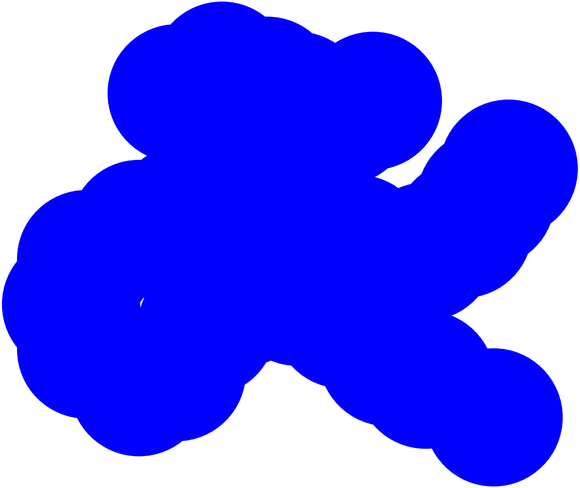

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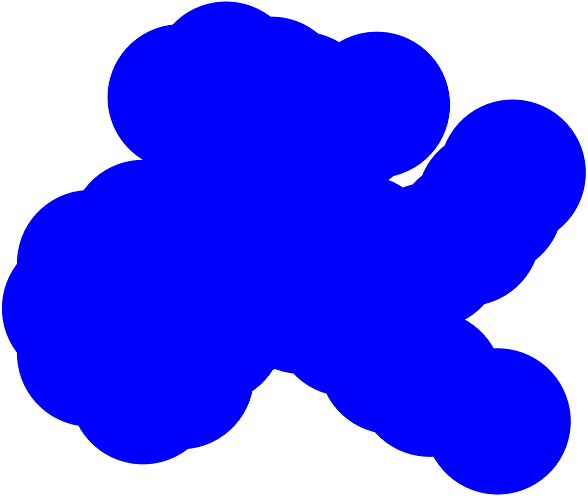

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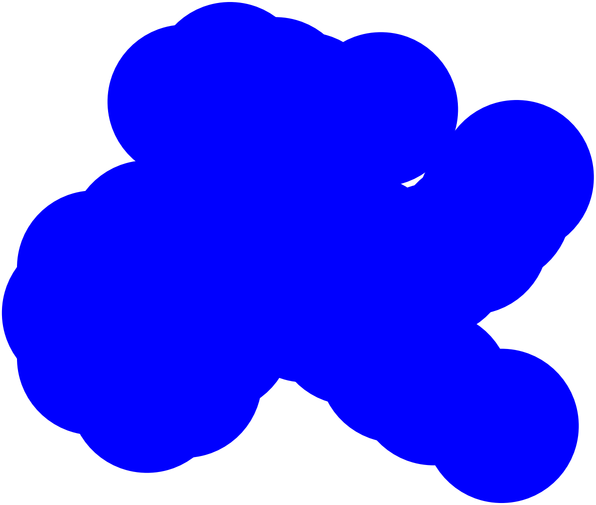

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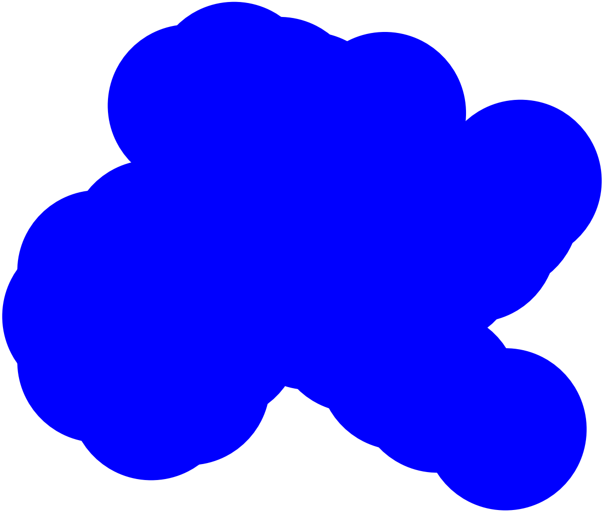

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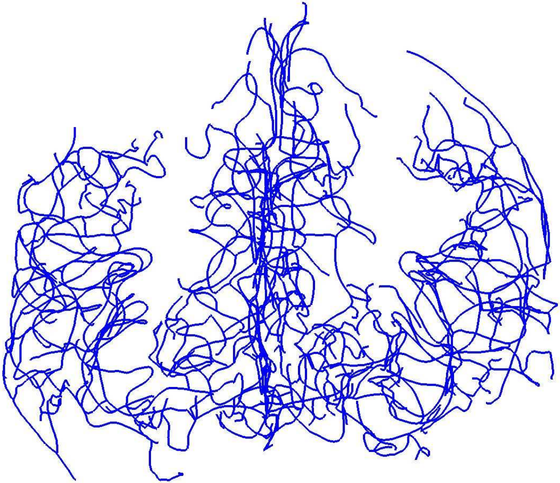

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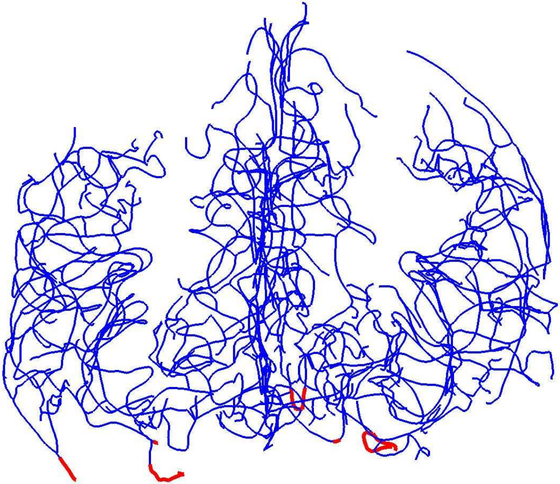

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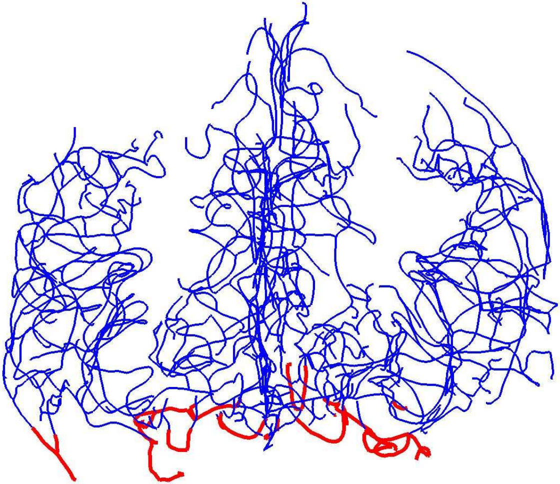

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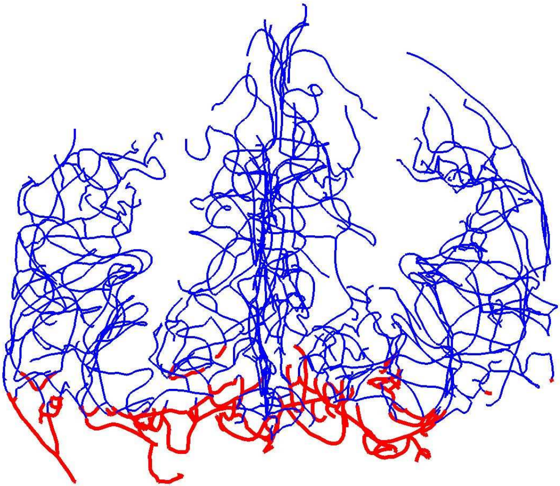

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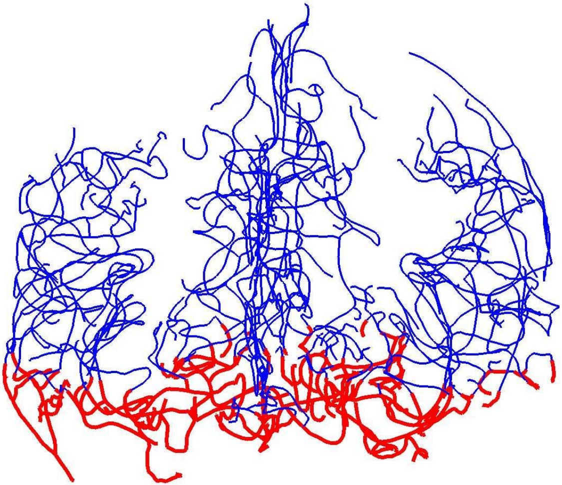

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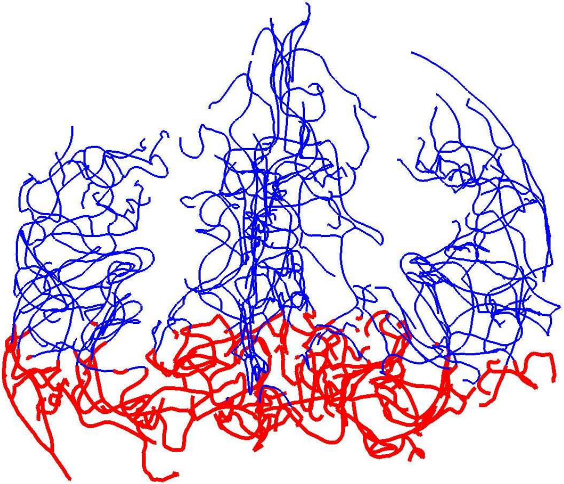

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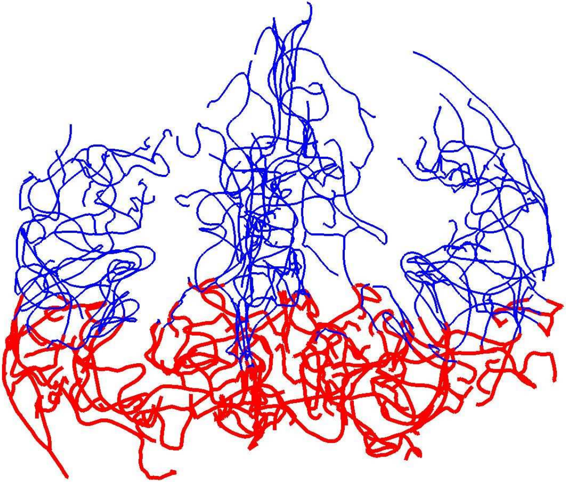

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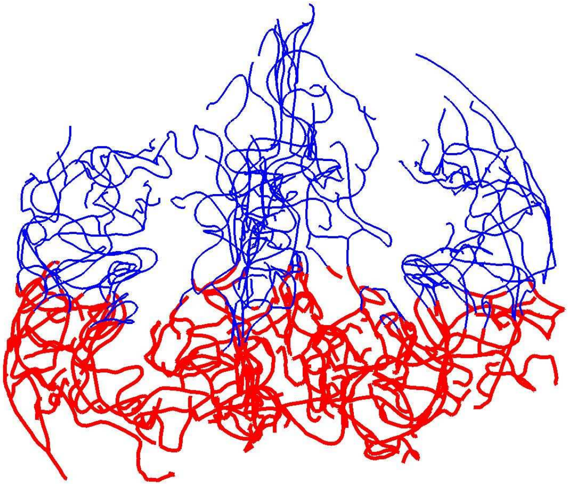

