Extracting bar lengths from multiparameter persistent homology

Ezra Miller

Duke University, Department of Mathematics
and Department of Statistical Science

ezra@math.duke.edu

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Outline

1. Persistent homology
2. Ordinary persistence: one parameter
3. Multiple parameters: fruit fly wings
4. Statistical analysis
5. Interval decomposition
6. Lifetime filtration
7. Tameness
8. Interleaving, matching, and bottleneck distances
9. Future directions

Persistent homology

Input. Topological space X filtered by set Q of subspaces: $X_q \subseteq X$ for $q \in Q$

$\Rightarrow Q$ is a partially ordered set: $X_q \subseteq X_{q'} \Leftrightarrow q \preceq q'$

Def. $\{X_q\}_{q \in Q}$ has persistent homology $\{H_q = H(X_q; \Bbbk)\}_{q \in Q}$.

Def. Q-module over the poset Q:

- family $H = \{H_q\}_{q \in Q}$ of vector spaces over the field $\Bbbk$ with
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Examples

- points in $\mathbb{R}^n$: $Q = \{0, \ldots, m\}$ or $\mathbb{R}$ 1-parameter ("ordinary") persistence
- brain arteries: $Q = \{0, \ldots, m\}$ or $\mathbb{R}$ 1-parameter ("ordinary") persistence
- wing veins: $Q = \mathbb{Z}^2$ or $\mathbb{R}^2$ 2 discrete or continuous parameters
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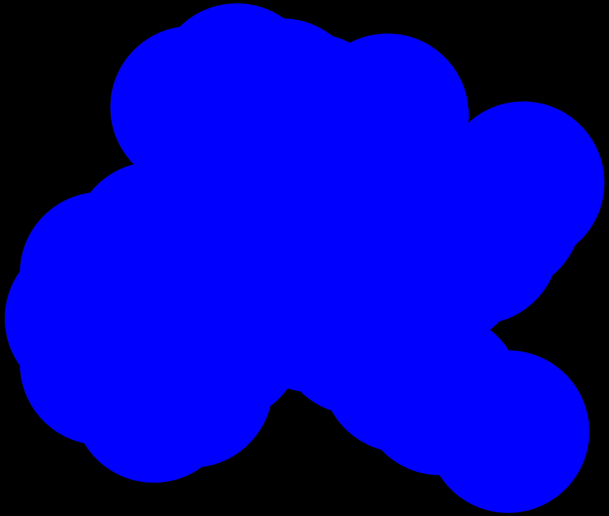
Example: expanding balls



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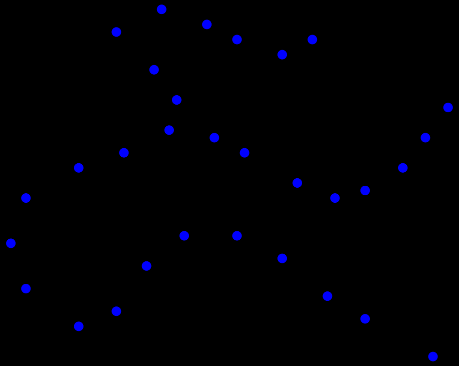

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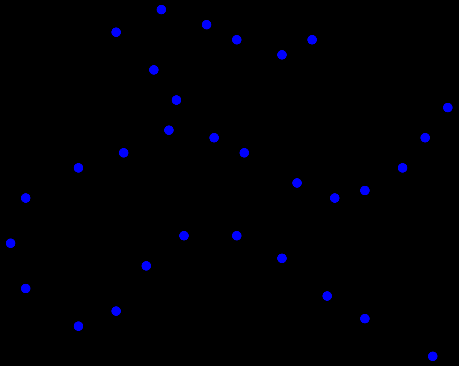

$$\dim(H_0) = 31$$

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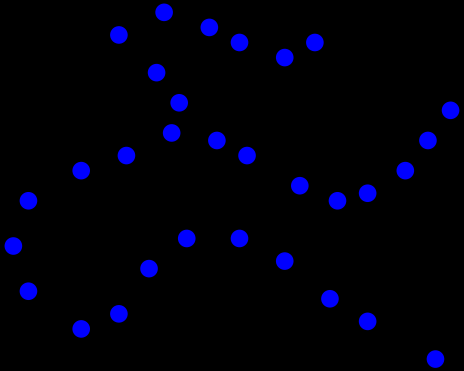

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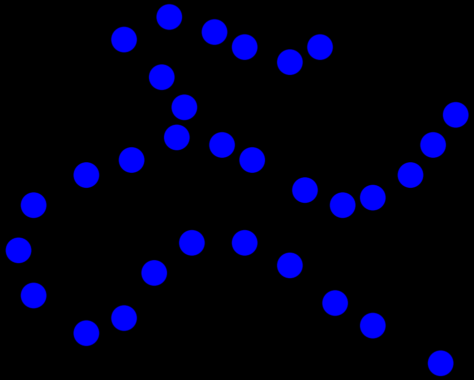

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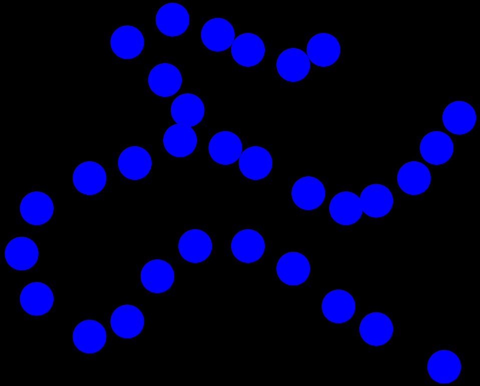
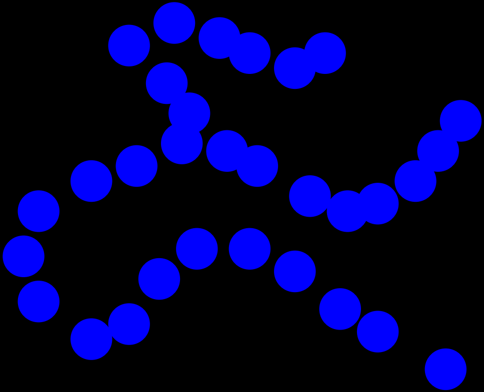


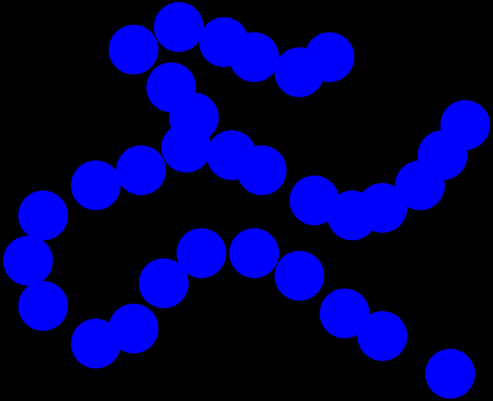
$\dim(H_0) = 26$

Example: expanding balls



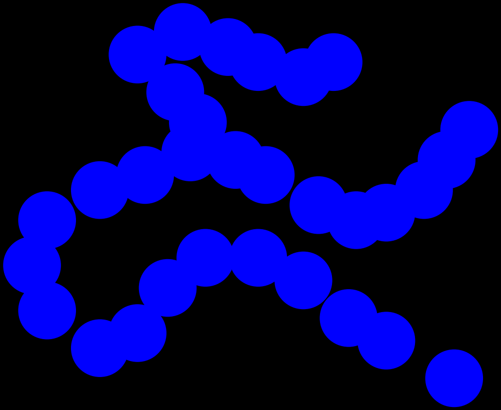
$\dim(H_0) = 21$

Example: expanding balls



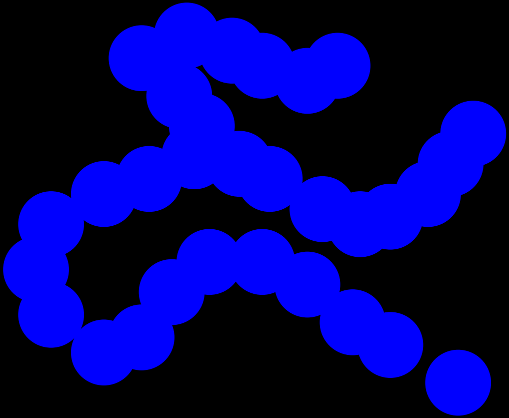
$\dim(H_0) = 12$

Example: expanding balls



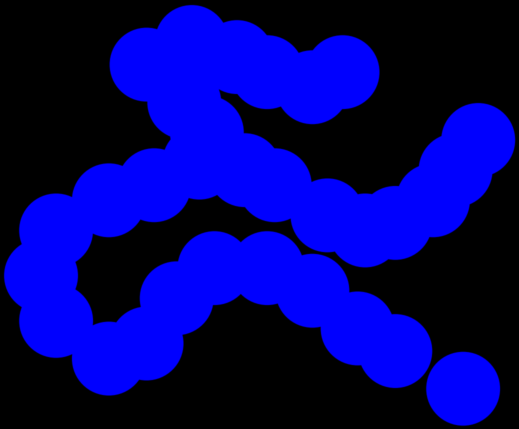
$\dim(H_0) = 6$

Example: expanding balls



$\dim(H_0) = 2$

Example: expanding balls



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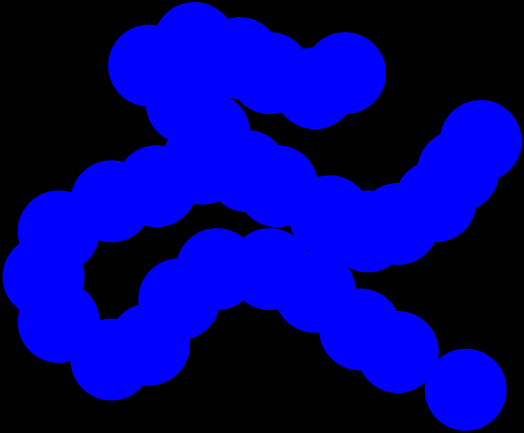
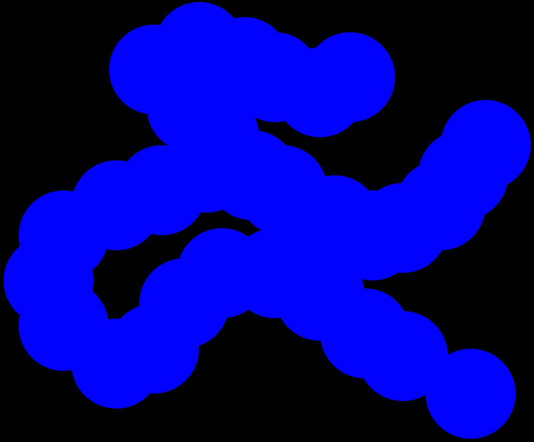


$\dim(H_0) = 1 \qquad \dim(H_1) = 2$

Example: expanding balls



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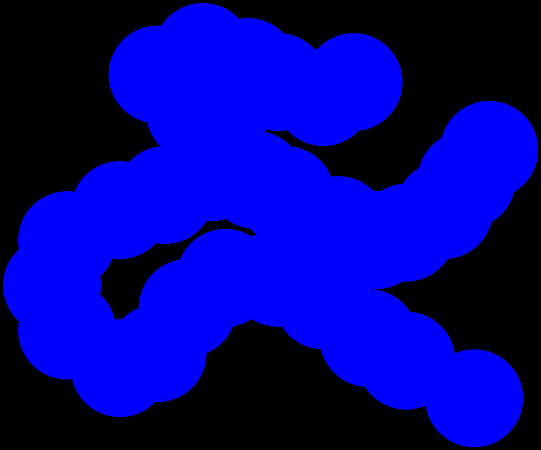
Example: expanding balls



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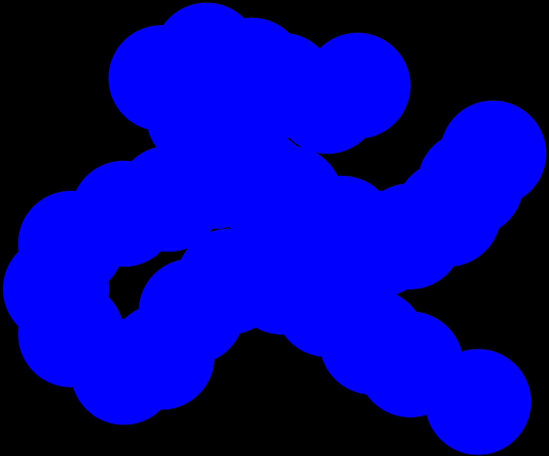

$\dim(H_0) = 1 \qquad \dim(H_1) = 3$

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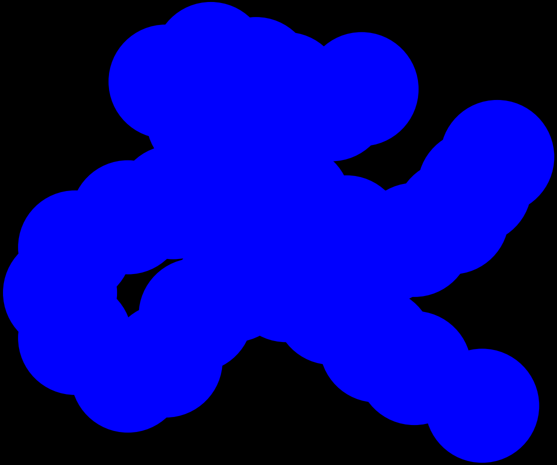

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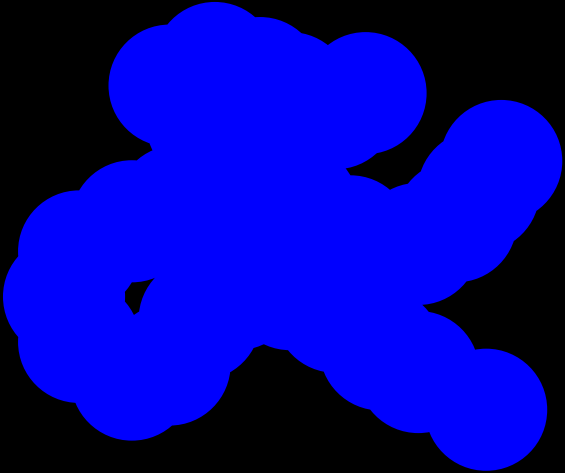

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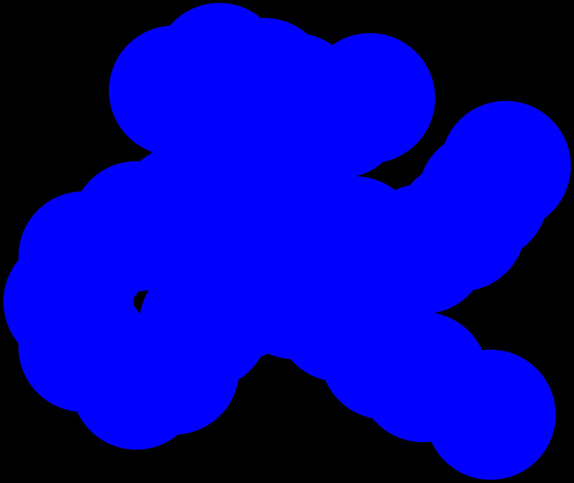

