GEOMETRY AND MEASURE ON CAT(κ) SPACES

EZRA MILLER
JOINT WITH JONATHAN MATTINGLY AND DO TRAN

INTERACTIONS OF STATISTICS AND GEOMETRY (ISAG) II

NATIONAL UNIVERSITY OF SINGAPORE
14–16 OCTOBER 2024

1. CAT(κ) spaces

For more on CAT(κ) spaces, consult a metric geometry text, such as [BBI01].

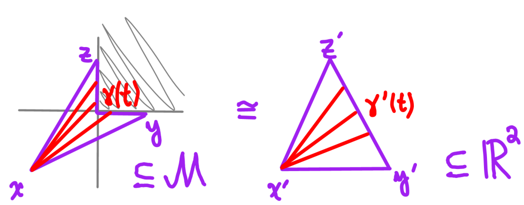
Definition 1.1 (Injectivity radius)**.** For any $\kappa \in \mathbb{R}$, a *model space of curvature* κ is a Riemannian manifold M_κ with geodesic distance $\mathbf{d}_\kappa$ and constant curvature κ. The *injectivity radius* of M_κ is $R_\kappa = \pi/\sqrt{\kappa}$ when $\kappa > 0$ and $R_\kappa = \infty$ if $\kappa < 0$.

Definition 1.2 (Comparison triangle)**.** Given a triangle $\triangle xyz$ (a union of geodesics x to y to z to x) in a metric space $(\mathcal{M}, \mathbf{d})$, a *comparison triangle* of $\triangle xyz$ in a model space $(M_\kappa, \mathbf{d}_\kappa)$ is a triangle $\triangle x'y'z'$ in M_κ such that $\{x', y', z'\}$ is an isometric copy of $\{x, y, z\}$.

Definition 1.3 (CAT(κ) metric space)**.** A metric space $(\mathcal{M}, \mathbf{d})$ is CAT(κ) if

1. any two points $x, y \in \mathcal{M}$ such that $\mathbf{d}(x, y) < R_\kappa$ can be joined by a unique geodesic of length $\mathbf{d}(x, y)$; and
2. for any triangle $\triangle xyz$ in $\mathcal{M}$ with $\mathbf{d}(x, y) + \mathbf{d}(y, z) + \mathbf{d}(z, x) < 2R_\kappa$, if $\triangle x'y'z'$ is a comparison triangle in M_κ of $\triangle xyz$, then the constant-speed geodesics $\gamma : [0, 1] \to \mathcal{M}$ from y to z and $\gamma' : [0, 1] \to M_\kappa$ from y' to z' satisfy, for all $t \in [0, 1]$,
$$\mathbf{d}\big(x, \gamma(t)\big) \le \mathbf{d}_\kappa\big(x', \gamma'(t)\big).$$

Example 1.4.



Remark 1.5. The definition of CAT(κ) space is never used directly in these lectures. The developments instead rest on certain consequences.

Date: 14–16 October 2024.

Definition 1.6. The *angle* between geodesics $\gamma_i : [0, \varepsilon_i) \to \mathcal{M}$ for $i = 1, 2$ emanating from $\bar{\mu}$ in a CAT(κ) space $(\mathcal{M}, \mathbf{d})$ and parametrized by arclength is characterized by

$$\cos\big(\angle(\gamma_1, \gamma_2)\big) = \lim_{t,s\to 0} \frac{s^2 + t^2 - \mathbf{d}^2(\gamma_1(s), \gamma_2(t))}{2st}.$$

The geodesics γ_i are *equivalent* if the angle between them is 0. The set $S_{\bar{\mu}}\mathcal{M}$ of equivalence classes is the *space of directions* at $\bar{\mu}$.

Remark 1.7. Definition 1.6 introduces angle by taking the law of cosines by fiat, namely $s^2 + t^2 = d^2 + 2st\cos(\angle)$.