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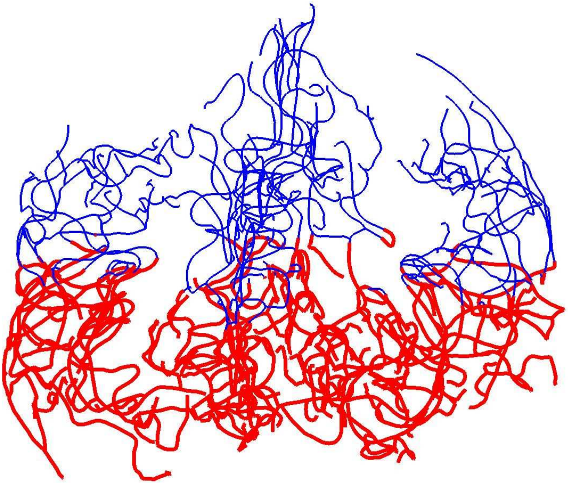

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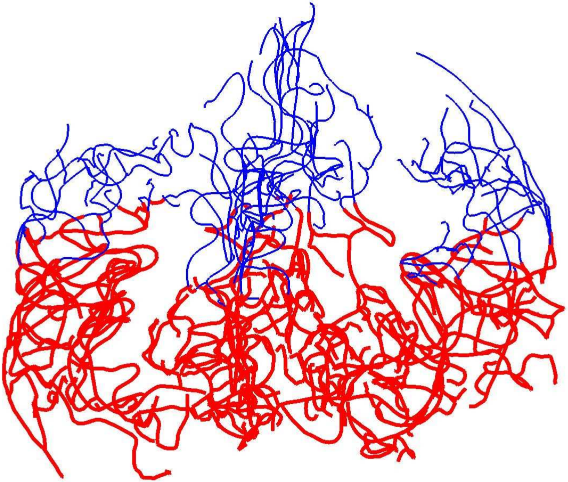

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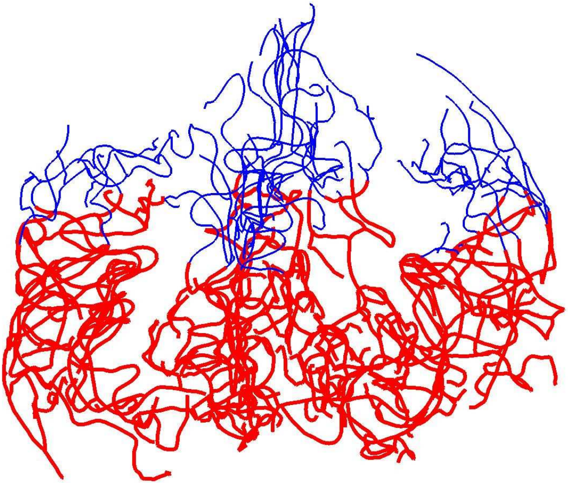

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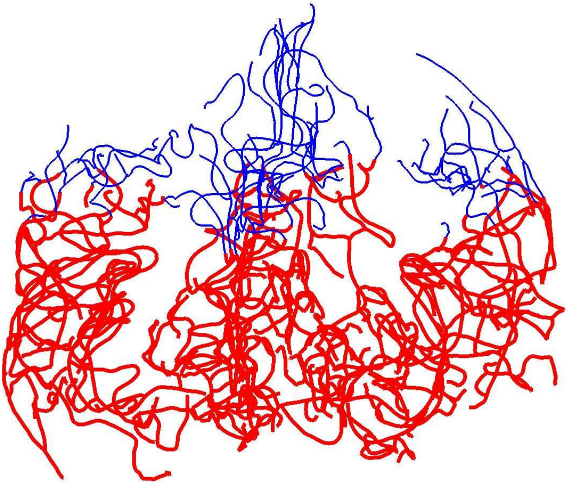

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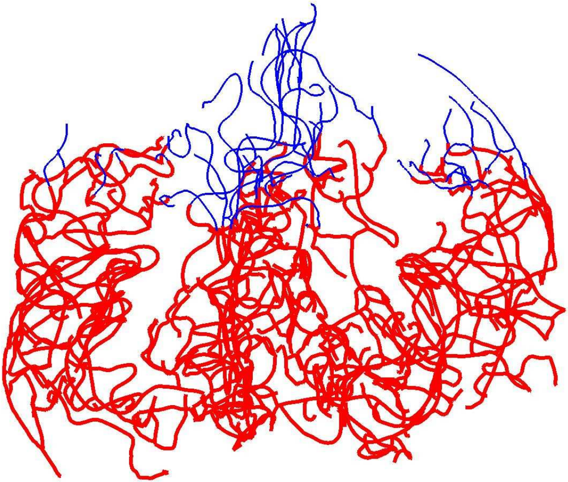

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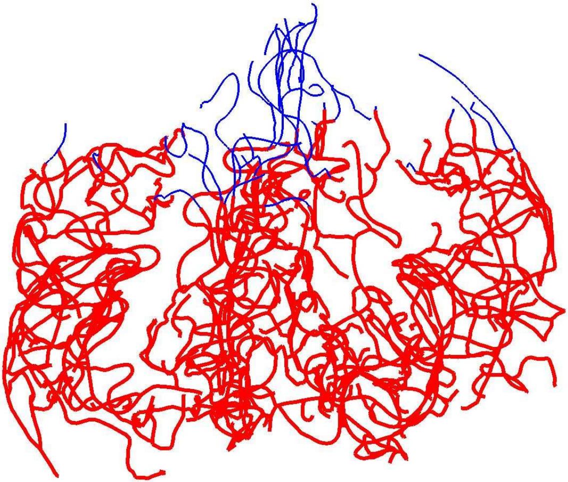

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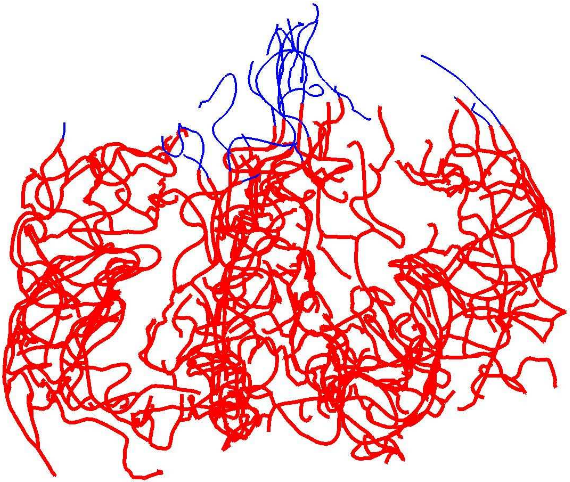

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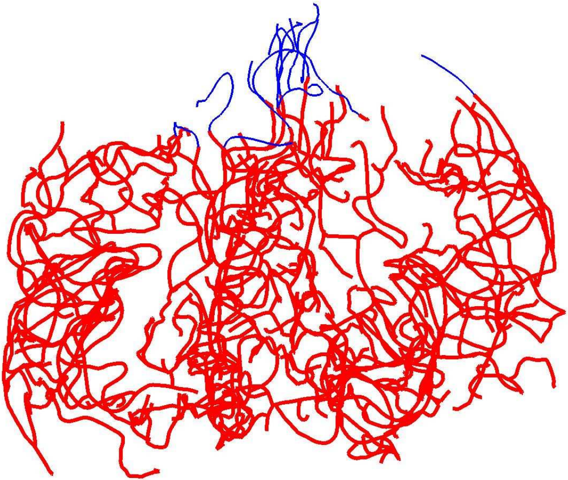

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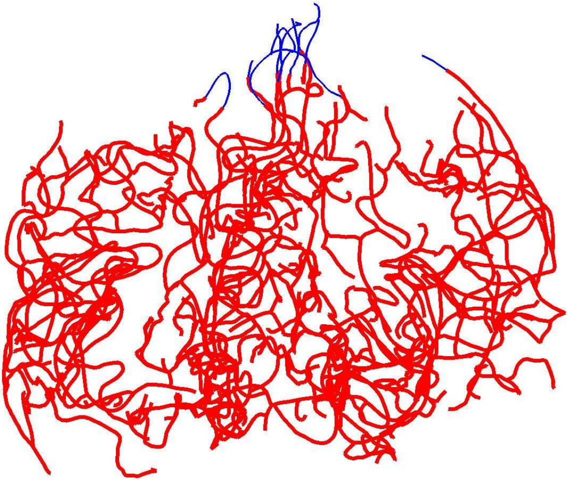

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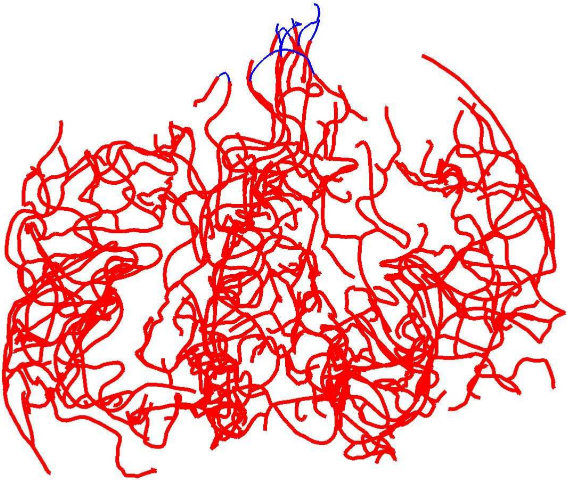

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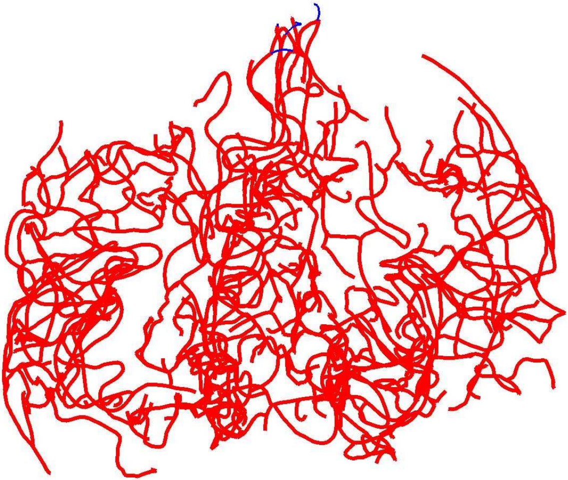