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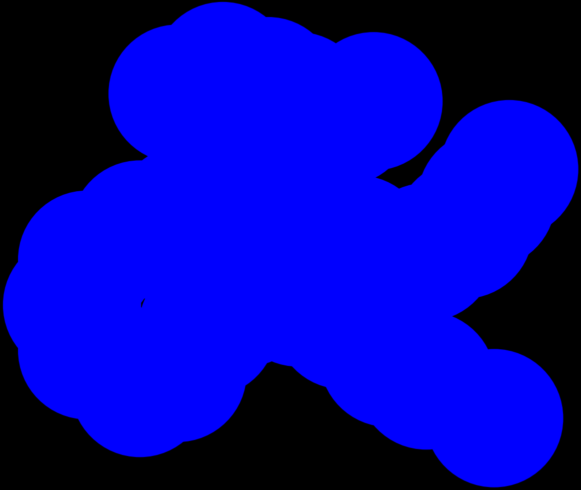

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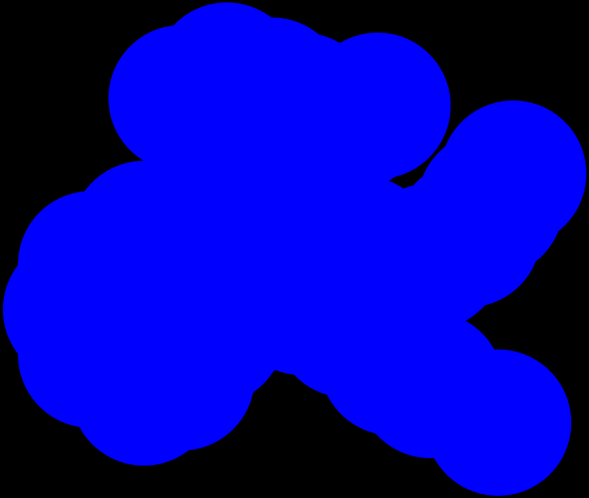

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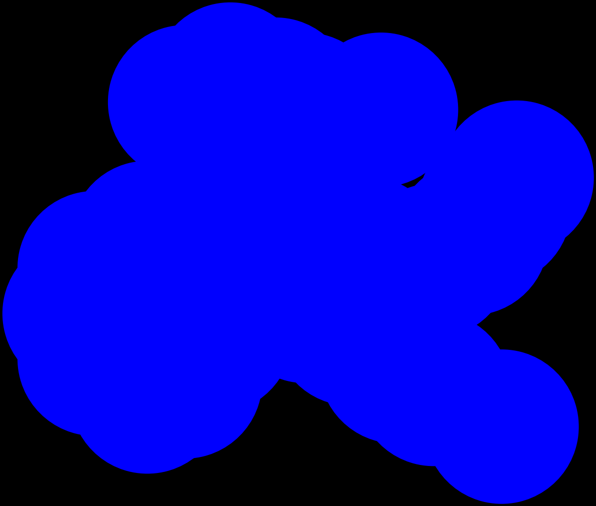

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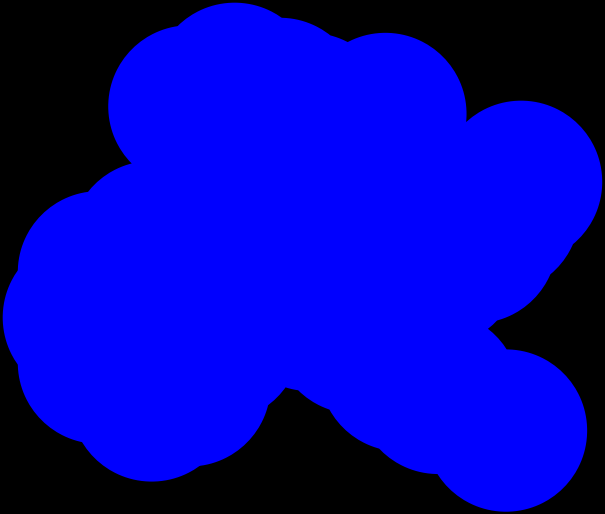

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Brain arteries

[Bullitt and Aylward, 2002]

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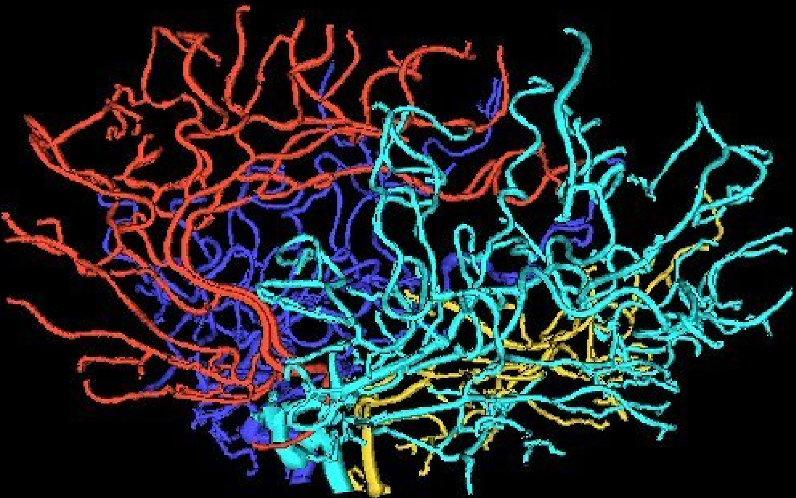

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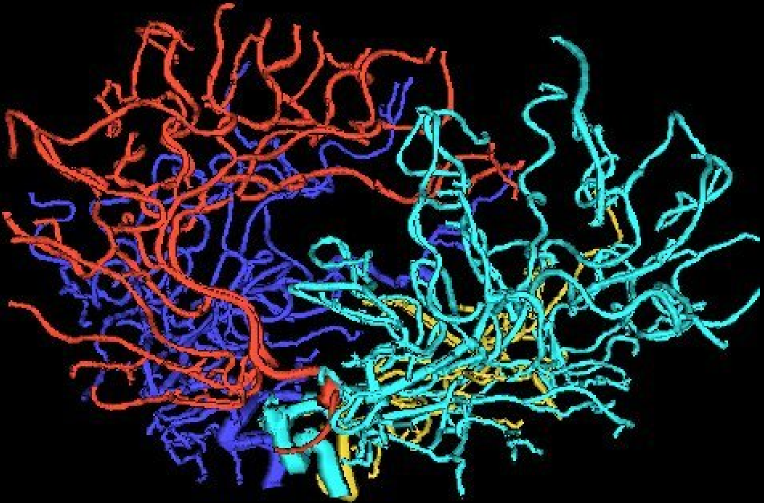

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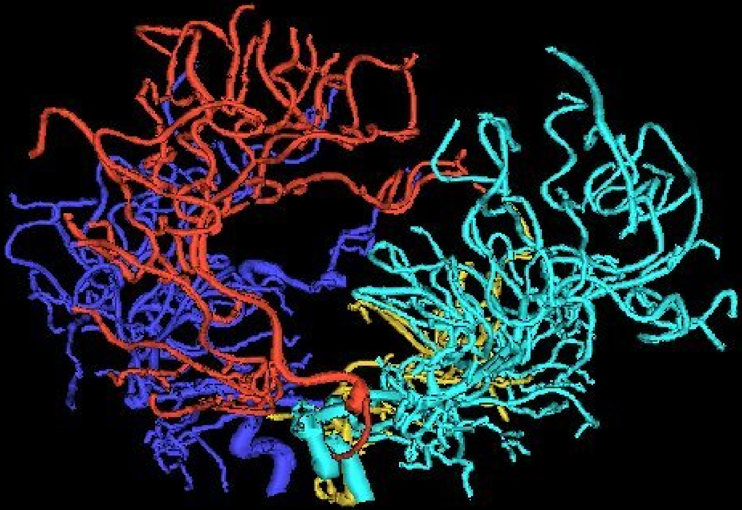

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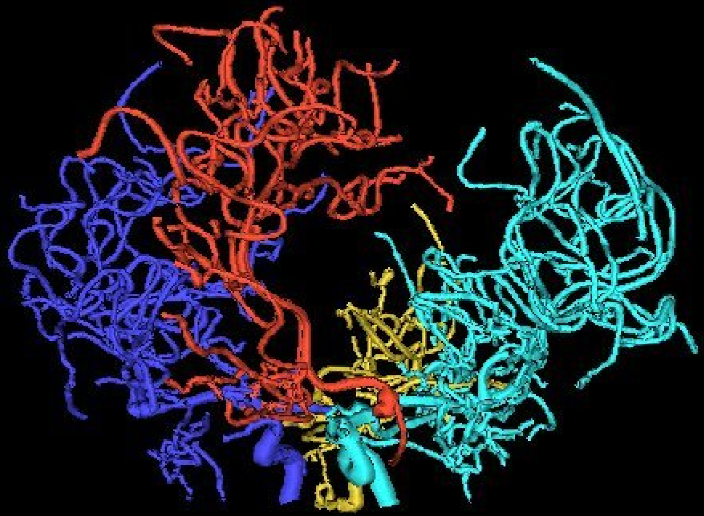

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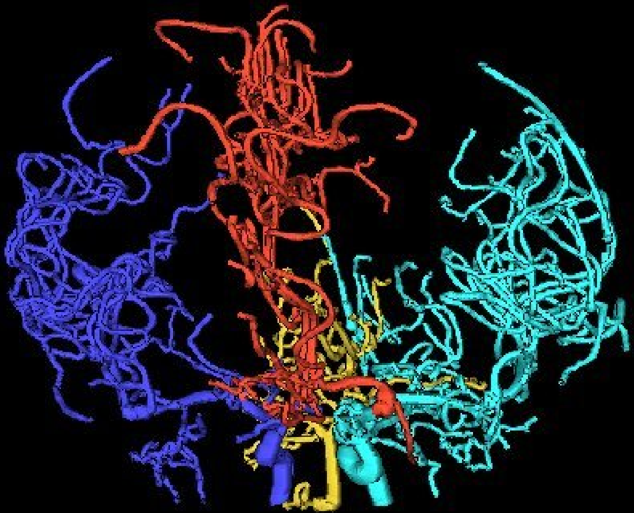

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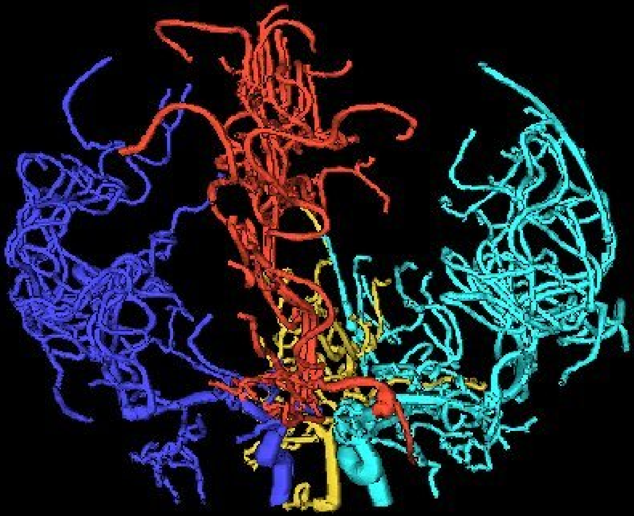
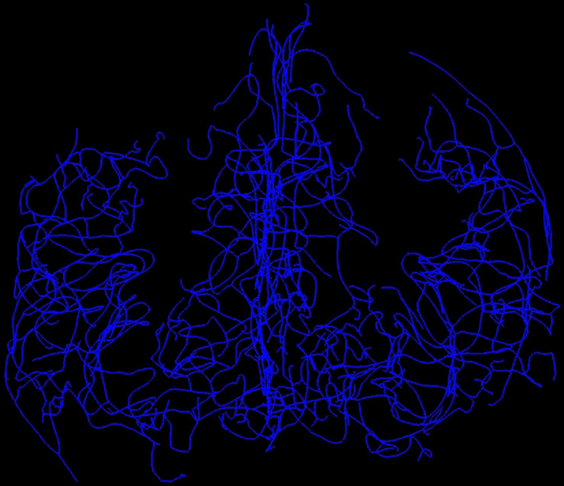


Goal: summary and statistical analysis [Bullitt and Aylward, 2002]

Example: filling brains [Bendich–Marron–M.–Pieloch–Skwerer 2014]



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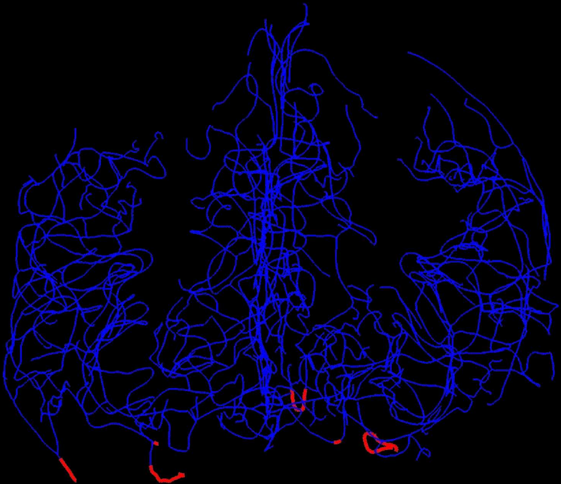

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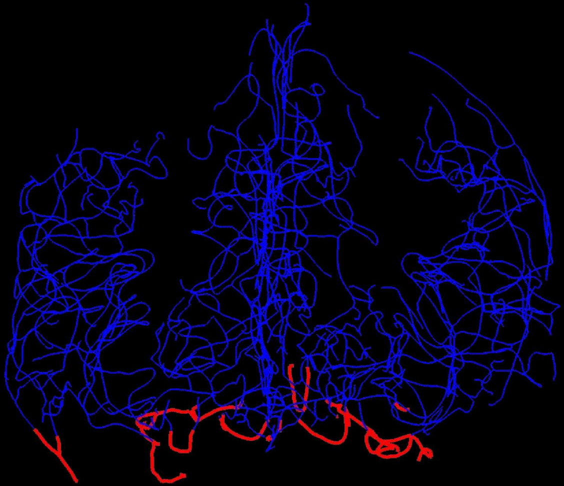

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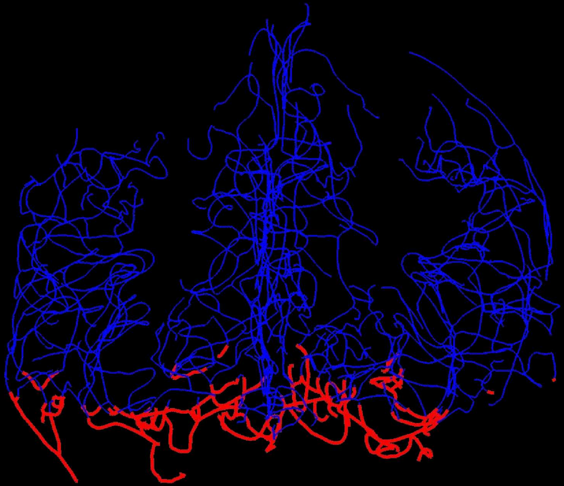