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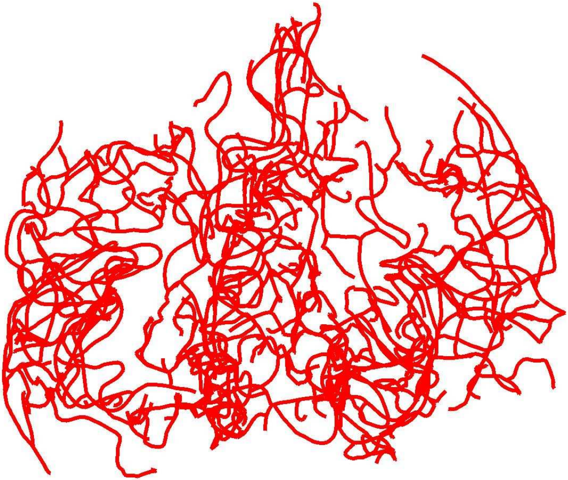

Persistent homology

Input. Topological space X filtered by set Q of subspaces: $X_q \subseteq X$ for $q \in Q$
$\Rightarrow Q$ is a partially ordered set: $X_q \subseteq X_{q'} \Leftrightarrow q \preceq q'$

Def. $\{X_q\}_{q \in Q}$ has persistent homology $\{H_q = H(X_q; \Bbbk)\}_{q \in Q}$. This is a

Def. Q-module over the poset Q:

- family $M = \{M_q\}_{q \in Q}$ of vector spaces over the field $\Bbbk$ with
- homomorphism $M_q \to M_{q'}$ whenever $q \preceq q'$ in Q such that
- $M_q \to M_{q''}$ equals the composite $M_q \to M_{q'} \to M_{q''}$ whenever $q \preceq q' \preceq q''$

Examples

- points in $\mathbb{R}^n$: $Q = \{0, \ldots, m\}$ or $\mathbb{R}$ — 1-parameter ("ordinary") persistence
- brain arteries: $Q = \{0, \ldots, m\}$ or $\mathbb{R}$ — 1-parameter ("ordinary") persistence
- wing veins: $Q = \mathbb{Z}^2$ or $\mathbb{R}^2$ — 2 discrete or continuous parameters
- probability distributions: $Q = \mathbb{R}^2$ — 2 continuous parameters
- $Q = \mathbb{Z}^n \Leftrightarrow M = \mathbb{Z}^n$-graded $\Bbbk[x_1, \ldots, x_n]$-module
- $Q = \mathbb{R}^n \Leftrightarrow M = \mathbb{R}^n$-graded $\Bbbk[\mathbb{R}^n_+]$-module

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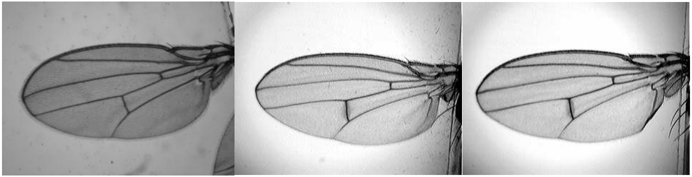
Fruit fly wings

Normal fly wings [images from David Houle's lab]:



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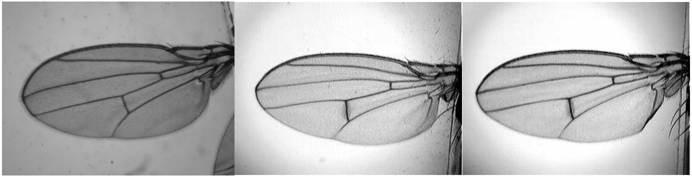
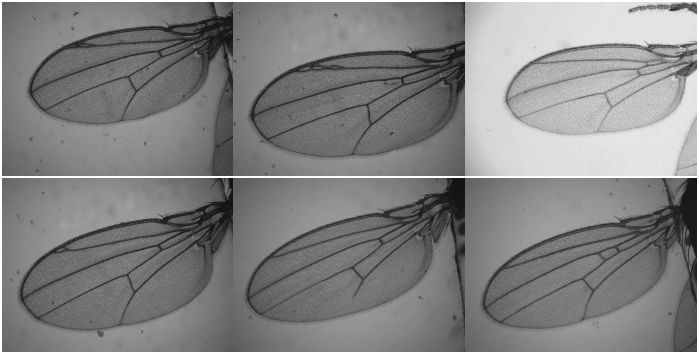
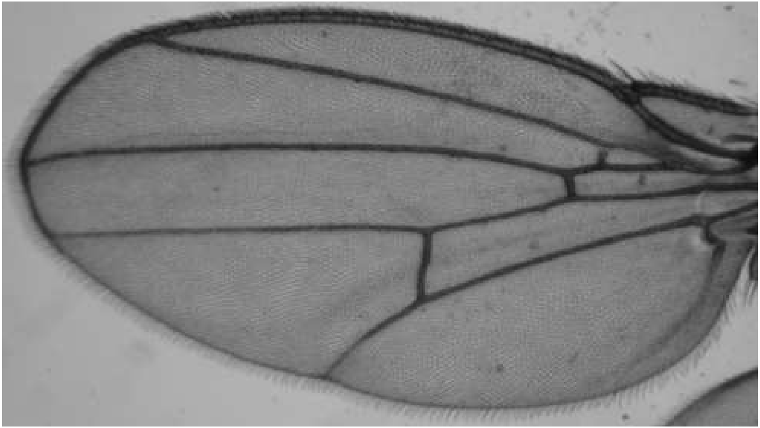


Topologically abnormal veins:



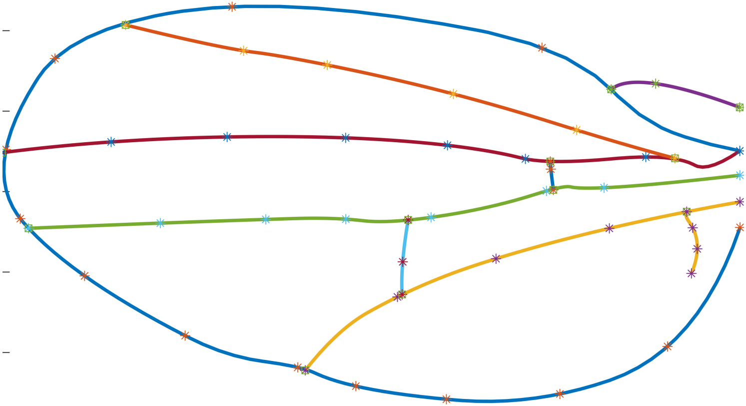
Fruit fly wings

photographic image



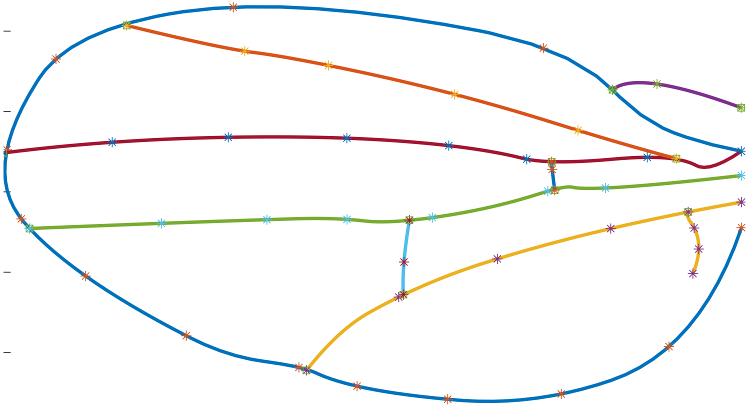
Fruit fly wings

spline



Fruit fly wings

spline

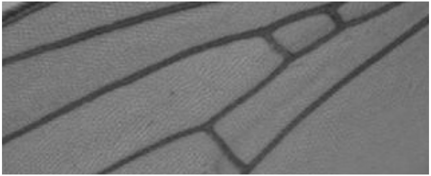


Hypothesis. Topological novelty arises when directional selection pushes continuous variation in a developmental program beyond a certain threshold.

Example: wing vein persistence [w/Houle, Thomas, Beriwal]

Example 1. Encode fruit fly wing with 2-parameter persistence

- 1st parameter: distance from vertex set
- 2nd parameter: distance from edge set



Sublevel set $W_{r,s}$ is near edges but far from vertices

Multiscale summary. Set $H_{r,s} = H_0(W_{r,s})$ or $H_1(W_{r,s})$

$\mathbb{Z}^2$-module:

$$\begin{array}{ccccccc} & \uparrow & & \uparrow & & \uparrow & \\ \to & H_{r-\varepsilon,s+\delta} & \to & H_{r,s+\delta} & \to & H_{r+\varepsilon,s+\delta} & \to \\ & \uparrow & & \uparrow & & \uparrow & \\ \to & H_{r-\varepsilon,s} & \to & H_{r,s} & \to & H_{r+\varepsilon,s} & \to \\ & \uparrow & & \uparrow & & \uparrow & \\ \to & H_{r-\varepsilon,s-\delta} & \to & H_{r,s-\delta} & \to & H_{r+\varepsilon,s-\delta} & \to \\ & \uparrow & & \uparrow & & \uparrow & \end{array}$$

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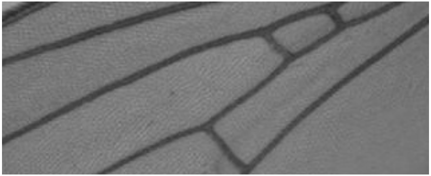

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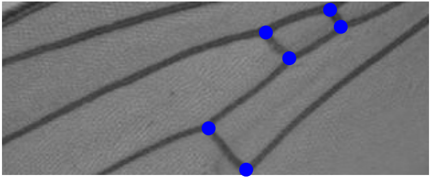

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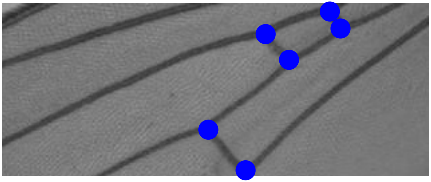

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- 1st parameter: distance from vertex set given as points in $\mathbb{R}^2$
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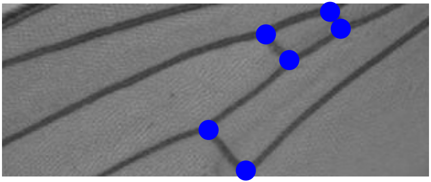

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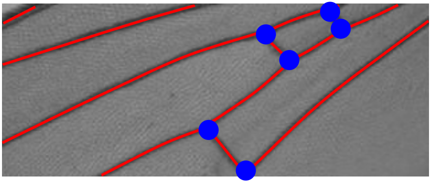

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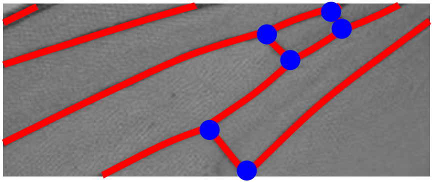

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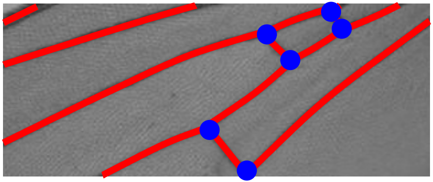
\end{array}$$