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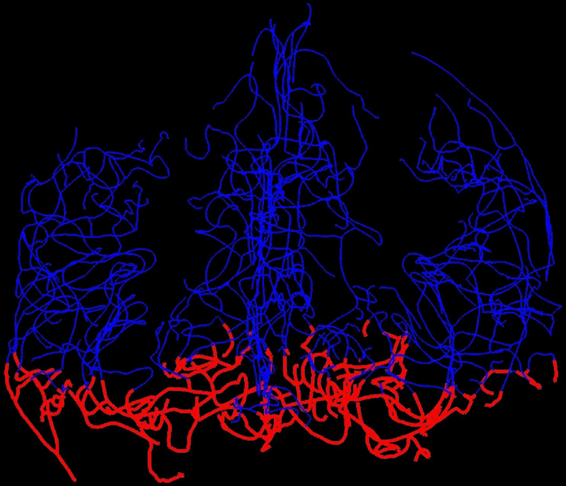

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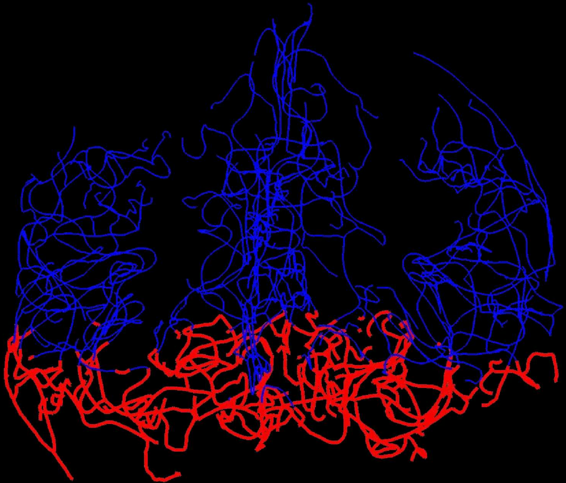

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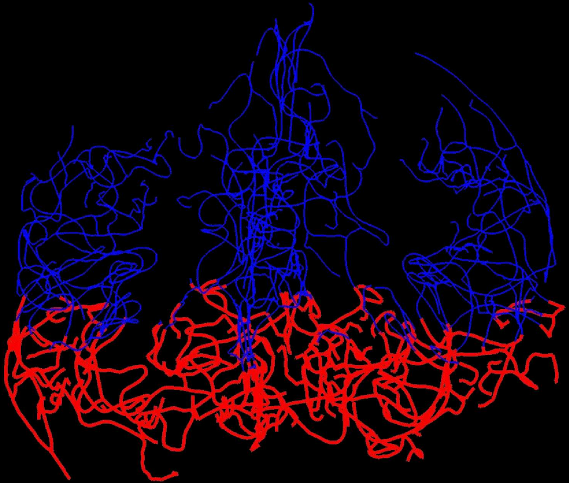

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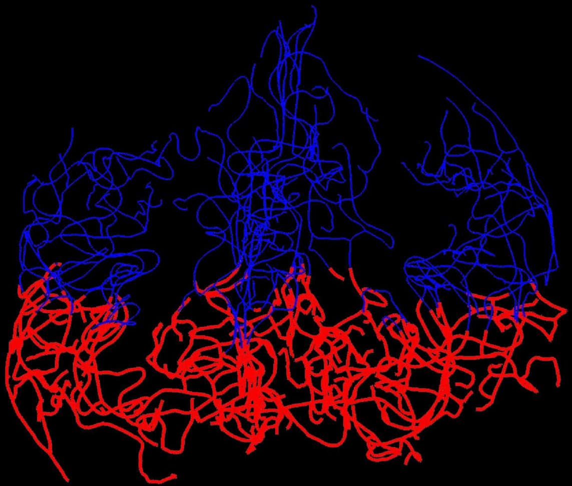

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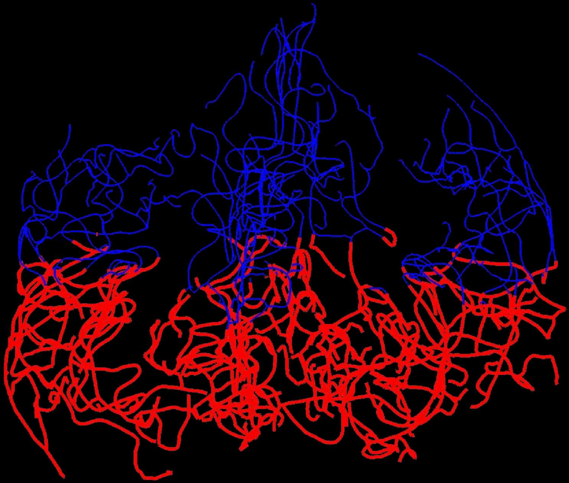

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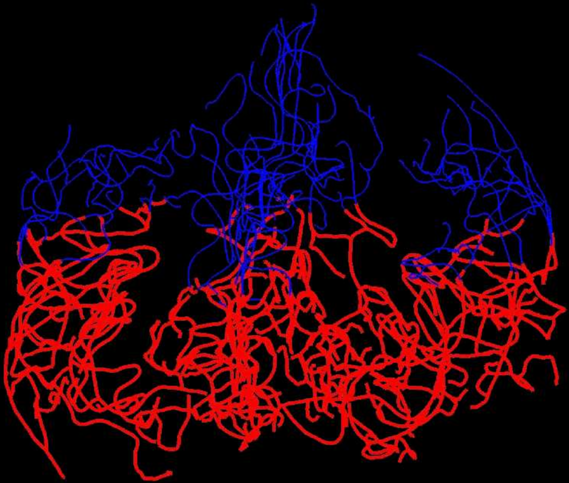

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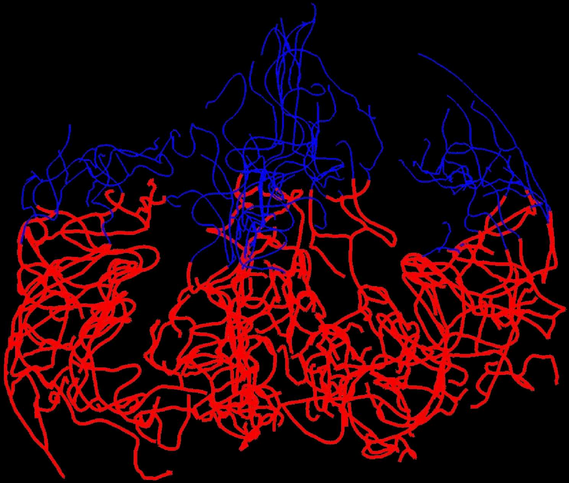

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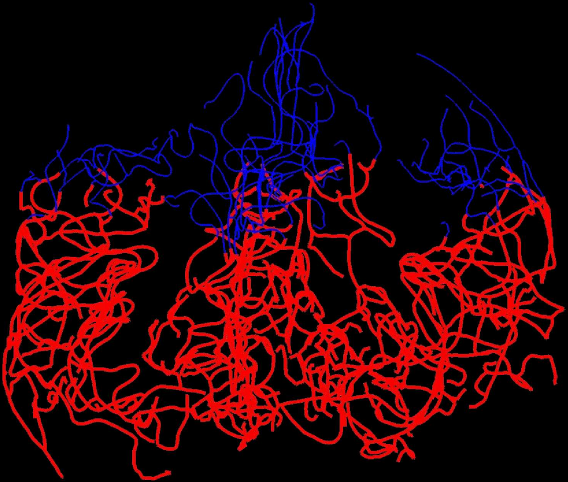

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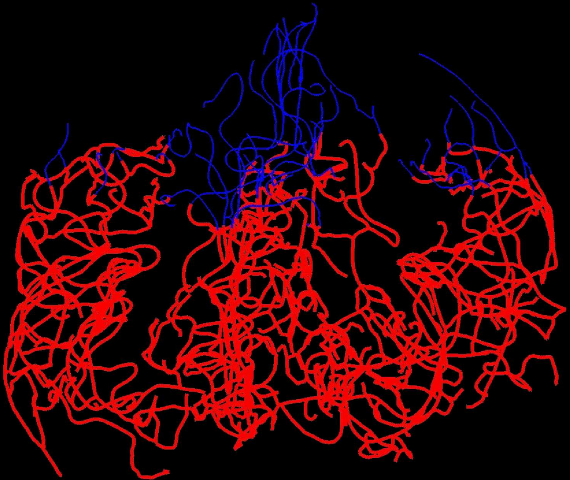

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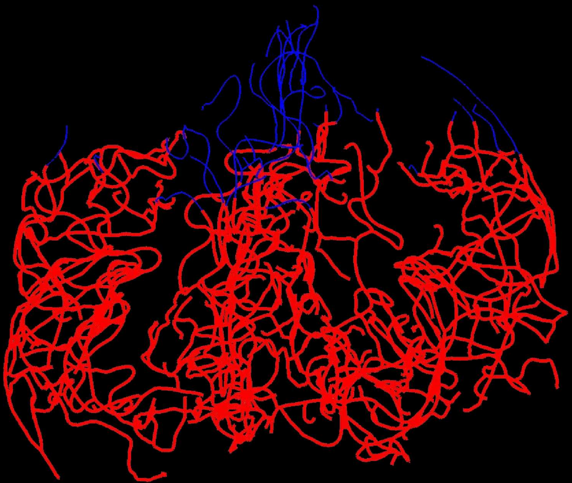

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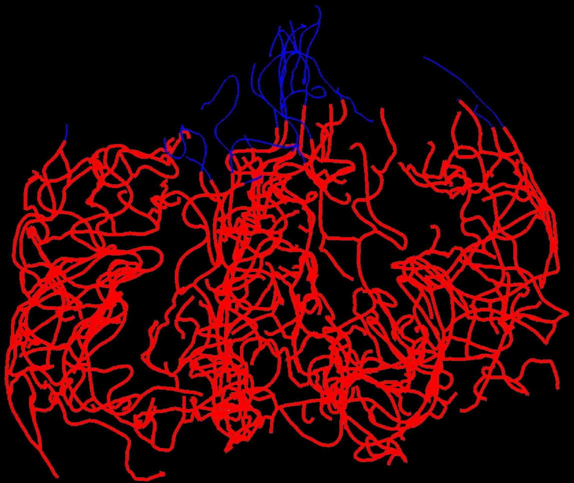

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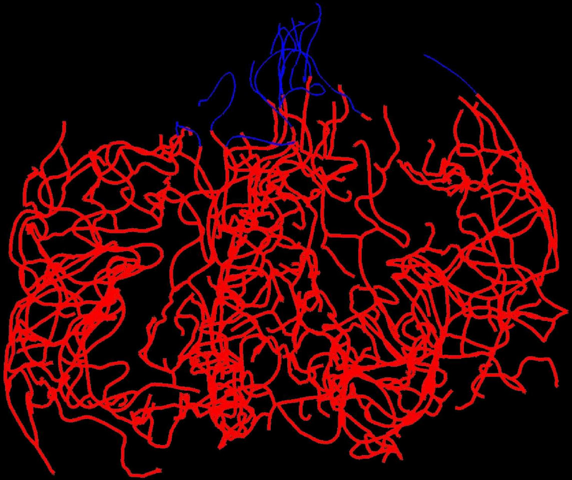

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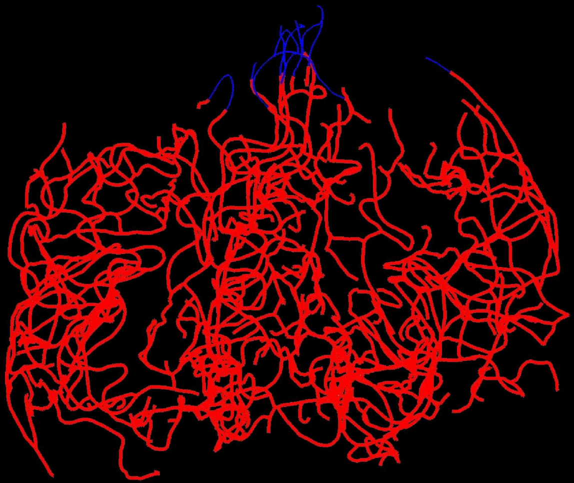

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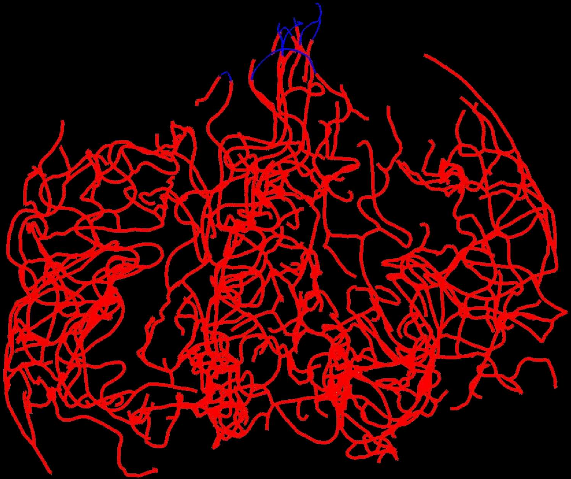

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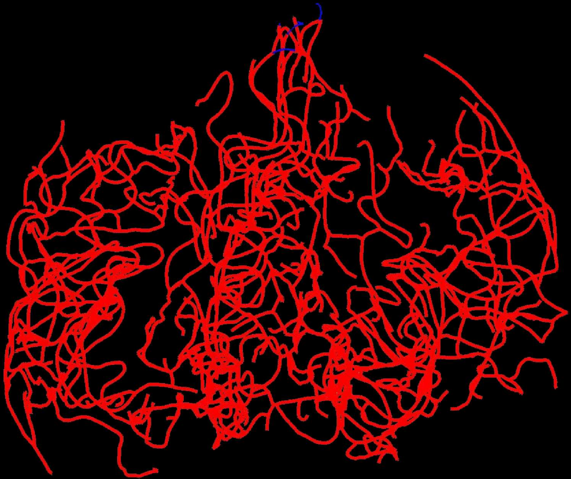

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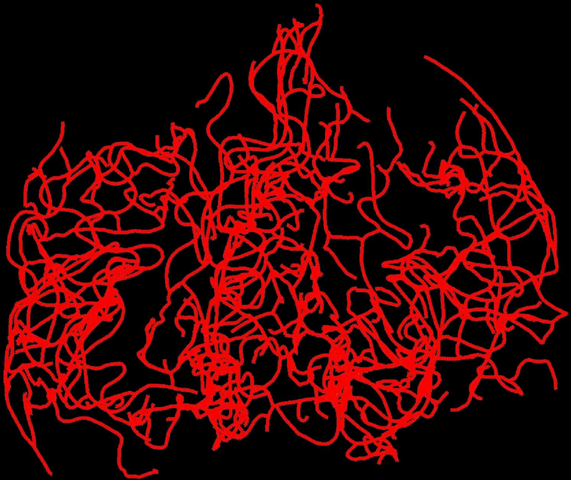