Example: wing vein persistence [w/Houle, Thomas, Beriwal]

Example 1. Encode fruit fly wing with 2-parameter persistence

- 1st parameter: distance from vertex set (require distance $\geq -r$)
- 2nd parameter: distance from edge set



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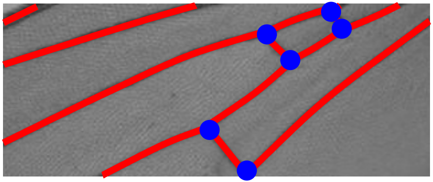

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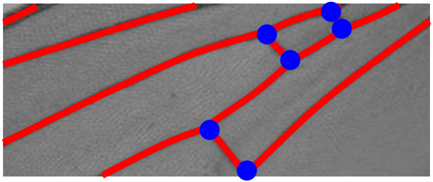

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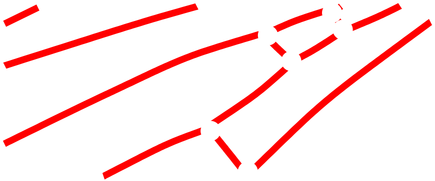

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Example: wing vein persistence [w/Houle, Thomas, Beriwal]

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- 1st parameter: distance from vertex set (require distance $\geq -r$)
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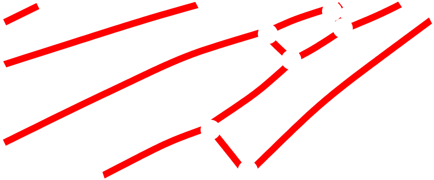

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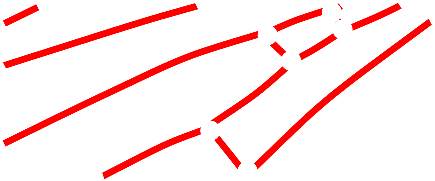

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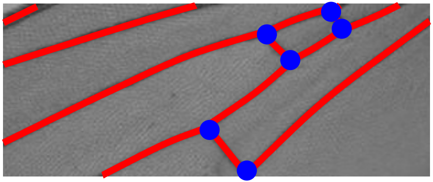

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Input. Topological space X filtered by set Q of subspaces: $X_q \subseteq X$ for $q \in Q$
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Examples

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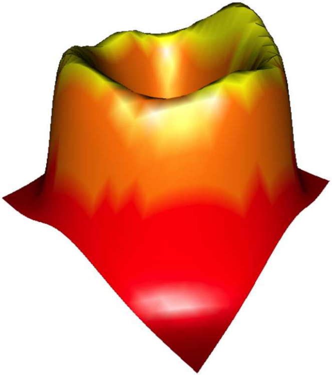

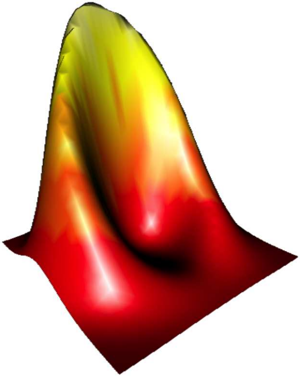

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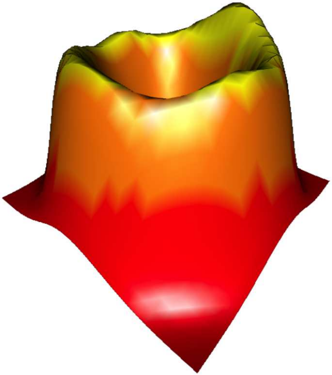

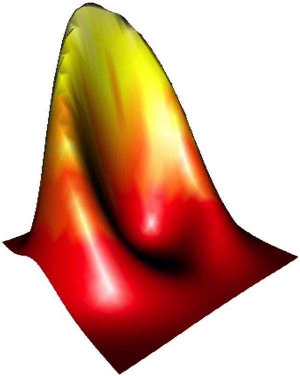

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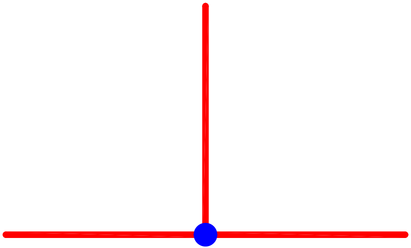
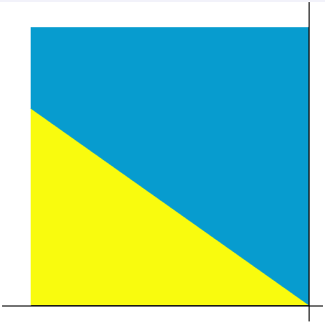
Example: toy model fly wings



A piece of fly wing vein

$\rightsquigarrow$



The (r, s)-plane $\mathbb{R}^2$

Observations

- finitely many regions
- boundaries between regions are (algebraic) curves
- discrete approximates algebraic

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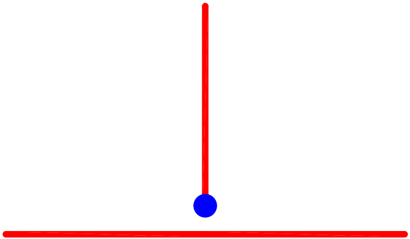

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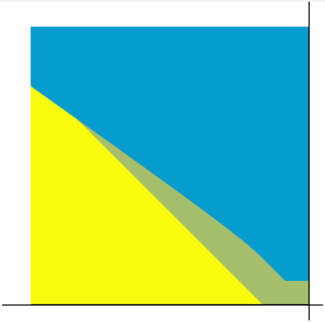

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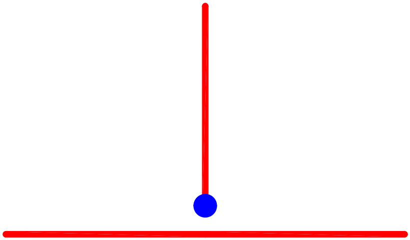

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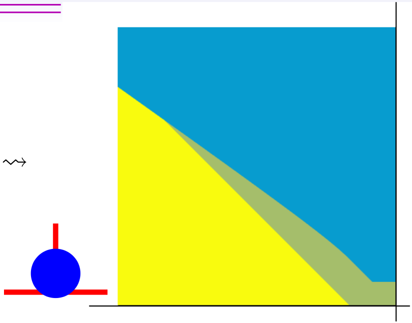

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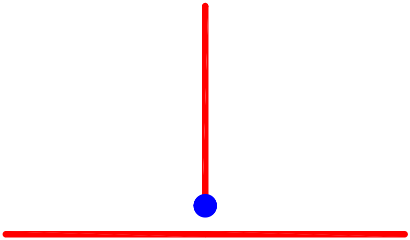

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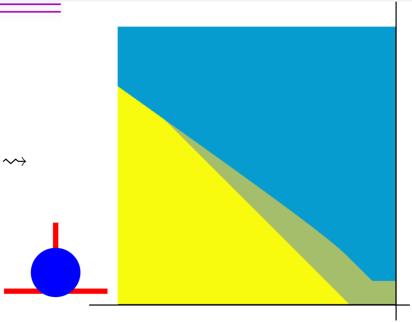

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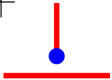

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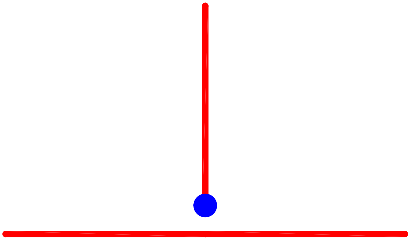

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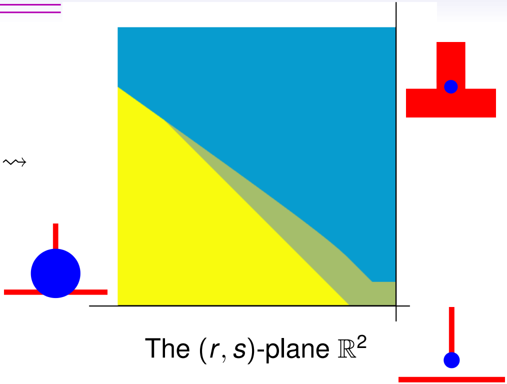

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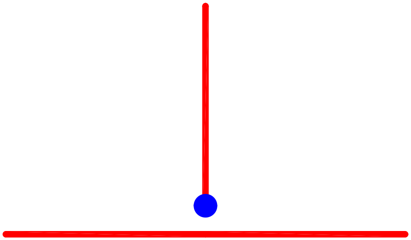

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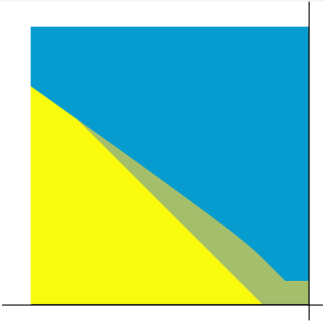

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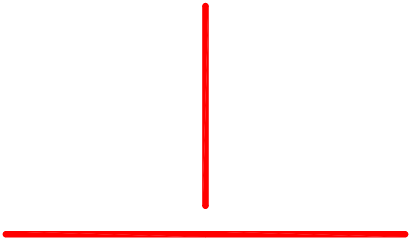

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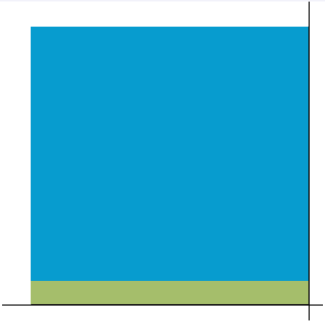

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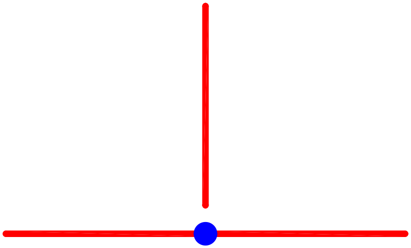

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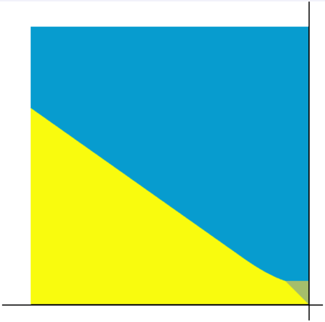

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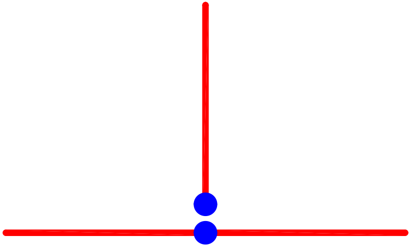

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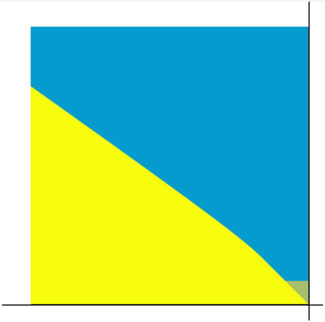

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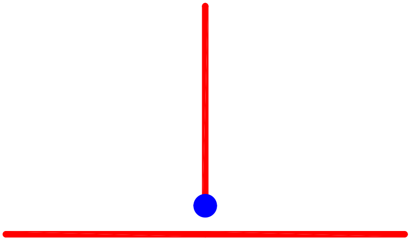

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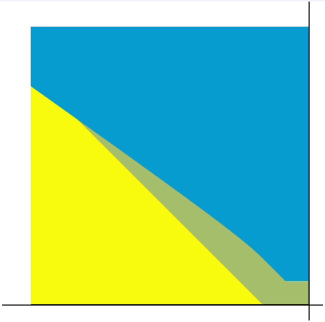

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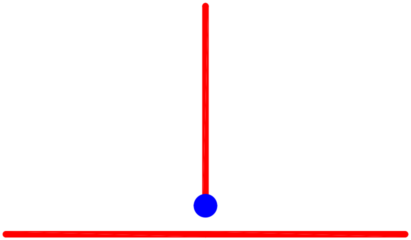

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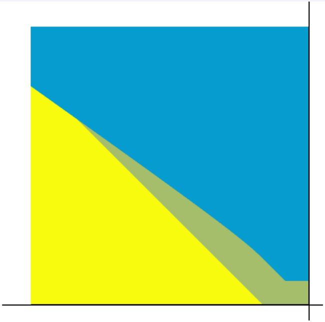

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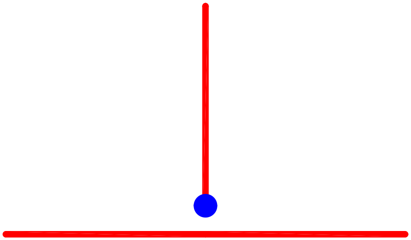

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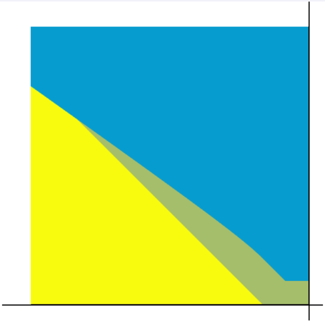

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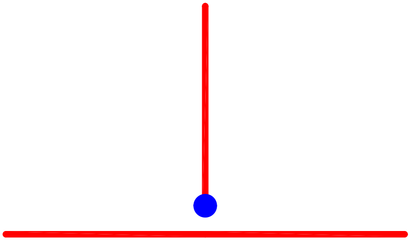

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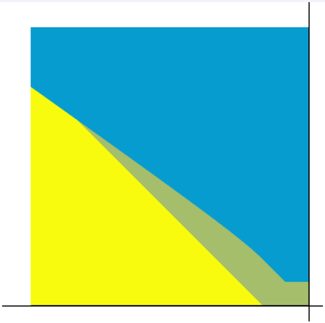

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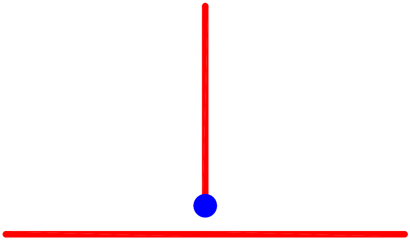

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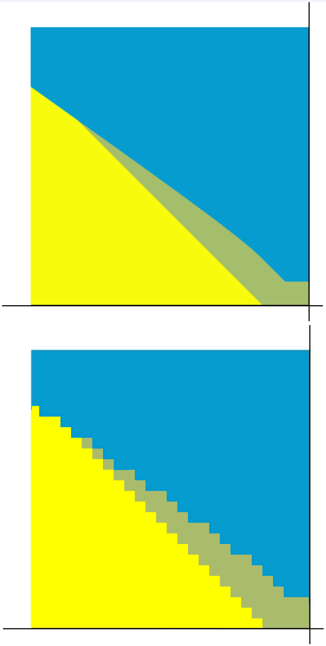

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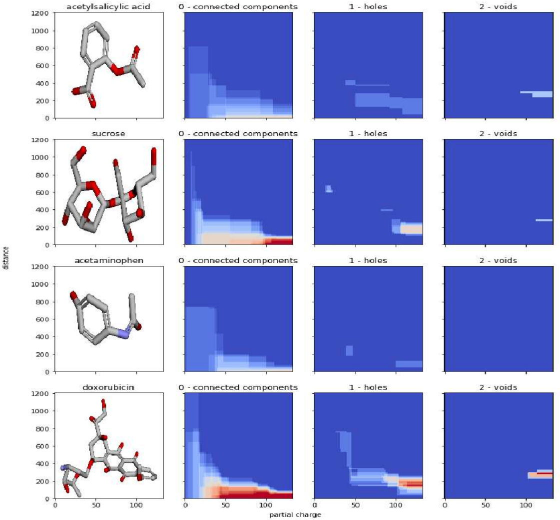
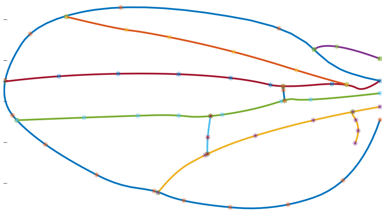
Example: molecular screening [Keller, Lesnick, Willke 2018]



Output computed by RIVET [Lesnick–Wright 2015–]

Example: fly wings, fully discretized



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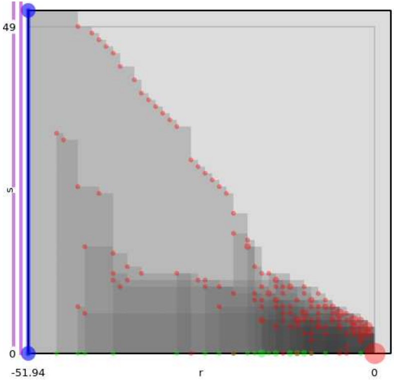

spline RIVET output

slow pipeline (spline $\rightsquigarrow$ RIVET output)
want instead:

- direct description of algebraic boundary curves for
- parallel computation—or better, single preprocessing step

Intervals in posets

Thm [Crawley–Boevey 2012]. $\mathbb{R}$-module $M \Rightarrow M \cong \bigoplus_{I \in \mathcal{I}} \Bbbk[I]$ with $\mathcal{I}$ a set of intervals

Def. An interval I in a poset Q is a convex connected subset: $a, b \in I \Rightarrow$

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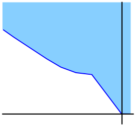
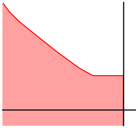
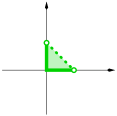
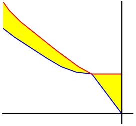
- In $\mathbb{R}^2$:  or  or  but not 
- In $\mathbb{R}^3$:

[Andrei Okounkov, *Limit shapes, real and imagined*, Bulletin of the AMS **53** (2016), no. 2, 187–216]

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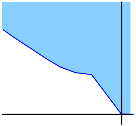
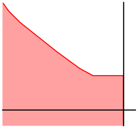
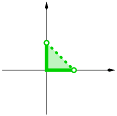
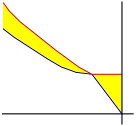
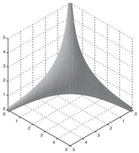
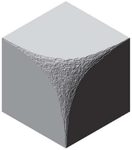
- In $\mathbb{R}^2$:  or  or  but not 

- In $\mathbb{R}^3$:



semialgebraic

or



piecewise linear

[Andrei Okounkov, *Limit shapes, real and imagined*, Bulletin of the AMS **53** (2016), no. 2, 187–216]

Intervals in posets

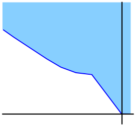
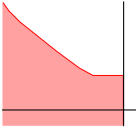
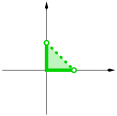
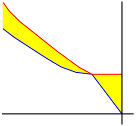
Thm [Crawley–Boevey 2012]. $\mathbb{R}$-module $M \Rightarrow M \cong \bigoplus_{I \in \mathcal{I}} \Bbbk[I]$ with $\mathcal{I}$ a set of intervals

Def. An interval I in a poset Q is a convex connected subset: $a, b \in I \Rightarrow$

- $q \in I$ whenever $a \preceq q \preceq b$ and
- there is a (zigzag) chain in I of comparable elements from a to b.

For any subset $S \subseteq Q$, let $\Bbbk[S] = \{\Bbbk_s\}_{s \in S}$.

Examples

- In $\mathbb{R}^2$:  or  or  but not 

However:

Thm fails in multiple parameters [Carlsson–Zomorodian 2009]

- every module $\cong \bigoplus$ indecomposables [Botnan–Crawley-Boevey 2020]
- indecomposable is arbitrarily worse than interval [Buchet–Escolar 2018]

Functorial endpoints over $\mathbb{R}$

$\mathbb{R}$-module interval decomposition works up to isomorphism:

"Bases can be chosen so that. . . "

but not in multipersistence $\Rightarrow$ crucial to extract endpoints functorially.

Right endpoint

- is a basis element for something, so
- look for "subspace of right endpoints"

Closed right endpoint at $a \in \mathbb{R}$

- element in M_a that dies when pushed up to $M_{>a}$
- submodule $\Bbbk_a \subseteq M$

Def. $\mathrm{Hom}(\Bbbk_a, M) = \sum(\text{submodules } \Bbbk_a \subseteq M)$
$= \text{vector space of all right endpoints}$
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Closed right endpoint at $a \in \mathbb{R}$ picture:

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- $\varinjlim_{a' < a} M_{a'}$ picture:

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$$(\delta^{\mathbb{R}_+} M)_a \to (\delta^{\{0\}} M)_a$$

Functorial endpoints over $\mathbb{R}$

Finite endpoints: unify

- upper boundary $\mathbb{R}$-module $\delta M = \bigoplus \delta^\sigma M$, the sum over faces σ of cone $\mathbb{R}_+$
- has natural maps $\delta^{\mathbb{R}_+} M \to \delta^{\{0\}} M \to 0$;
- take kernels of these and closed socles $\underline{\text{Hom}}(\Bbbk, -) = \bigoplus_a \text{Hom}(\Bbbk_a, -)$
- $\rightsquigarrow$ finite endpoint vector spaces: open in $\delta^{\mathbb{R}_+} M$ and closed in $M = \delta^{\{0\}} M$

Infinite right endpoint

- submodule $\Bbbk[a + \mathbb{R}_+]$ for any $a \gg 0$
- submodule $\Bbbk[a + \mathbb{R}_+] \supseteq \Bbbk[a' + \mathbb{R}_+]$ for any $a \leq a' \leftrightarrow$ same ∞-endpoint
- need *injective* homomorphisms $\varphi : \Bbbk[a + \mathbb{R}_+] \hookrightarrow M$, with
- $\varphi \sim \varphi'$ if one is an $\mathbb{R}$-translate of the other
- start with $\text{Hom}(\Bbbk[a + \mathbb{R}_+], M)$, but
  - declare translation along $\mathbb{R}$ to be invertible by localizing: $H \rightsquigarrow H_{\mathbb{R}_+}$
  - mod out by translation: $H_{\mathbb{R}_+} \rightsquigarrow H_{\mathbb{R}_+}/\mathbb{R}$.