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Sweep filtration

Filter brain arteries by sweeping across with a plane:

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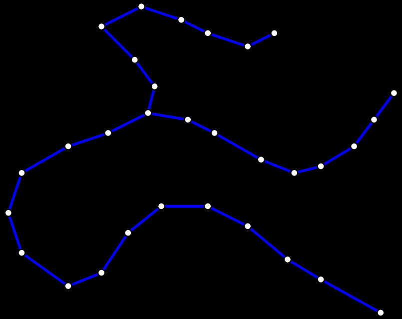

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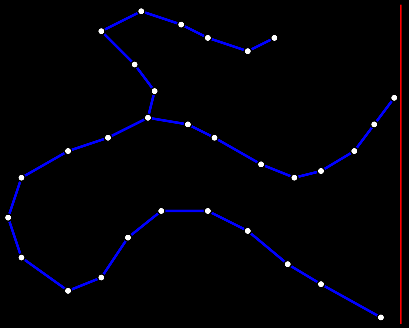

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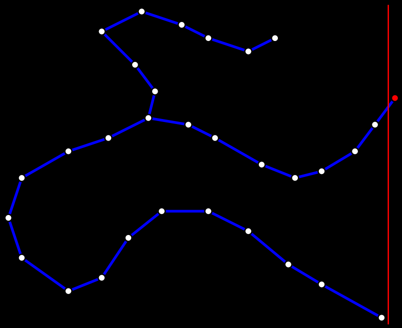

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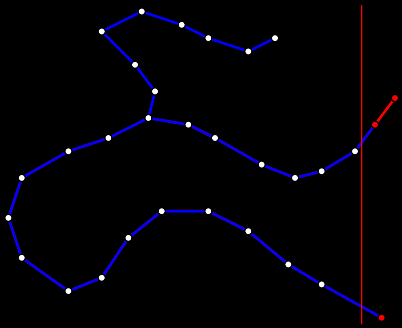

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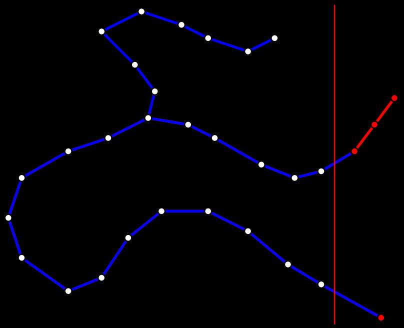

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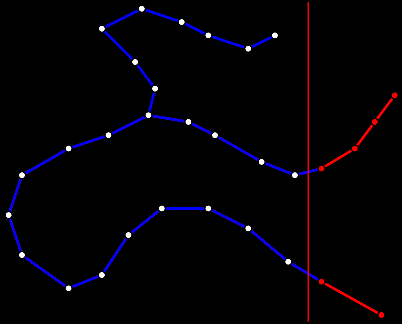

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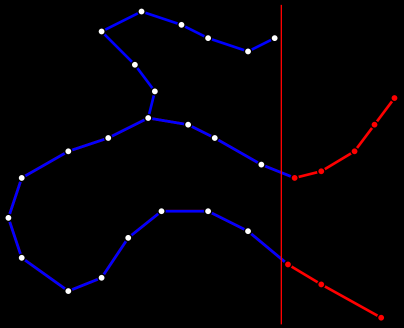

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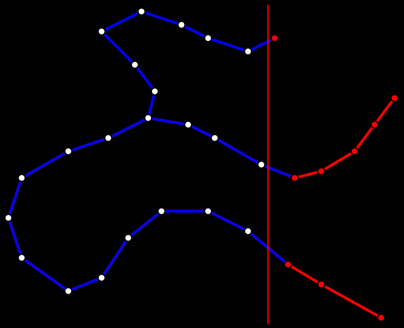

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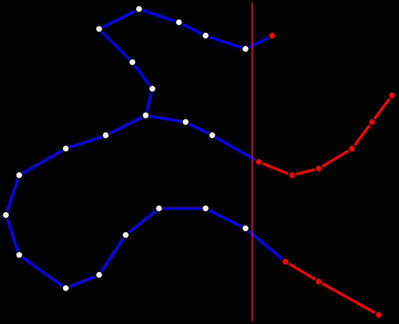

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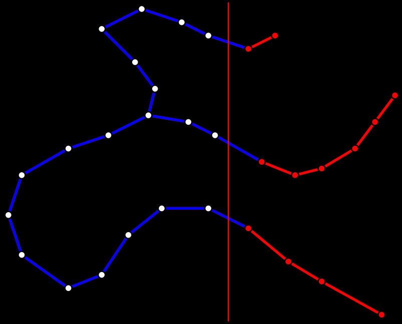

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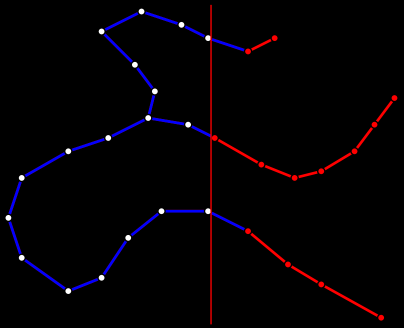

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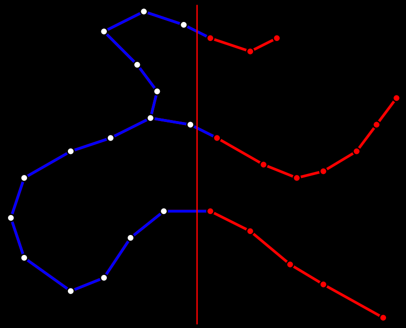

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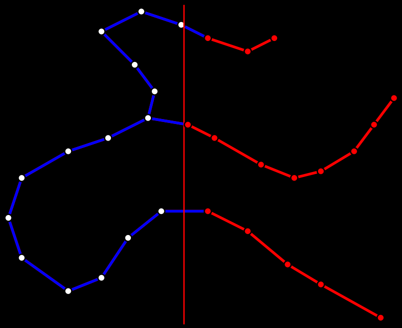

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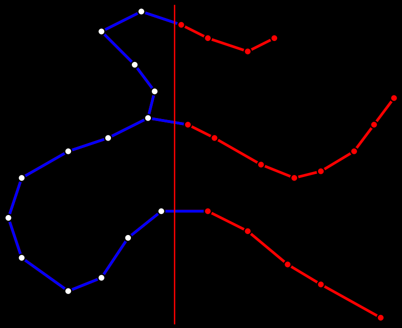

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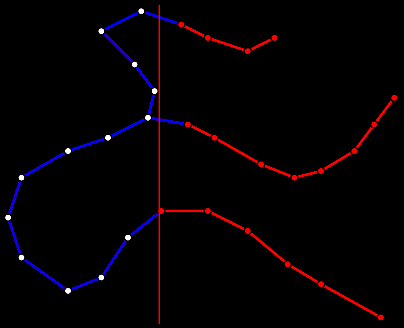

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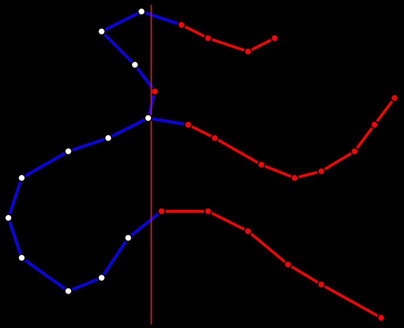

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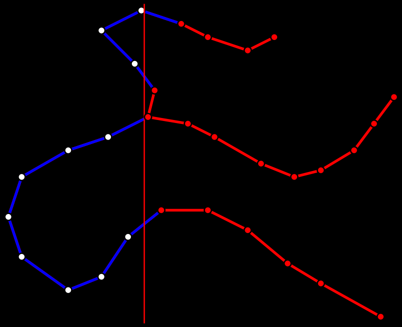

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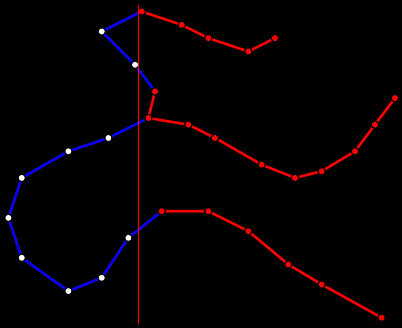

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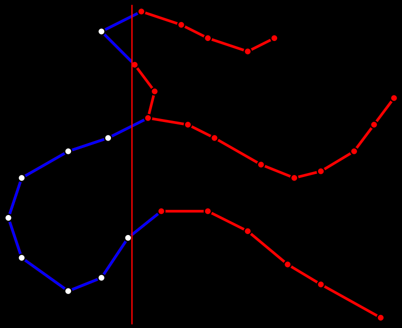

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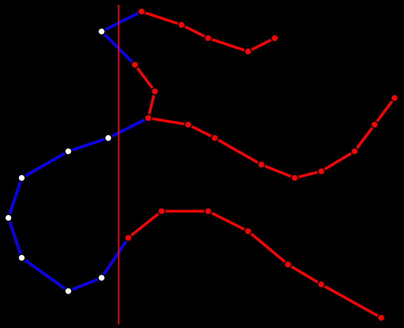

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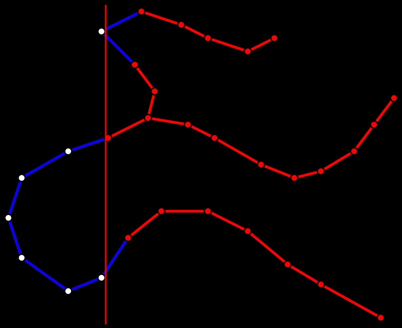

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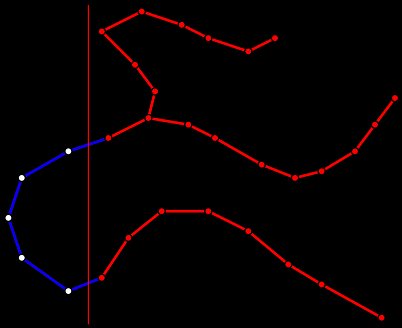

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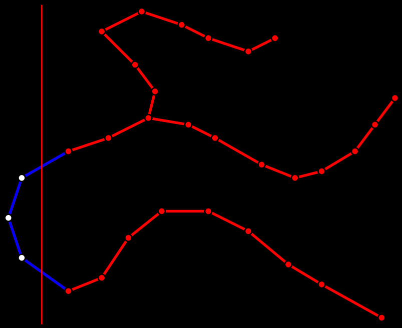

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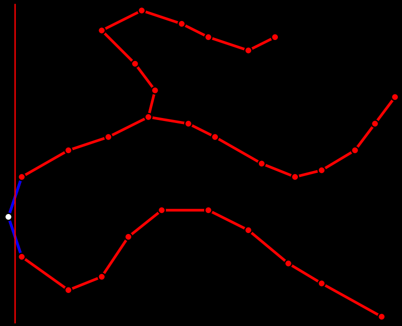

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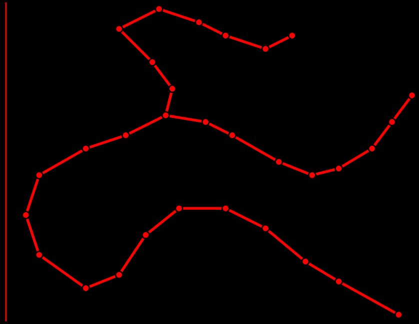

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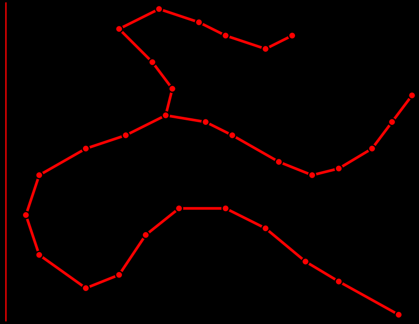

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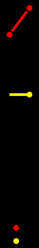

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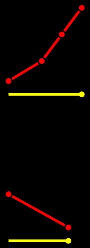

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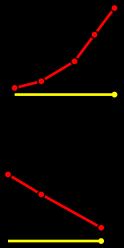

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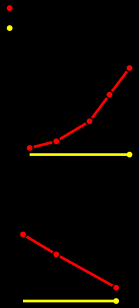

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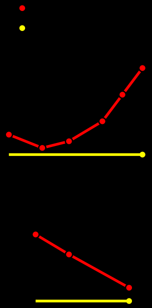

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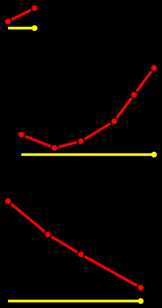

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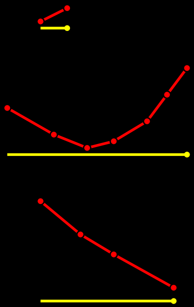

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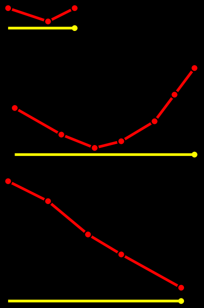

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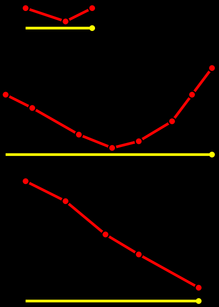

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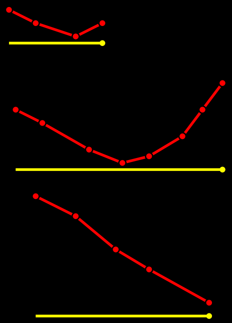

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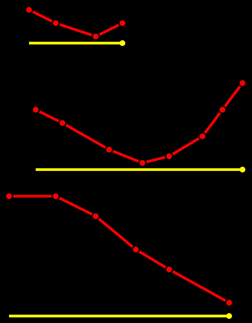

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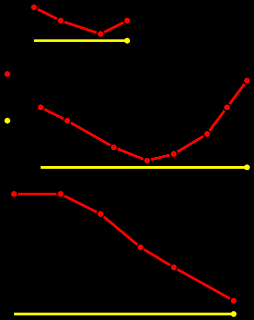

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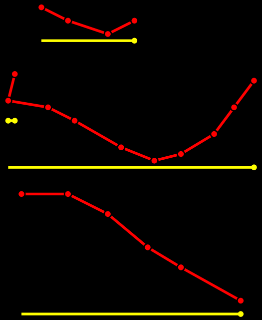

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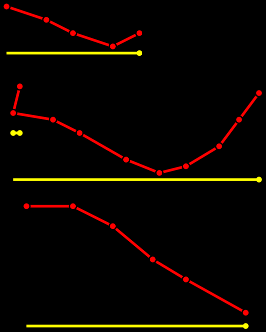