Note. $H \rightsquigarrow H_{\mathbb{R}_+}$

- kills any interval with a finite right endpoint (open or closed)
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Def. $H \rightsquigarrow H_{\mathbb{R}_+}/\mathbb{R}$ is quotient-restriction.

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- upper boundary $\mathbb{R}$-module $\delta M = \bigoplus \delta^\sigma M$, the sum over faces σ of cone $\mathbb{R}_+$
- has natural maps $\delta^{\mathbb{R}_+} M \to \delta^{\{0\}} M \to 0$;
- take kernels of these and closed socles $\underline{\mathrm{Hom}}(\Bbbk, -) = \bigoplus_a \mathrm{Hom}(\Bbbk_a, -)$
- $\rightsquigarrow$ finite endpoint vector spaces: open in $\delta^{\mathbb{R}_+} M$ and closed in $M = \delta^{\{0\}} M$

Infinite right endpoint

- submodule $\Bbbk[a + \mathbb{R}_+]$ for any $a \gg 0$
- submodule $\Bbbk[a + \mathbb{R}_+] \supseteq \Bbbk[a' + \mathbb{R}_+]$ for any $a \leq a' \leftrightarrow$ same ∞-endpoint
- need *injective* homomorphisms $\varphi : \Bbbk[a + \mathbb{R}_+] \hookrightarrow M$, with
- $\varphi \sim \varphi'$ if one is an $\mathbb{R}$-translate of the other
- start with $\mathrm{Hom}(\Bbbk[a + \mathbb{R}_+], M)$, but
  - declare translation along $\mathbb{R}$ to be invertible by localizing: $H \rightsquigarrow H_{\mathbb{R}_+}$
  - mod out by translation: $H_{\mathbb{R}_+} \rightsquigarrow H_{\mathbb{R}_+}/\mathbb{R}$.

Note. $H \rightsquigarrow H_{\mathbb{R}_+}$

- kills any interval with a finite right endpoint (open or closed)
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Def. $H \rightsquigarrow H_{\mathbb{R}_+}/\mathbb{R}$ is quotient-restriction.

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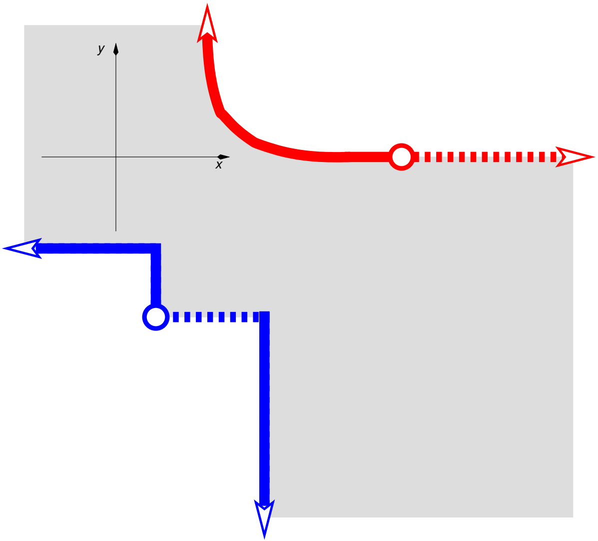

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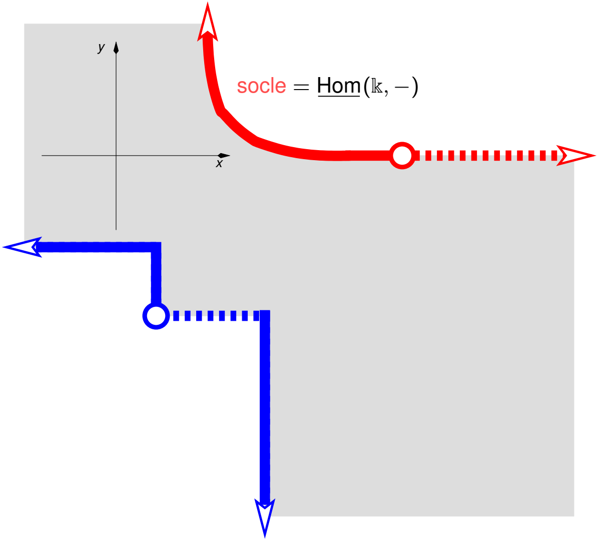

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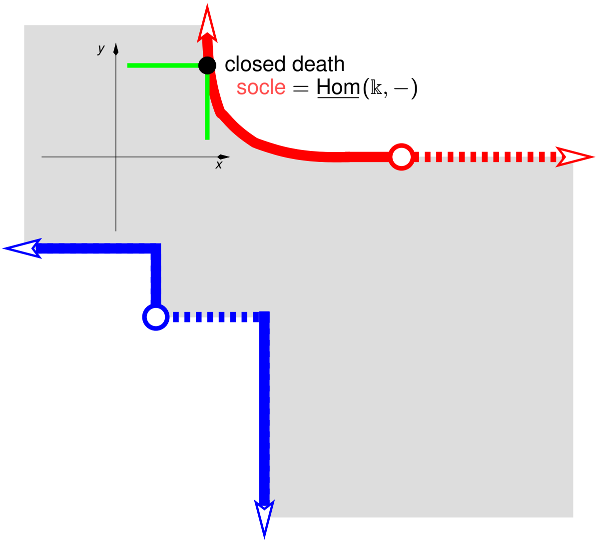

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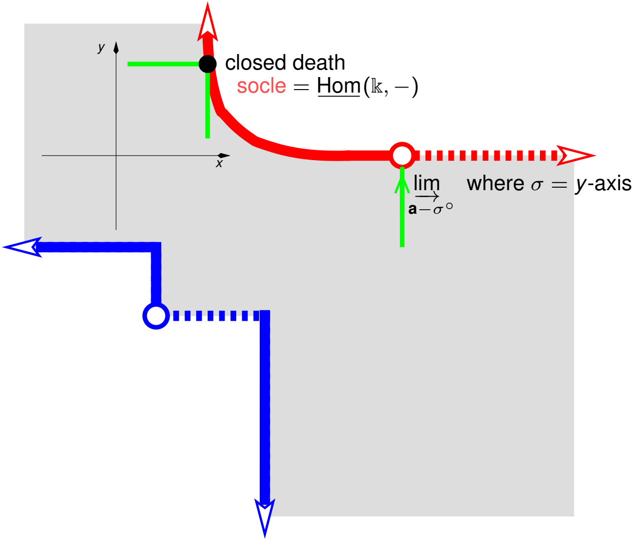

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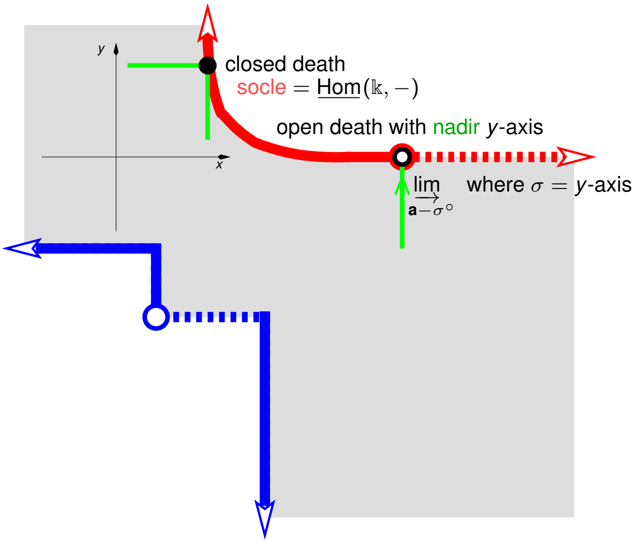

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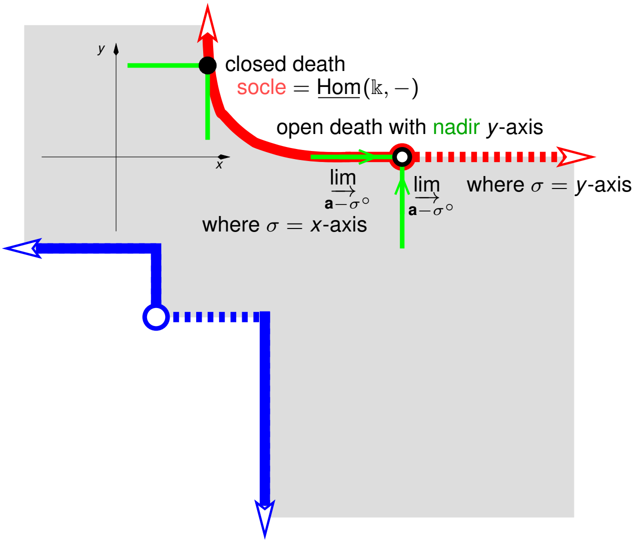

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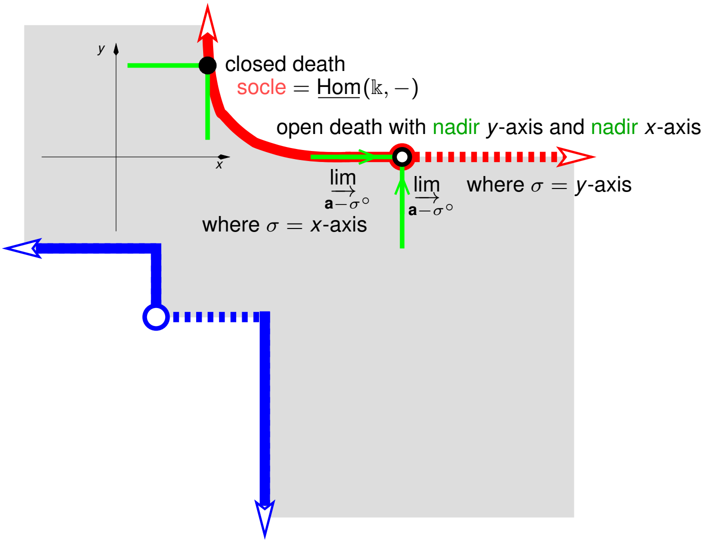

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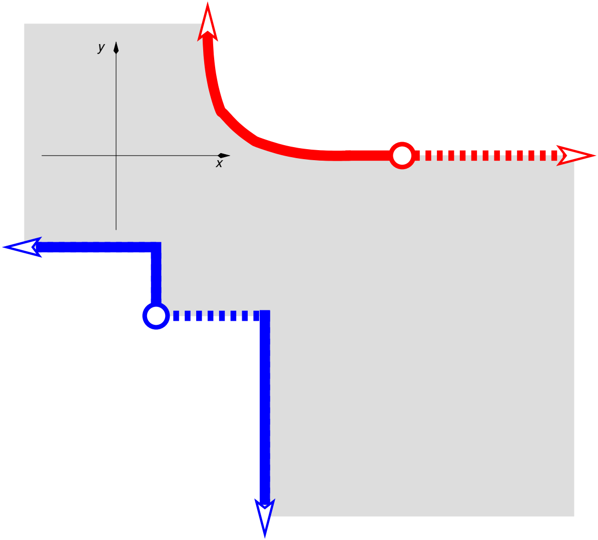

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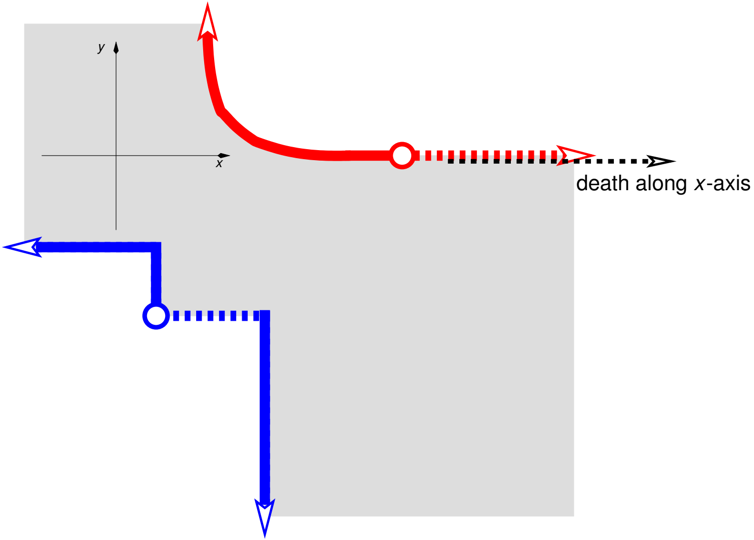

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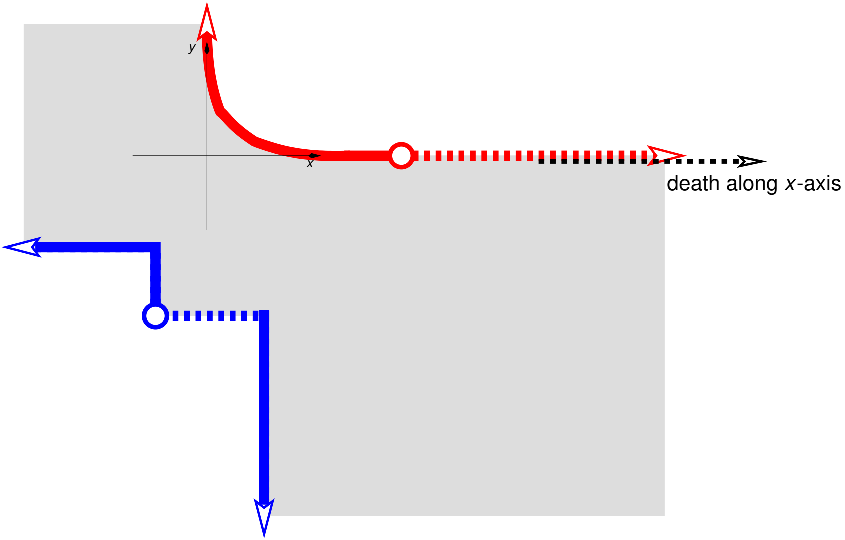

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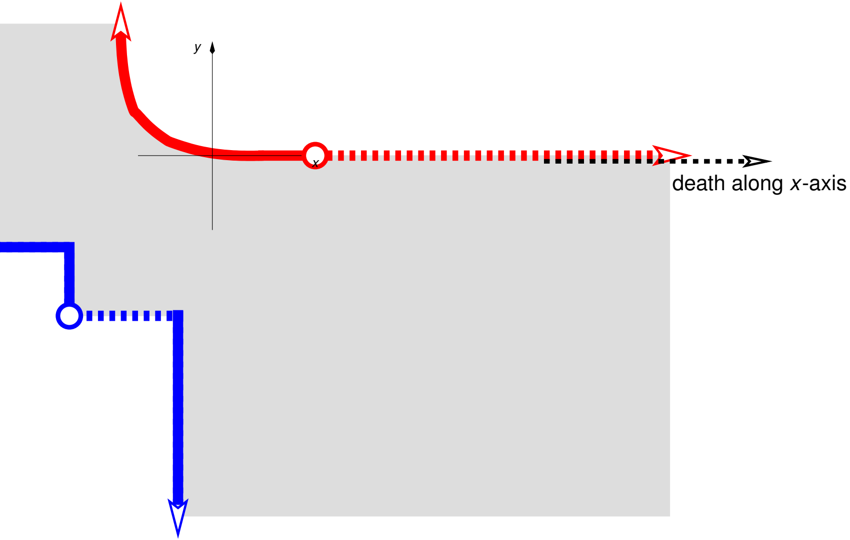

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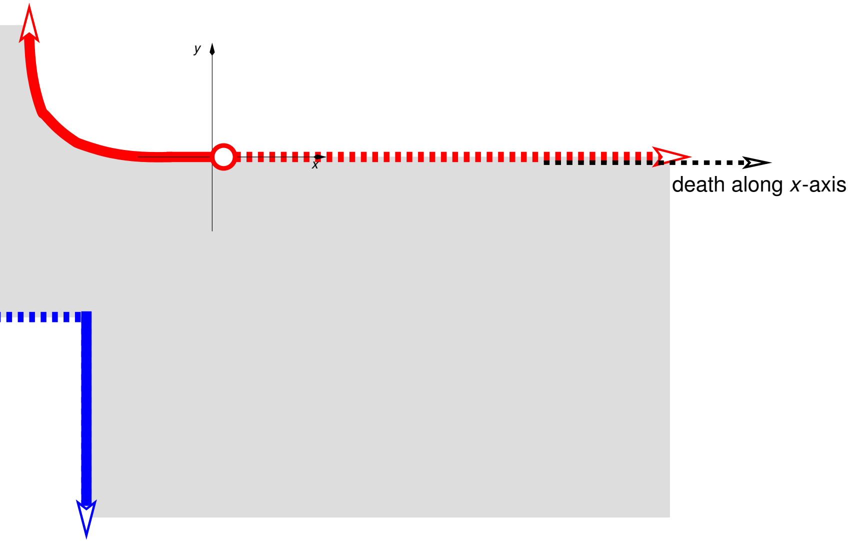

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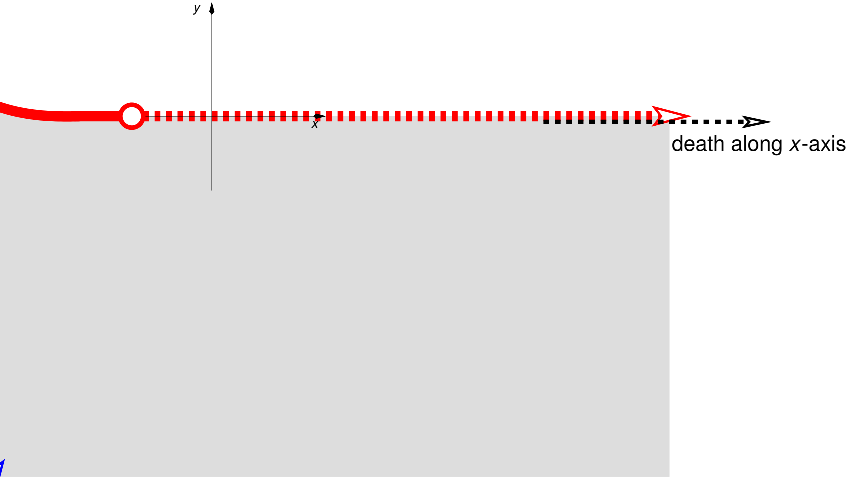

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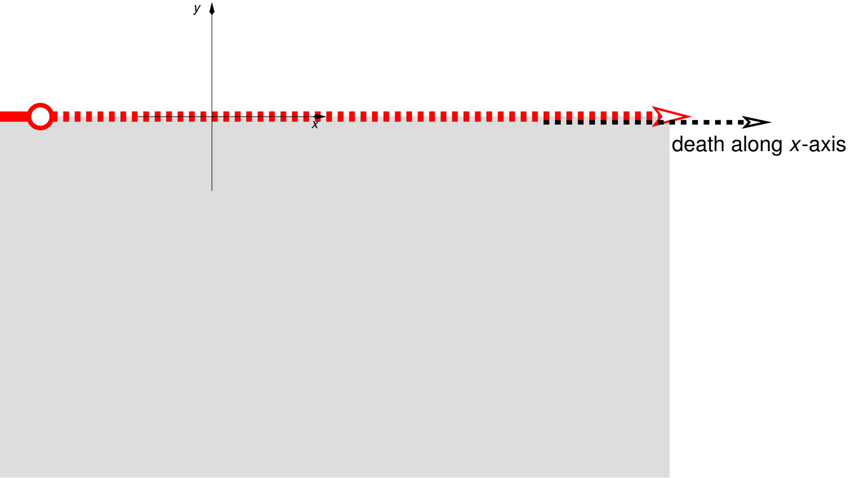

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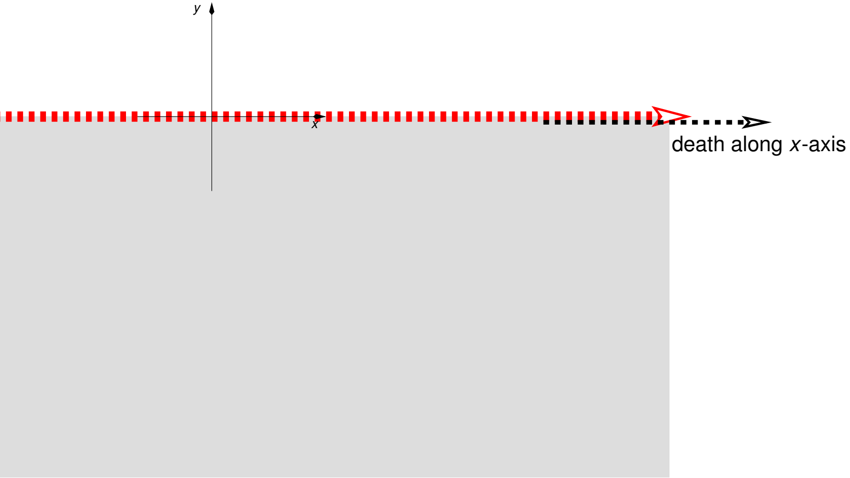