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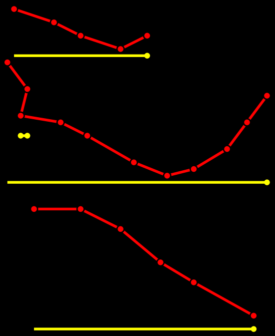

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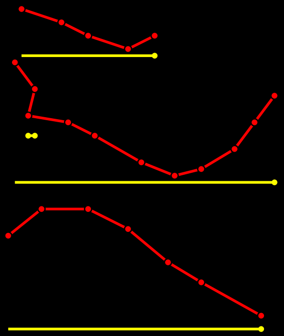

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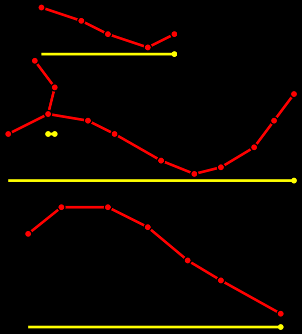
- death of each component (when it joins to an older component)

Method

Sweep filtration

Filter brain arteries by sweeping across with a plane:



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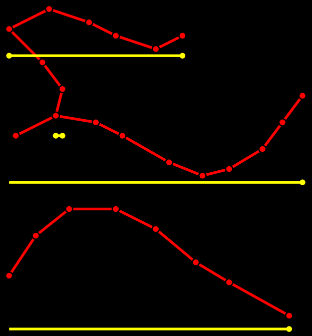

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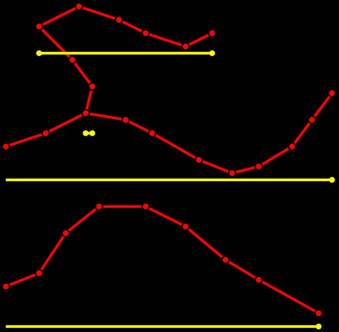

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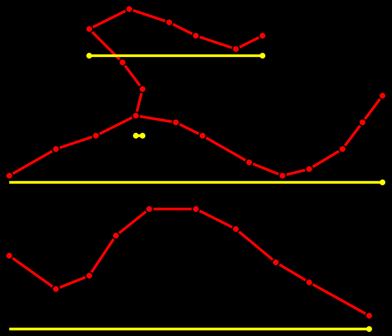

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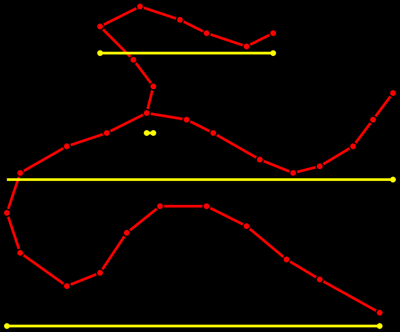

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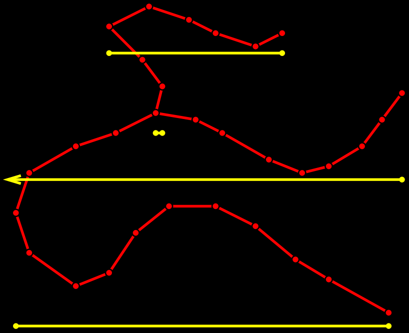

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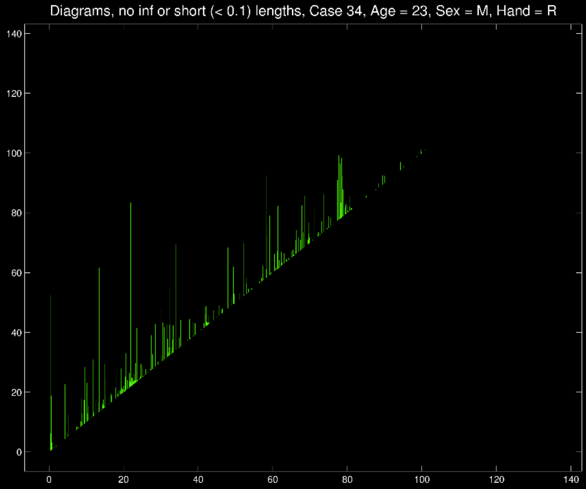
Bar codes

Data structure: 3D tree $\rightsquigarrow$ bar code / lace array / persistence diagram:



- multiset of (vertical) line segments $[t, t']$ (plotted at x-coordinate t)
- one for each class with birth time t and death time t'.

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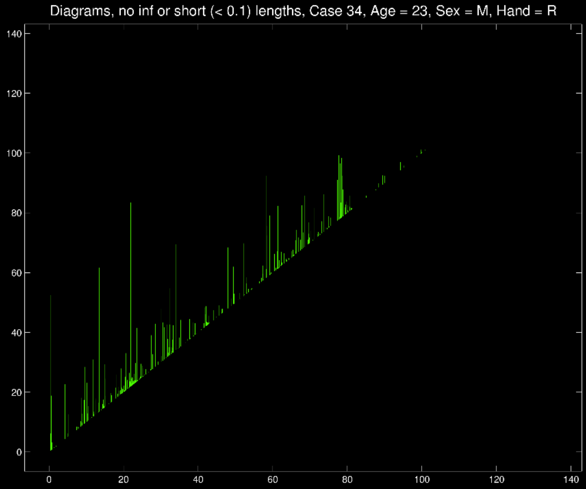

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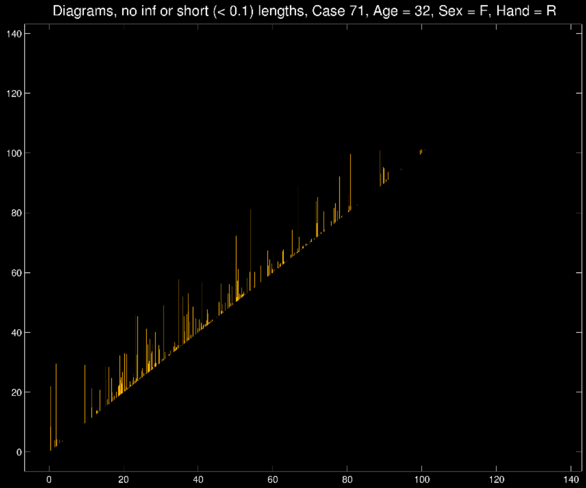

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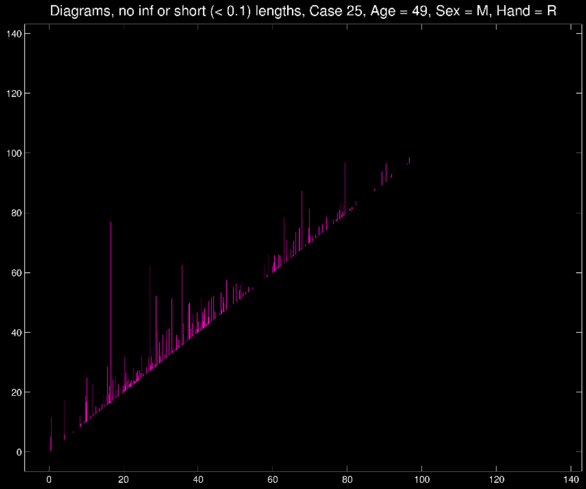

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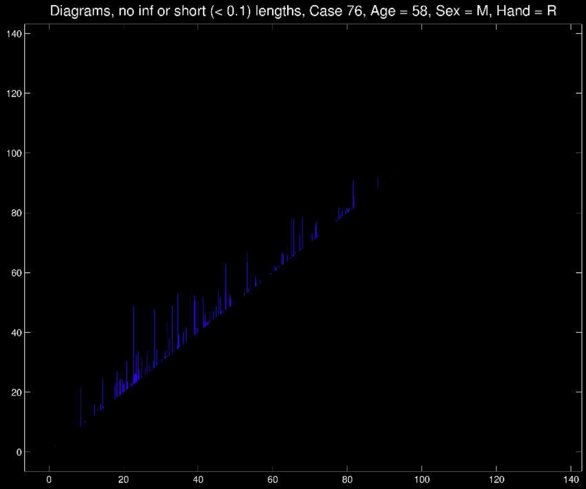

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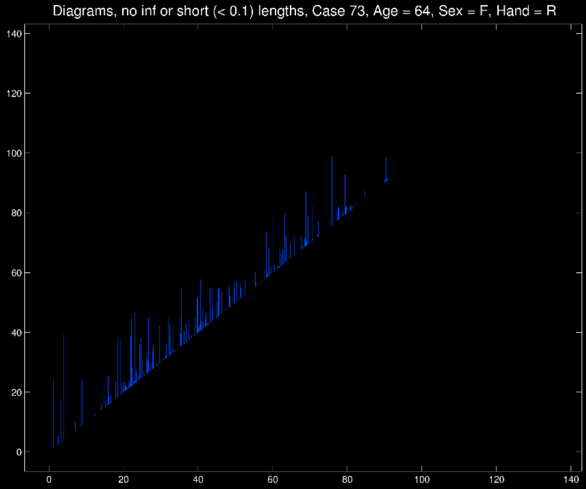

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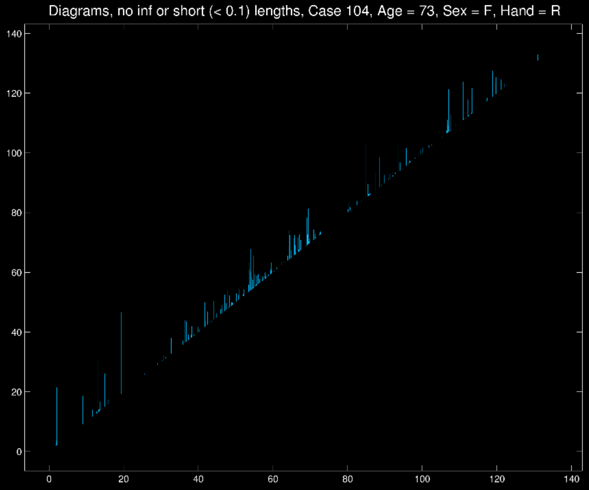

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Persistent homology

Input. Topological space X filtered by set Q of subspaces: $X_q \subseteq X$ for $q \in Q$
$\Rightarrow Q$ is a partially ordered set: $X_q \subseteq X_{q'} \Leftrightarrow q \preceq q'$

Def. $\{X_q\}_{q \in Q}$ has persistent homology $\{H_q = H(X_q; \Bbbk)\}_{q \in Q}$. This is a

Def. Q-module over the poset Q:

- family $H = \{H_q\}_{q \in Q}$ of vector spaces over the field $\Bbbk$ with
- homomorphism $H_q \to H_{q'}$ whenever $q \prec q'$ in Q such that
- $H_q \to H_{q''}$ equals the composite $H_q \to H_{q'} \to H_{q''}$ whenever $q \prec q' \prec q''$

Examples

- points in $\mathbb{R}^n$: $Q = \{0, \ldots, m\}$ or $\mathbb{R}$ 1-parameter ("ordinary") persistence
- brain arteries: $Q = \{0, \ldots, m\}$ or $\mathbb{R}$ 1-parameter ("ordinary") persistence
- wing veins: $Q = \mathbb{Z}^2$ or $\mathbb{R}^2$ 2 discrete or continuous parameters
- probability distributions: $Q = \mathbb{R}^2$ 2 continuous parameters
- $Q = \mathbb{Z}^n \Leftrightarrow H = \mathbb{Z}^n$-graded $\Bbbk[x_1, \ldots, x_n]$-module
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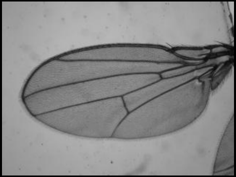
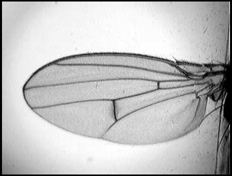
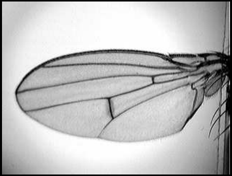
Fruit fly wings

Normal fly wings [images from David Houle's lab]:

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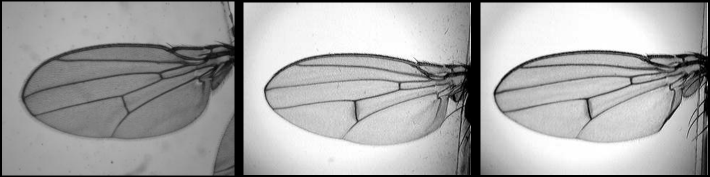
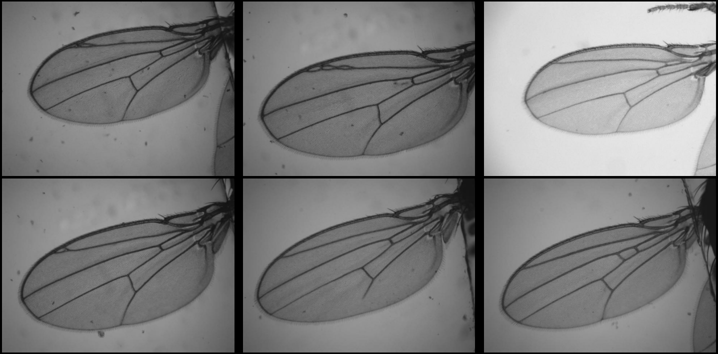
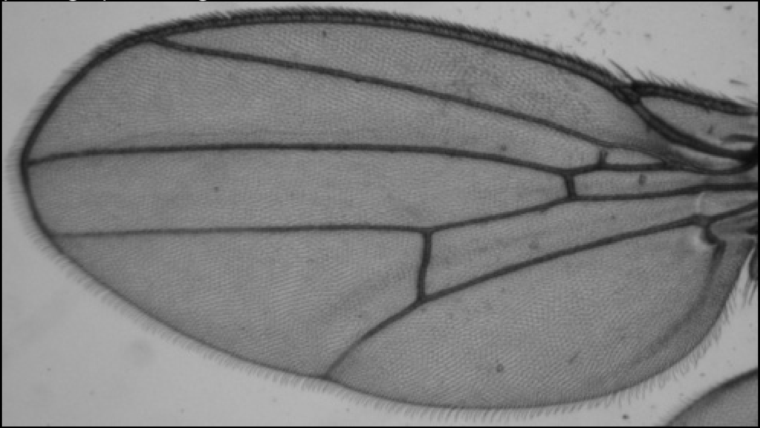


Topologically abnormal veins:



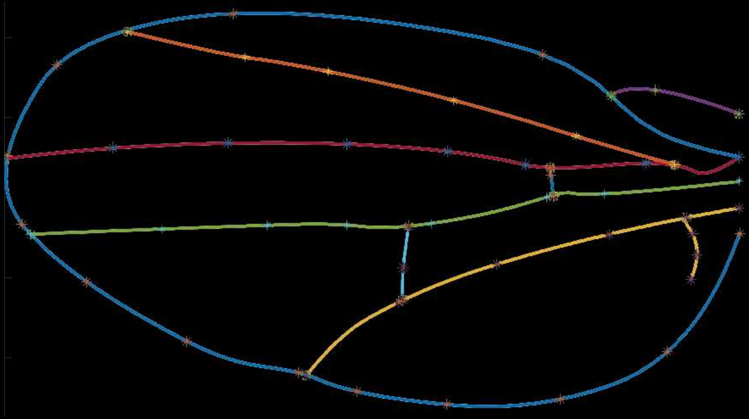
Fruit fly wings

photographic image



Fruit fly wings

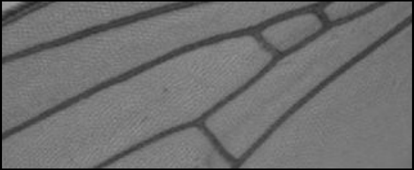
spline



Wing vein persistence [w/Houle, et al., ongoing]

Example. Encode fruit fly wing with 2-parameter persistence

- 1st parameter: distance from vertex set
- 2nd parameter: distance from edge set



Sublevel set $W_{r,s}$ is near edges but far from vertices $\Rightarrow H_{r,s} = H_i(W_{r,s})$

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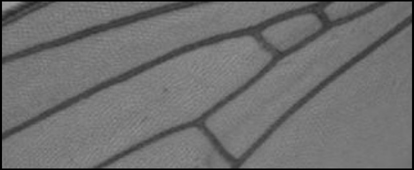

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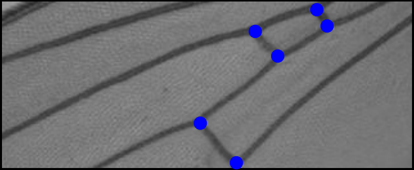

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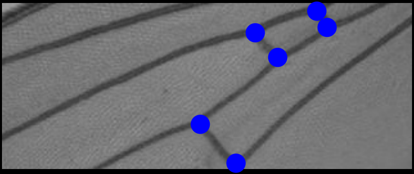

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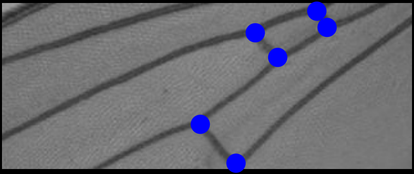

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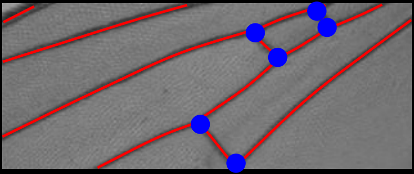

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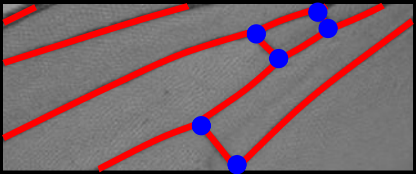

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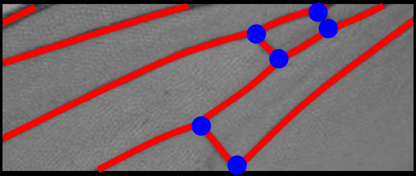

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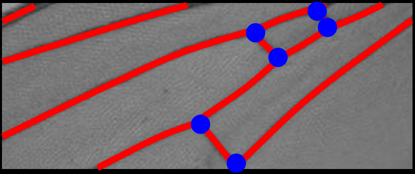

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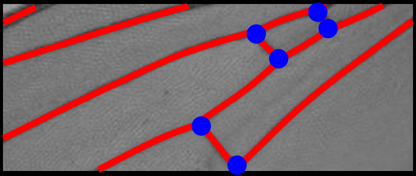

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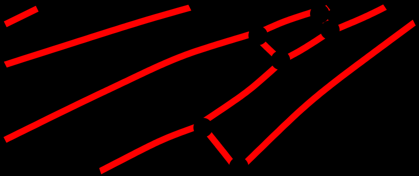

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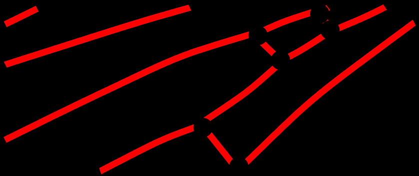

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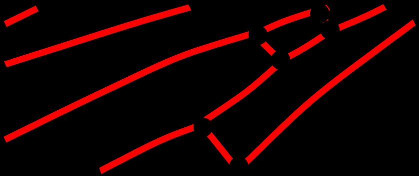

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Multiscale summary

Wing vein persistence [w/Houle, et al., ongoing]

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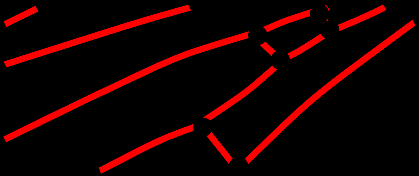

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Multiscale summary

$\mathbb{Z}^2$-module:

$$
\begin{array}{ccccccc}
 & \uparrow & & \uparrow & & \uparrow & \\
\rightarrow & H_{r-\varepsilon,s+\delta} & \rightarrow & H_{r,s+\delta} & \rightarrow & H_{r+\varepsilon,s+\delta} & \rightarrow \\
 & \uparrow & & \uparrow & & \uparrow & \\
\rightarrow & H_{r-\varepsilon,s} & \rightarrow & H_{r,s} & \rightarrow & H_{r+\varepsilon,s} & \rightarrow \\
 & \uparrow & & \uparrow & & \uparrow & \\
\rightarrow & H_{r-\varepsilon,s-\delta} & \rightarrow & H_{r,s-\delta} & \rightarrow & H_{r+\varepsilon,s-\delta} & \rightarrow \\
 & \uparrow & & \uparrow & & \uparrow &
\end{array}
$$

Wing vein persistence [w/Houle, et al., ongoing]

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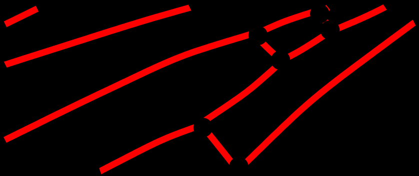

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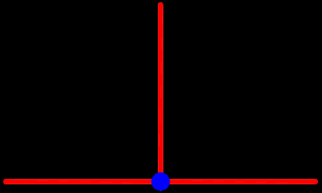
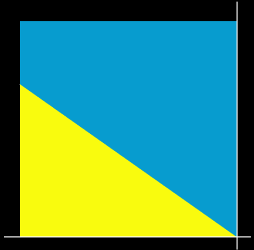


A piece of fly wing vein

$\rightsquigarrow$



The (r,s)-plane $\mathbb{R}^2$

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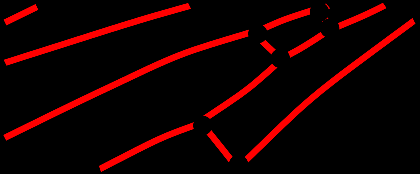

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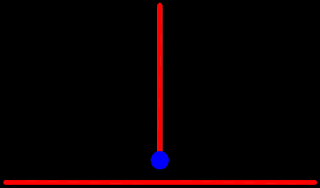

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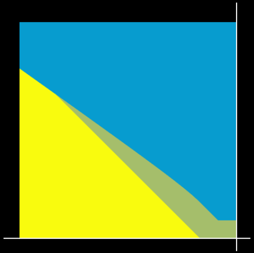

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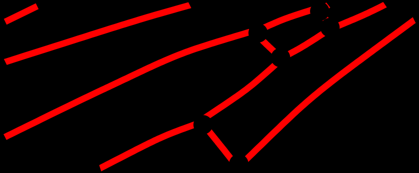

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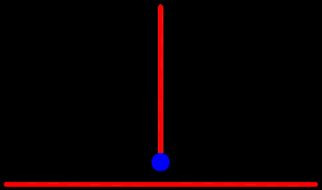

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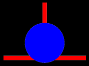

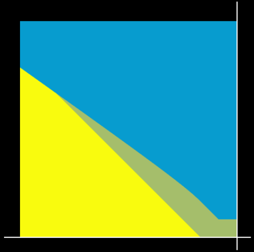

The (r, s)-plane $\mathbb{R}^2$

Wing vein persistence [w/Houle, et al., ongoing]

Example. Encode fruit fly wing with 2-parameter persistence

- 1st parameter: distance from vertex set (require distance $\geq -r$)
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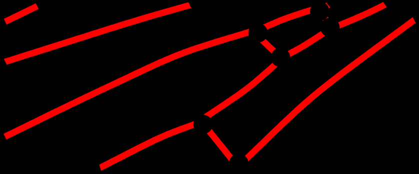

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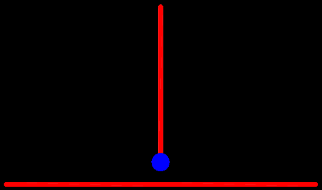

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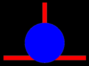

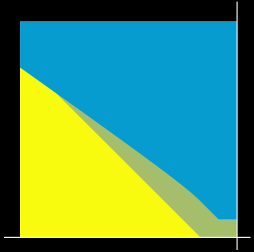

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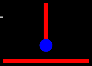

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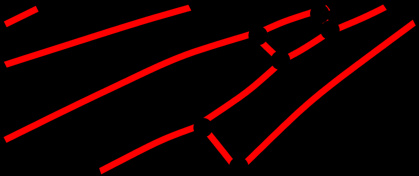

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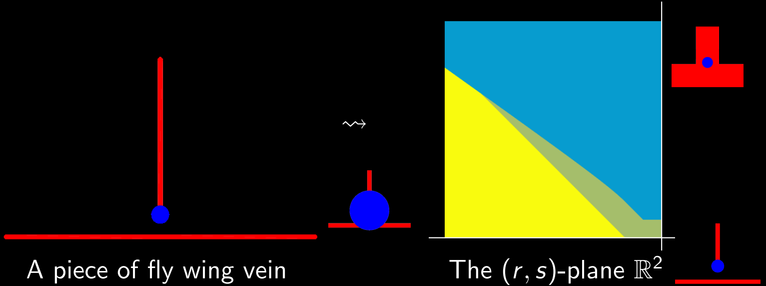

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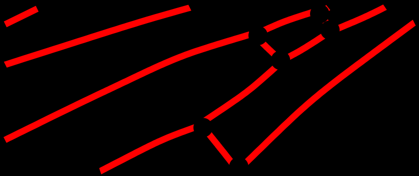

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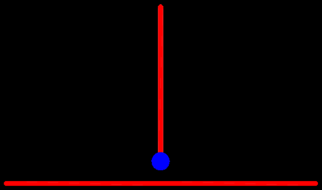

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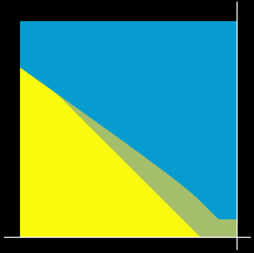

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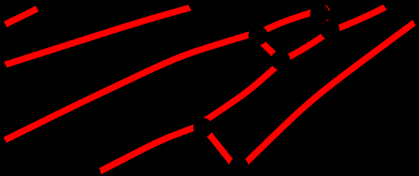

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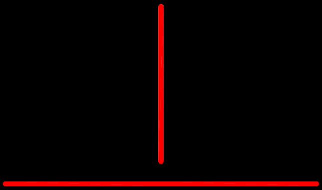

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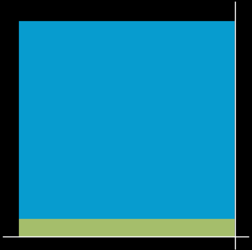

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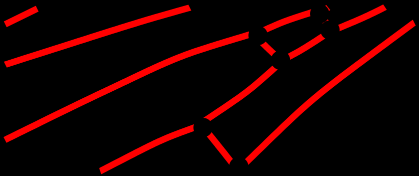

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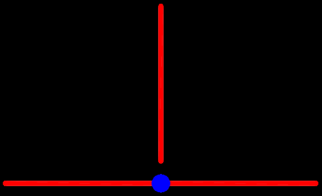

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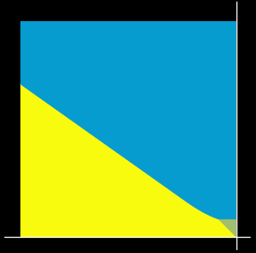

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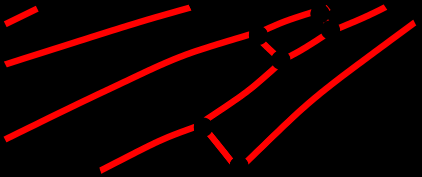

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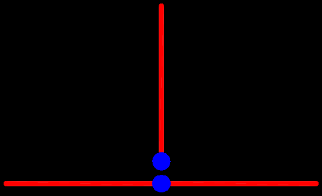

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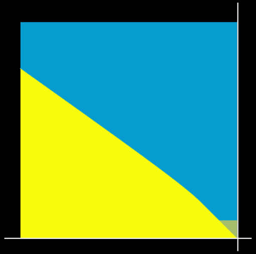

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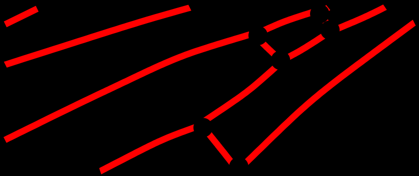

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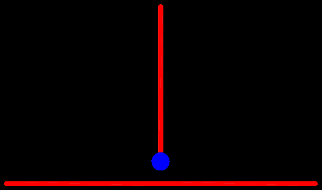

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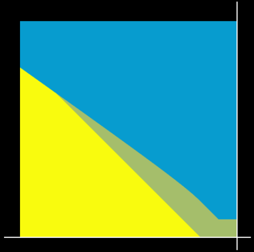

The (r, s)-plane $\mathbb{R}^2$

Persistent homology

Input. Topological space X filtered by set Q of subspaces: $X_q \subseteq X$ for $q \in Q$
$\Rightarrow Q$ is a partially ordered set: $X_q \subseteq X_{q'} \Leftrightarrow q \preceq q'$

Def. $\{X_q\}_{q \in Q}$ has persistent homology $\{H_q = H(X_q; \Bbbk)\}_{q \in Q}$. This is a

Def. Q-module over the poset Q:

- family $H = \{H_q\}_{q \in Q}$ of vector spaces over the field $\Bbbk$ with
- homomorphism $H_q \to H_{q'}$ whenever $q \prec q'$ in Q such that
- $H_q \to H_{q''}$ equals the composite $H_q \to H_{q'} \to H_{q''}$ whenever $q \prec q' \prec q''$

Examples

- points in $\mathbb{R}^n$: $Q = \{0, \ldots, m\}$ or $\mathbb{R}$ 1-parameter ("ordinary") persistence
- brain arteries: $Q = \{0, \ldots, m\}$ or $\mathbb{R}$ 1-parameter ("ordinary") persistence
- wing veins: $Q = \mathbb{Z}^2$ or $\mathbb{R}^2$ 2 discrete or continuous parameters
- probability distributions: $Q = \mathbb{R}^2$ 2 continuous parameters
- $Q = \mathbb{Z}^n \Leftrightarrow H = \mathbb{Z}^n$-graded $\Bbbk[x_1, \ldots, x_n]$-module
- $Q = \mathbb{R}^n \Leftrightarrow H = \mathbb{R}^n$-graded $\Bbbk[\mathbb{R}^n_+]$-module

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Example: topology of probability distributions

Given probability measure μ on a space M and kernel function of bandwidth r

e.g. • K_r = Gaussian (normal distribution) of variance r on $\mathbb{R}^d$

• K_r = uniform measure on ball of radius r on $\mathbb{R}^d$

Def. Convolution with kernel K_r yields bandwidth r expansion $B_r(\mu) = K_r * \mu$.