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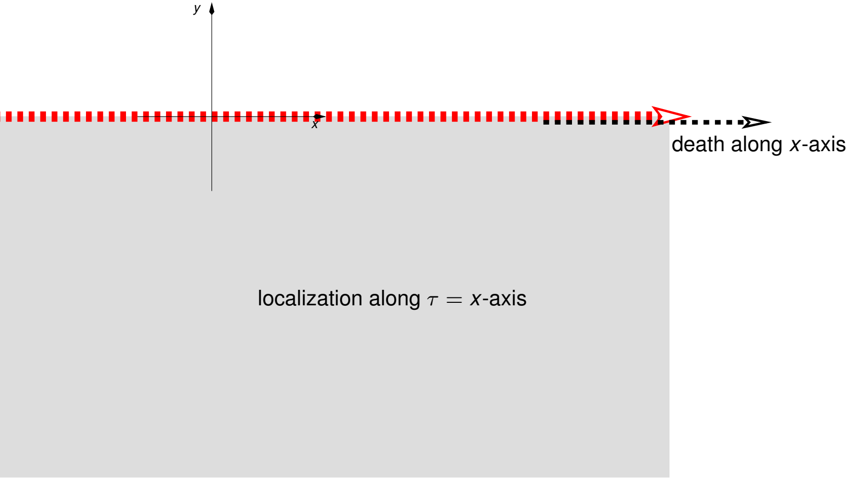

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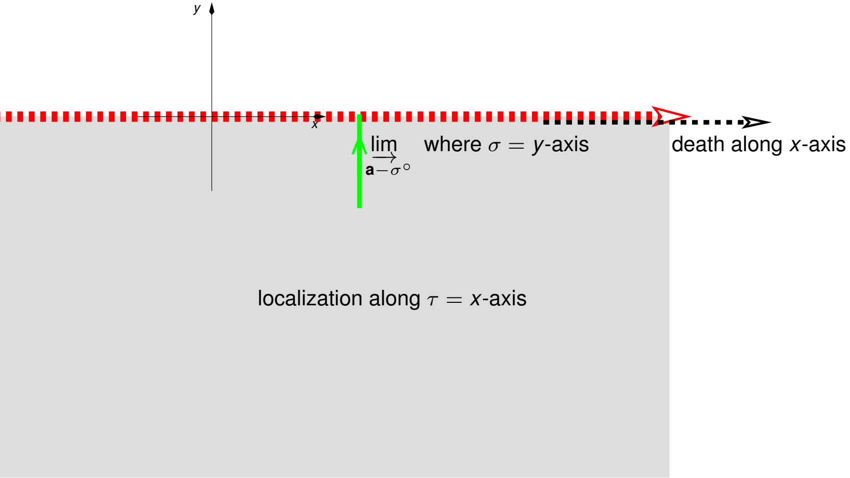

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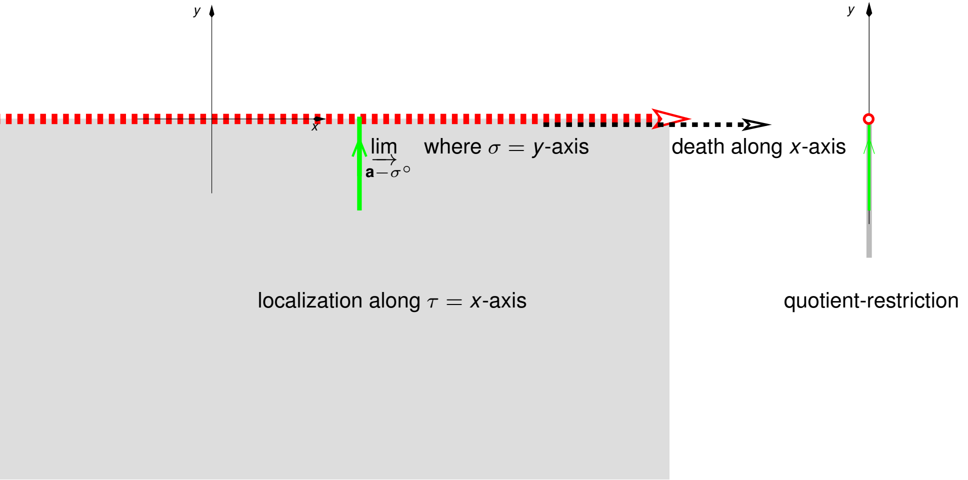

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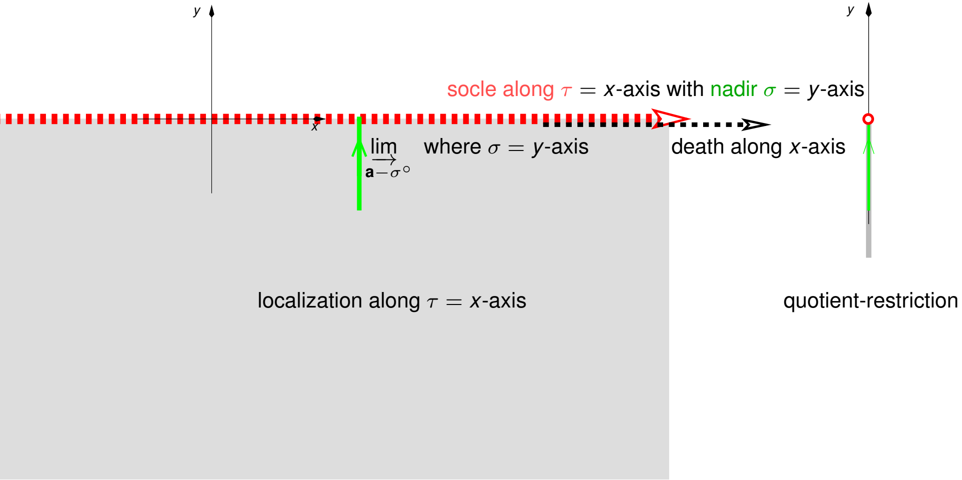

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Def. Fix a face σ of the cone $\mathbb{R}^n_+$. The upper boundary of M atop σ is the $\mathbb{R}^n$-module $\delta^\sigma M$ whose fiber over $\mathbf{a} \in \mathbb{R}^n$ is the vector space

$$(\delta^\sigma M)_\mathbf{a} = M_{\mathbf{a}-\sigma} = \varinjlim_{\mathbf{a}' \in \mathbf{a}-\sigma^\circ} M_{\mathbf{a}'}$$

Def. Fix a face τ of the cone $\mathbb{R}^n_+$.

- Let $\nabla\tau$ = poset of faces containing τ.
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$$\delta_\tau M = \bigoplus_{\sigma \in \nabla\tau} \delta^\sigma M$$

Def. The socle of M along τ is

$$\operatorname{soc}_\tau M = \underline{\operatorname{Hom}}_{\mathbb{R}_+ \times \nabla\tau}\big(\Bbbk[\tau], \delta_\tau M\big)/\tau$$

- take upper boundary of M along various faces of $\mathbb{R}^n_+$
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Comments on endpoints

Left endpoints are dual to right

- closed left: quotient module $M \otimes \Bbbk_a = (M/M_{<a})_a$
- open left: $\operatorname{coker}\Big(M_a \to \varprojlim_{a<a'} M_{a'}\Big)$
- infinite left: quotient module $\Bbbk[a - \mathbb{R}_+]$

Question. Is there a distinction between open and closed endpoints?

Answers

- For metrics and distances, no: [Berkouk–Petit 2021]; cf. Berkouk's talk at least for metrics based on γ-topology [Kashiwara–Schapira 2018–]
- But a module isn't handed to you as "closed" or "open"; often, it turns out, it has closed left and open right endpoints (from free presentation or from γ-topology, for instance).
- Boundary components, present or not, crucial for derived category computation: module $\rightsquigarrow$ concrete complex representing it.
- Closed vs. open endpoints have very different meanings for topological calculations with constructible sheaves and functions via pushforwards.

Note. Socles, minimal generators, resolutions, etc.: completely new fundamental commutative algebra of real-exponent polynomial rings.

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Note. Socles, minimal generators, resolutions, etc.: completely new fundamental commutative algebra of real-exponent polynomial rings.

Comments on endpoints

Left endpoints are dual to right

- closed left: quotient module $M \otimes \Bbbk_a = (M/M_{<a})_a$
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- infinite left: quotient module $\Bbbk[a - \mathbb{R}_+]$

Question. Is there a distinction between open and closed endpoints?

Answers

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Birth and death posets

Def. An $\mathbb{R}^n$-module M has

- birth poset B_M = parameters indexing left endpoints (births) and
- death poset D_M = parameters indexing right endpoints (deaths).

Note. A death degree $\alpha \in D_M$ records

- parameter $\mathbf{a} \in \mathbb{R}^n_+$
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- nadir σ.

To visualize: Let $\mathcal{O}(\mathbb{R}^n)$ be the poset of orthants in $\mathbb{R}^n$ partially ordered by inclusion, where an orthant is a translate of $\mathbb{R}^n_+$ missing a set of closed faces. Then B_M is a subposet of the disjoint union $B_n = \bigcup_{\text{faces}\,\tau} \mathcal{O}(\mathbb{R}^n/\mathbb{R}\tau)$.

Examples.

- B_1 is the union of $\{-\infty\}$ with the set of positive-pointing rays in $\mathbb{R}$ totally ordered by inclusion.
- $\mathbb{Z}^n$-module M with only finitely many nonzero $M_{\mathbf{a}} \Rightarrow B_M$ and D_M are $\subseteq \mathbb{Z}^n$: no nadirs ($\mathbb{Z}^n$ is discrete) or quotient-restriction (every element is mortal).

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QR-codes

Prop. M has natural vector spaces

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with natural maps $M_\gamma \to M_{\gamma'}$ when $\gamma \preceq \gamma'$ in the life poset $B_M \,\dot\cup\, D_M$.

Def. The quotient-restriction code (or QR code) $\mathrm{QR}(M)$ is the module

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Remark. Better intuition: QR code $\leftrightarrow$ "morphism" $\mathrm{Birth}(M) \to \mathrm{Death}(M)$, where $\mathrm{Birth}(M) = \mathrm{QR}(M)|_{B_M}$ and $\mathrm{Death}(M) = \mathrm{QR}(M)|_{D_M}$.

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Applications

1. Direct computation with real parameters
   - multipersistence often uses semialgebraic or even PL geometry
   - Thm: QR codes inherit such geometry, plus tameness
   - Current project: use this to implement semialgebraic computation
   - potential payoff 1: single preprocessing step for many similar multipersistence computations; e.g., fly wings
   - potential payoff 2: data structure for multipersistence is amenable to distance computation of the sort we typically see; in essence it thinks like Lebesgue instead of Riemann
2. Algorithmic indecomposable decomposition
   - functorial reduction to QR codes $\Rightarrow$ decompose QR codes
   - birth-death perspective lends insight into how to extend [Dey–Xin 2018] to work for arbitrary birth and death degrees
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3. Distances
   - use just the endpoints, or
   - "corresponding parts" of the endpoints, e.g. same associated prime (history or mortality type)

Thank You