Example. • $B_r(\mu_n) \sim B_r(\mu)$ if μ_n is uniform on an n-sample from μ

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Def. ν with density function F has support at sensitivity s:

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Def. The expansion of μ to bandwidth r and sensitivity s is $B_r(\mu)_{r^d s} \subseteq M$.

Prop. $\{B_r(\mu)_{r^d s} \mid r \in \mathbb{R}_{\geq 0}$ and $s \in \mathbb{R}_{\geq 1}\} \subseteq M$ nested as r and s increase.

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Def. μ has i^{th} bipersistent homology $H_i^{rs}(\mu) = H_i\big(B_r(\mu)_{r^d s}\big)$, an invariant of μ

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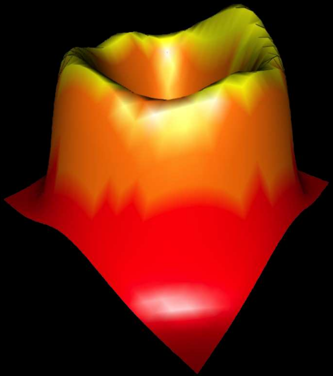

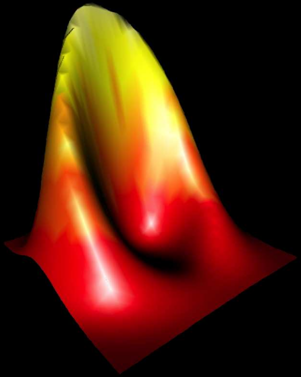

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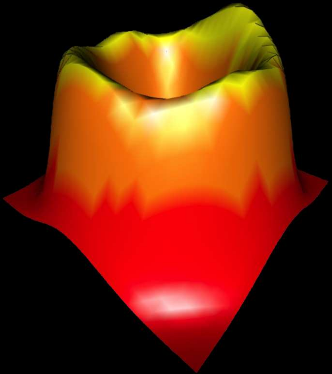

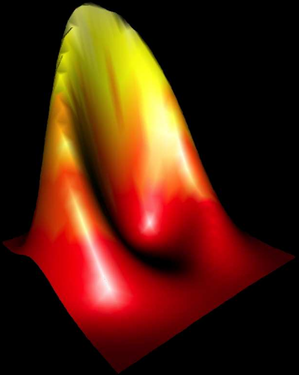

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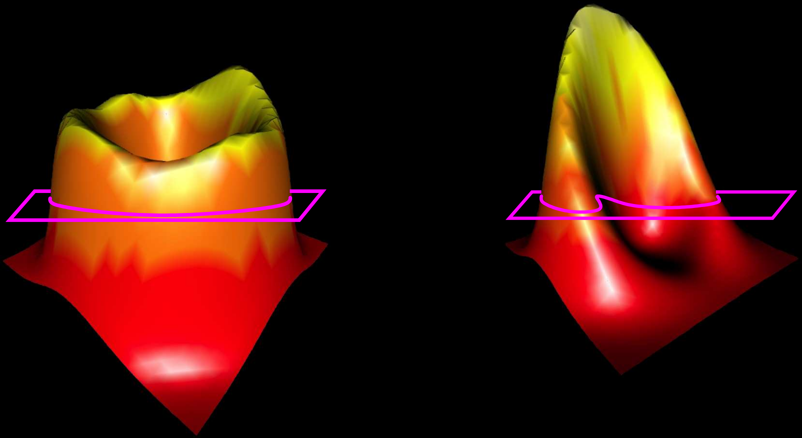

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$$\nu_s = \{x \in M \mid F(x) \geq 1/s\}.$$

Def. The expansion of μ to bandwidth r and sensitivity s is $B_r(\mu)_{r^d s} \subseteq M$.

Prop. $\{B_r(\mu)_{r^d s} \mid r \in \mathbb{R}_{\geq 0}$ and $s \in \mathbb{R}_{\geq 1}\} \subseteq M$ nested as r and s increase.

Persistent homology: $B_r(\mu)_{r^d s} \rightsquigarrow$ homology $H_*(B_r(\mu)_{r^d s})$

Def. μ has i^{th} bipersistent homology $H_i^{rs}(\mu) = H_i\big(B_r(\mu)_{r^d s}\big)$, an invariant of μ

algebra, geometry, combinatorics of $H_*^{rs}(\nu) \leftrightarrow$ statistics of ν

Example: topology of probability distributions

Given probability measure μ on a space M and kernel function of bandwidth r

e.g. • $K_r =$ Gaussian (normal distribution) of variance r on $\mathbb{R}^d$

• $K_r =$ uniform measure on ball of radius r on $\mathbb{R}^d$

Def. Convolution with kernel K_r yields bandwidth r expansion $B_r(\mu) = K_r * \mu$.

Example. • $B_r(\mu_n) \sim B_r(\mu)$ if μ_n is uniform on an n-sample from μ

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Def. $\{X_q\}_{q \in Q}$ has persistent homology $\{H_q = H(X_q; \Bbbk)\}_{q \in Q}$. This is a

Def. Q-module over the poset Q:

- family $H = \{H_q\}_{q \in Q}$ of vector spaces over the field $\Bbbk$ with
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Examples

- points in $\mathbb{R}^n$: $Q = \{0, \ldots, m\}$ or $\mathbb{R}$ 1-parameter ("ordinary") persistence
- brain arteries: $Q = \{0, \ldots, m\}$ or $\mathbb{R}$ 1-parameter ("ordinary") persistence
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Statistical analysis [Bendich–Marron–M.–Pieloch–Skwerer 2014]

Reduce to linear methods. 3D tree $\rightsquigarrow$ bar code $\rightsquigarrow$ vector in $\mathbb{R}^{100}$:

- top 100 bar lengths, in decreasing order, log scale
- correlate first principal component score vs. age

Conclusions

Longest bars in older brains tend to be shorter and later.

- Pearson correlation 0.52663
- p-value 3.0127×10^{-8} strongly significant

Remarks. Results essentially unchanged after

- rescaling to account for natural variation in overall brain size (force standard deviation of the set of bar lengths to equal 1)
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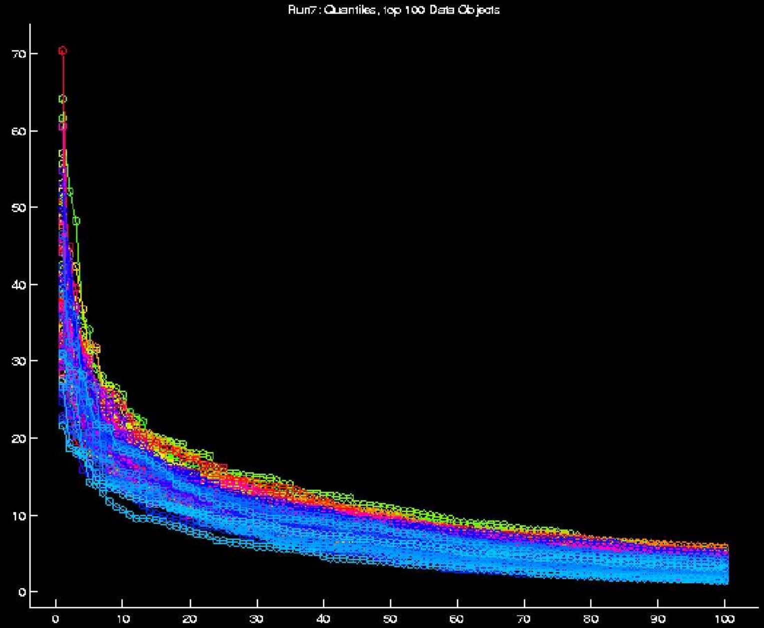
- repeating the analysis with residuals from regression between feature vector and total length.

Moral. Persistent homology can topologically detect statistically significant geometric motifs

Top 100 bars


Run7: Quantiles, top 100 Data Objects
70
60
50
40
30
20
10
0
0
10
20
30
40
50
60
70
80
90
100

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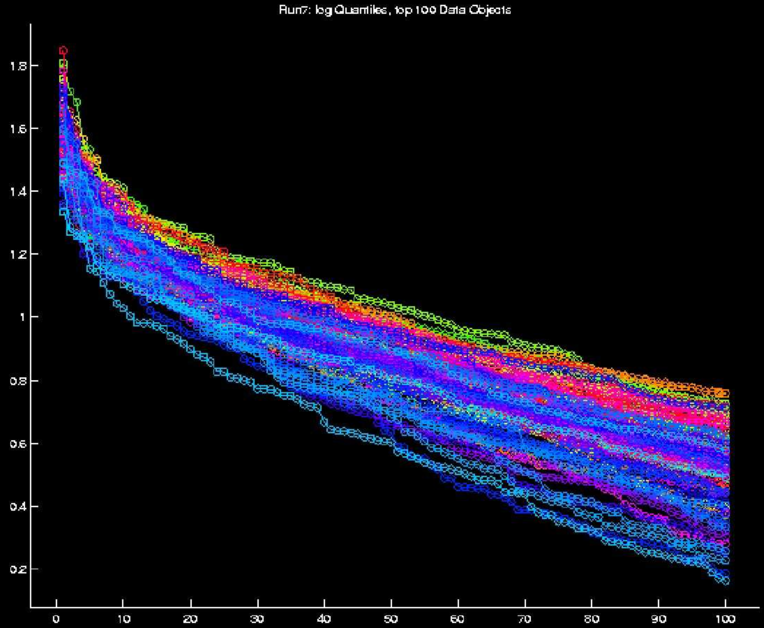
Top 100 bars: log scale



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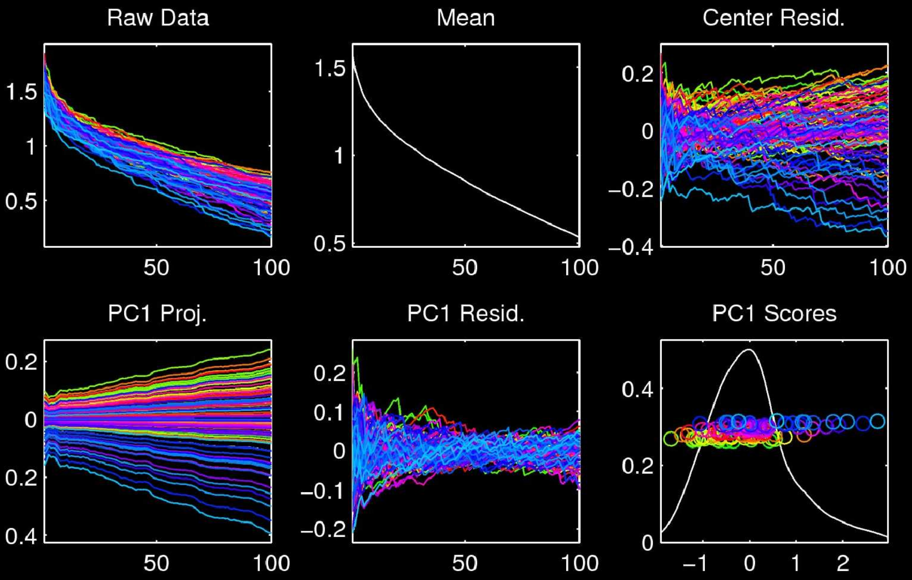
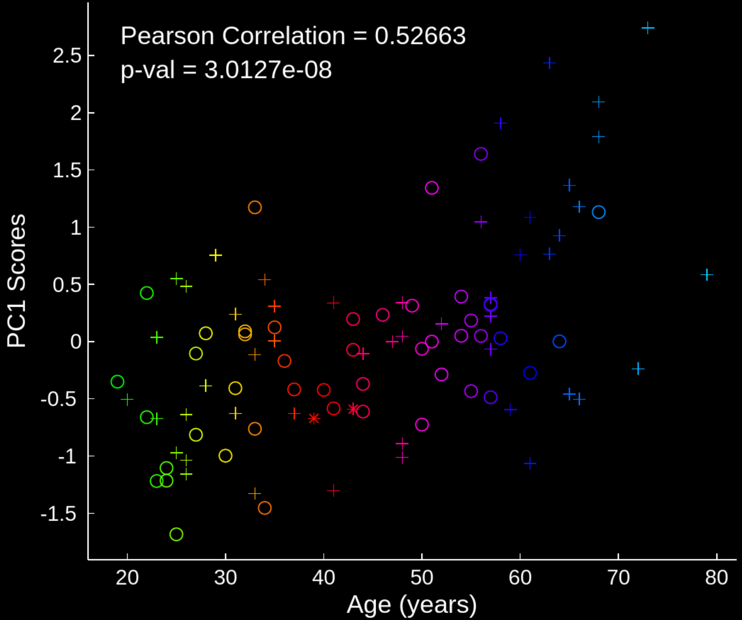
Age vs. PC1


Raw Data
1.5
1
0.5
50
100
Mean
1.5
1
0.5
50
100
Center Resid.
0.2
0
−0.2
−0.4
50
100
PC1 Proj.
0.2
0
0.2
0.4
50
100
PC1 Resid.
0.2
0.1
0
−0.1
−0.2
50
100
PC1 Scores
0.4
0.2
0
−1
0
1
2

Age vs. PC1


Pearson Correlation = 0.52663
p-val = 3.0127e-08
PC1 Scores
Age (years)
2.5
2
1.5
1
0.5
0
-0.5
-1
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20
30
40
50
60
70
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Interval decomposition

Thm [Crawley-Boevey 2015]. $\mathbb{R}$-module $M \Rightarrow M \cong \bigoplus_{I \in \mathcal{I}} \Bbbk\{I\}$ with $\mathcal{I}$ a set of intervals

Consequence over $\mathbb{R}$: $M \rightsquigarrow$ bar code / lace array / persistence diagram

- reinvented a number of times
- earliest: algebraic geometry of representation theory [Abeasis–Del Fra 1980]
  - explicitly drawn bars
  - Möbius inversion formulas

Def. An interval I in a poset Q is a convex connected subset: $a, b \in I \Rightarrow$

- $q \in I$ whenever $a \preceq q \preceq b$ and
- there is a (zigzag) chain in I of comparable elements from a to b.

For any subset $S \subseteq Q$, let $\Bbbk\{S\} = \bigoplus_{s \in S} \Bbbk_s$ be its indicator module.

Examples. In $\mathbb{R}^2$, intervals can look like

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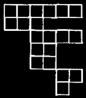
Old bar codes

It is convenient to represent λ^A as a "*diagram of boxes*", each row starting at i and ending at j stands for one indecomposable factor of type $E_{(i,j)}$.
E.g. the following diagram represents λ^A for A isomorphic to

$$E_{(1,6)} \oplus E_{(1,3)} \oplus E_{(3,6)} \oplus E_{(3,4)} \oplus E_{(3,4)} \oplus E_{(5,6)} \oplus E_{(5,5)}:$$



2.4. Conversely any indexed set $\lambda = (\lambda_{(i,j)})_{1 \leqq i \leqq j \leqq m}$ of natural numbers determines an orbit in $L(V_1, V_2, \ldots, V_m)$ provided $\dim V_i = \hat{\lambda}_i := \sum_{r \leqq i \leqq s} \lambda_{(r,s)}$ ($= \#$ boxes in the i^{th} column of λ). We will shortly call such an indexed set a *diagram*, define
[...]
Let us introduce now the set of non-negative integers $n^A = \{n^A_{rs}\}_{1 \leqslant r \leqslant s \leqslant m}$ associated to A and defined by

$$n^A_{rs} := \sum_{p \leqslant r \leqslant s \leqslant q} e^A_{pq}. \tag{2.3}$$

n^A_{rs} is the number of the segments of the diagram of $|A|$ which contain the integers r, s. It follows that we have

$$e^A_{pq} = n^A_{pq} - n^A_{p-1,q} - n^A_{p,q+1} + n^A_{p-1,q+1} \tag{2.4}$$

where we set $n^A_{rs} = 0$ if $r < 0$ or $s > m + 1$.

[Abeasis–Del Fra 1980, Abeasis–Del Fra–Kraft 1981]

Old bar codes

Example 1.5. Consider the rank array $\mathbf{r} = (r_{ij})$, its lace array $\mathbf{s} = (s_{ij})$, and its rectangle array $\mathbf{R} = (R_{ij})$, which we depict as follows.

$$
\mathbf{r} = \begin{array}{cccc|c}
3 & 2 & 1 & 0 & i \diagup j \\
\hline
 & & & 2 & 0 \\
 & & 3 & 2 & 1 \\
 & 4 & 2 & 1 & 2 \\
3 & 2 & 1 & 0 & 3
\end{array}
\qquad
\mathbf{s} = \begin{array}{cccc|c}
3 & 2 & 1 & 0 & i \diagup j \\
\hline
 & & & 0 & 0 \\
 & & 0 & 1 & 1 \\
 & 1 & 0 & 1 & 2 \\
1 & 1 & 1 & 0 & 3
\end{array}
\qquad
\mathbf{R} = \begin{array}{ccc|c}
2 & 1 & 0 & i \diagup j \\
\hline
 & & & 1 \\
 & \square\!\square & \square & 2 \\
\substack{\square \\ \square} & \square & \square & 3
\end{array}
$$

The relation (1.2) says that an entry of $\mathbf{r}$ is the sum of the entries in $\mathbf{s}$ that are weakly southeast of the corresponding location. The height of R_{ij} is obtained by subtracting the entry r_{ij} from the one above it, while the width of R_{ij} is obtained by subtracting the entry r_{ij} from the one to its left.

It follows from the definition of R_{ij} that

$$\sum_{k \geq j} \operatorname{height}(R_{ik}) = r_{i,j-1} - r_{i,n} \leq r_{i,j-1} \qquad \text{for all } i \tag{1.3}$$

$$\sum_{\ell \leq i} \operatorname{width}(R_{\ell j}) = r_{i+1,j} - r_{0,j} \leq r_{i+1,j} \qquad \text{for all } j. \tag{1.4}$$

(This will be applied in Proposition 8.12.) The relation (1.2) can be inverted to obtain

$$s_{ij} = r_{ij} - r_{i-1,j} - r_{i,j+1} + r_{i-1,j+1} \tag{1.5}$$

[Knutson–M.–Shimozono 2005]

Interval decomposition

Thm [Crawley-Boevey 2015]. $\mathbb{R}$-module $M \Rightarrow M \cong \bigoplus_{I \in \mathcal{I}} \Bbbk\{I\}$ with $\mathcal{I}$ a set of intervals

Consequence over $\mathbb{R}$: $M \rightsquigarrow$ bar code / lace array / persistence diagram

- reinvented a number of times
- earliest: algebraic geometry of representation theory [Abeasis–Del Fra 1980]
  - explicitly drawn bars
  - Möbius inversion formulas

Def. An interval I in a poset Q is a convex connected subset: $a, b \in I \Rightarrow$

- $q \in I$ whenever $a \preceq q \preceq b$ and
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For any subset $S \subseteq Q$, let $\Bbbk\{S\} = \bigoplus_{s \in S} \Bbbk_s$ be its indicator module.

Examples. In $\mathbb{R}^2$, intervals can look like

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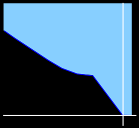
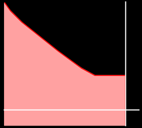
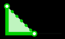
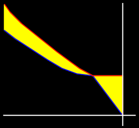
 or  or  but not 

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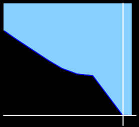
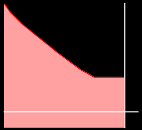
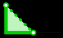
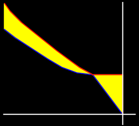
 or  or  but not 

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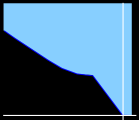

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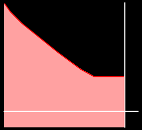

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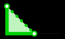

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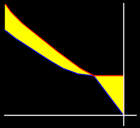

Def. Q-module has interval decomposition $M \cong \bigoplus_{I \in \mathcal{I}} \Bbbk\{I\}$ with $\mathcal{I}$ a set of intervals (but M need not have such a decomposition!)

Arbitrary posets

Thm [Botnan–Crawley-Boevey 2020], cf. [Gabriel–Roĭter 1992]. Over arbitrary poset Q, M has indecomposable decomposition: $M \cong \bigoplus_{\alpha \in A} M_\alpha$ with M_α indecomposable.

Essentially unique: multiset $\{M_\alpha\}_{\alpha \in A}$ of isomorphism classes is invariant.

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Positivity. $M = \bigoplus_{\alpha \in A} M_\alpha$ expresses M positively in term of the M_α. Choose:

1. retain positivity or
2. retain description in terms of intervals.

Question. Can both be achieved?

Proposal. Pipeline:

$$\begin{array}{ccccc} \text{data} \rightsquigarrow \text{filtered topological spaces} \rightsquigarrow & \text{algebraic objects} & & & \\ & \rightsquigarrow & & & \\ & \text{“nice” algebraic objects} & \rightsquigarrow \text{invariants} & \rightsquigarrow & \text{statistics} \end{array}$$

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data $\rightsquigarrow$ filtered topological spaces $\rightsquigarrow$ algebraic objects

$\wr$ old: decompose

"nice" algebraic objects $\rightsquigarrow$ invariants $\rightsquigarrow$ statistics

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$\wr\downarrow$ new: filter

"nice" algebraic objects $\rightsquigarrow$ invariants $\rightsquigarrow$ statistics

Lifetime filtration

Idea [Cunha–M.–Zhang 2024–2025]. Filter with indicator modules $\Bbbk\{S\}$:

- find a "maximally persistent" element $x \in M$
- L = lifetime of $x \Rightarrow$ lifetime submodule $\Bbbk\{L\} \subseteq M$
- $\Bbbk\{L\} \cong \Bbbk\{U \cap D\}$ for upset U and downset D
- replace M with $M/\Bbbk\{L\}$
- iterate: view M as "stack of lifetimes"

Motivation. What could "top 100 bar lengths" mean in multipersistence?

Input. Q-module M for arbitrary poset Q

Output. (noncanonical) filtration

- $F_\bullet : M = M_\ell \supseteq M_{\ell-1} \supseteq \cdots \supseteq M_1 \supseteq M_0 = 0$
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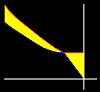

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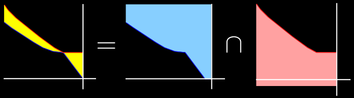

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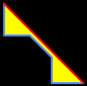

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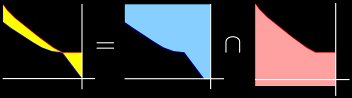

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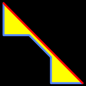

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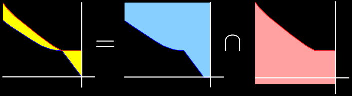

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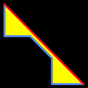
Idea [Cunha–M.–Zhang 2024–2025]. Filter with indicator modules $\Bbbk\{S\}$: 

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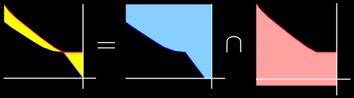

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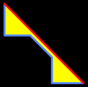

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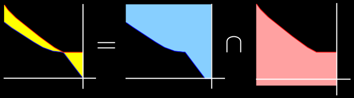

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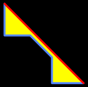

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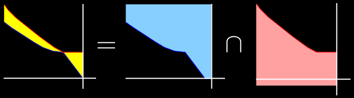

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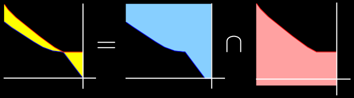

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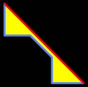

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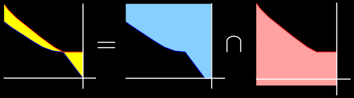

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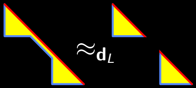

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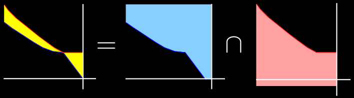

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M is tame if it admits a finite constant subdivision and $\dim_{\Bbbk} M_q < \infty$ for all q.

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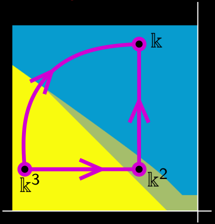

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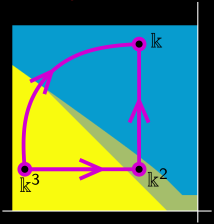

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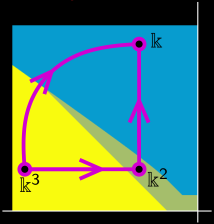

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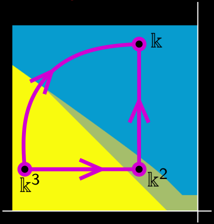

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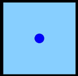
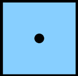
 $=$  $\cup$ 

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1. bottleneck distance $\mathbf{d}_B$ from $\mathcal{D}(N) = \{\text{indecomposable decompositions of } N\}$ if $K = K_1 \oplus \cdots \oplus K_k$ is a direct sum of indecomposables
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Def [Cunha–M.–Zhang 2024–2025]. The lifetime displacement from M to N is

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Distances

Def [Lesnick 2015]. Lifetime modules K, L are ε-interleaved if "K fits ε-almost in L" and vice versa. Similar for any modules M and N. The interleaving distance is

$$\mathbf{d}_{\mathcal{I}}(M, N) = \inf\{\varepsilon \mid M \text{ and } N \text{ are } \varepsilon\text{-interleaved}\}$$

Def. $\bigoplus_{\alpha\in A} M_\alpha$ and $\bigoplus_{\alpha\in A} N_\alpha$ are ε-matched if M_α and N_α are ε-interleaved $\forall\alpha \in A$.

Def. Let $\mathcal{D}$: Q-mods $\to$ families of finitely decomposed Q-modules with ordered summands, so each element of $\mathcal{D}(N)$ is a direct sum $L = L_1 \oplus \cdots \oplus L_\ell$. Assume $K = K_1 \oplus \cdots \oplus K_k$. The bottleneck distance determined by $\mathcal{D}$ is

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Looking forward

Question. What could “top 100 bar lengths” mean in multipersistence?

- Locate “maximally persistent” elements
- What is meant by “maximally persistent”?
  - length, width, area, volume
  - “size” is crucial when parameters have incomparable scientific meanings
  - primary distances: separate classes according to birth and death types
  - note: primary decomposition is really another filtration!

Compare Bjerkevik’s pruning distance stability/Lipschitz conjecture [Bjerkevik 2023]

- Must an indecomposable possess a big individual element?
- Is every indecomposable close to interval decomposing? If not, how likely is it?
- How likely is M to break into interpretable small pieces by perturbation?

Implementation

- Locate maximally persistent elements algorithmically
- Certify lower bounds for approximating $\Delta_{\mathcal{L}}$

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2. Silvana Abeasis, Alberto Del Fra, and Hanspeter Kraft, *The geometry of representations of A_m*, Math. Ann. **256** (1981), no. 3, 401–418.
3. Stephen Aylward and Elizabeth Bullitt, *Initialization, noise, singularities, and scale in height ridge traversal for tubular object centerline extraction*, IEEE Trans. on Medical Imaging **21**, no. 2 (2002), 61–75.
4. Ulrich Bauer and Luis Scoccola, *Generic multi-parameter persistence modules are nearly indecomposable*, preprint, 2022. `arXiv:math.RT/2211.15306`
5. Paul Bendich, Steve Marron, Ezra Miller, Alex Pieloch, and Sean Skwerer, *Persistent homology analysis of brain artery trees*, Ann. Appl. Stat. **10** (2016), #1, 198–218.
6. Håvard Bjerkevik, *Stabilizing decomposition of multiparameter persistence modules*, preprint, 2023. `arXiv:math.RT/2305.15550`
7. Magnus Botnan and William Crawley-Boevey, *Decomposition of persistence modules*, Proc. Amer. Math. Soc. **148** (2020), 4581–4596.
8. Mickaël Buchet, Emerson Escolar, *Every 1D persistence module is a restriction of some 2D indecomposable*, J. Appl. Comput. Top. **4** (2020), no. 3, 387–424.
9. Gunnar Carlsson and Afra Zomorodian, *The theory of multidimensional persistence*, Discrete and Comput. Geom. **42** (2009), 71–93.
10. William Crawley-Boevey, *Decomposition of pointwise finite-dimensional persistence modules*, J. Algebra Appl. **14** (2015), no. 5, 1550066, 8 pp.
11. Ricardo Prado Cunha, Ezra Miller, and Jiaxi (Jesse) Zhang, *Lifetime filtration of multiparameter persistence modules*, draft, 2025.
12. Allen Knutson, Ezra Miller, and Mark Shimozono, *Four positive formulae for type A quiver polynomials*, Invent. Math. **166** (2006), no. 2, 229–325.
13. Michael Lesnick, *The theory of the interleaving distance on multidimensional persistence modules*, Found. Comput. Math. **15** (2015), 613–650.
14. Ezra Miller, *Data structures for real multiparameter persistence modules*, 107 pages. `arXiv:math.AT/1709.08155`
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16. Ezra Miller and Bernd Sturmfels, *Combinatorial commutative algebra*, Graduate Texts in Mathematics, vol. 227, Springer-Verlag, New York, 2005.

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