Lemma 1.8. *The notion of angle makes the space $S_{\bar{\mu}}\mathcal{M}$ of directions into a length space whose* angular metric $\mathbf{d}_s$ *satisfies*

$$\mathbf{d}_s(V, W) = \angle(V, W) \text{ whenever } V, W \in S_{\bar{\mu}}\mathcal{M} \text{ with } \angle(V, W) < \pi$$

Proof. This is [MMT23a, Proposition 1.7], which in turn is [BBI01, Lemma 9.1.39]. □

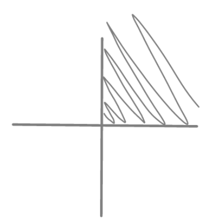
Definition 1.9. The *tangent cone* at a point $\bar{\mu}$ in a CAT(κ) space $\mathcal{M}$ is

$$T_{\bar{\mu}}\mathcal{M} = S_{\bar{\mu}}\mathcal{M} \times [0, \infty)/\big(S_{\bar{\mu}}\mathcal{M} \times \{0\}\big),$$

whose *apex* is often also called $\bar{\mu}$ (although it can be called $\mathcal{O}$ if necessary for clarity). The *length* of a vector $W = W_{\bar{\mu}} \times t$ with $W_{\bar{\mu}} \in S_{\bar{\mu}}\mathcal{M}$ is $\|W\| = t$ in $T_{\bar{\mu}}\mathcal{M}$. The *unit tangent sphere* $S_{\bar{\mu}}\mathcal{M}$ of length 1 vectors in $T_{\bar{\mu}}\mathcal{M}$ is identified with the space of directions.

Example 1.10.

1. three quadrants:



2. kale: see slides for many pictures
3. open book: see slides for many pictures

Definition 1.11. Given tangent vectors $V, W \in T_{\bar{\mu}}\mathcal{M}$, their *angular pairing* is

$$\langle V, W\rangle_{\bar{\mu}} = \|V\|\|W\| \cos\big(\angle(V, W)\big).$$

The subscript $\bar{\mu}$ is suppressed when the point $\bar{\mu}$ is clear from context.

Lemma 1.12. *For a fixed basepoint in a* CAT(κ) *space, the angular pairing function* $\langle\,\cdot\,,\,\cdot\,\rangle_{\bar{\mu}} : T_{\bar{\mu}}\mathcal{M} \times T_{\bar{\mu}}\mathcal{M} \to \mathbb{R}$ *is continuous.*

Proof. This is [MMT23a, Lemma 1.21], where it is derived from Lemma 1.8. □

The angular metric $\mathbf{d}_s$ induces a metric on the tangent cone $T_{\bar{\mu}}\mathcal{M}$ which makes $T_{\bar{\mu}}\mathcal{M}$ into a length space.

Definition 1.13. The *conical metric* on the tangent cone $T_{\bar{\mu}}\mathcal{M}$ of a CAT(κ) space is

$$\mathbf{d}_{\bar{\mu}}(V,W) = \sqrt{\|V\|^2 + \|W\|^2 - 2\langle V,W\rangle} \quad \text{for } V,W \in T_{\bar{\mu}}\mathcal{M}.$$

Lemma 1.14 ([BBI01, Lemma 3.6.15])**.** *Any geodesic triangle in $T_{\bar{\mu}}\mathcal{M}$ with one vertex at the apex is isometric to a triangle in $\mathbb{R}^2$.*

Definition 1.15. Fix a point $\bar{\mu}$ in a CAT(κ) space $(\mathcal{M},\mathbf{d})$. For each point v in the set $\mathcal{M}' \subseteq \mathcal{M}$ of points with a unique shortest path to $\bar{\mu}$, write γ_v for the unit-speed shortest path from $\bar{\mu}$ to v and $V = \gamma_v'(0)$ for its tangent vector at $\bar{\mu}$. Define the *log map* by

$$\begin{aligned} \log_{\bar{\mu}} : \mathcal{M}' &\to T_{\bar{\mu}}\mathcal{M} \\ v &\mapsto \mathbf{d}(\bar{\mu},v)V. \end{aligned}$$

$\mathcal{M}$ is *conical* with *apex* $\bar{\mu}$ if $\mathcal{M}' = \mathcal{M}$ and $\log_{\bar{\mu}} : \mathcal{M} \to T_{\bar{\mu}}\mathcal{M}$ is an isometry.

2. Measures on CAT(κ) spaces

Definition 2.1 (Localized measure)**.** A measure μ on CAT(κ) $\mathcal{M}$ is *localized* if

- μ has unique Fréchet mean $\bar{\mu}$,
- μ has locally convex Fréchet function in a neighborhood of $\bar{\mu}$, and
- the logarithm map $\log_{\bar{\mu}} : \mathcal{M} \to T_{\bar{\mu}}\mathcal{M}$ is μ-almost surely uniquely defined.

Denote the pushforward of a localized measure μ to $T_{\bar{\mu}}\mathcal{M}$ by

$$\widehat{\mu} = (\log_{\bar{\mu}})_\sharp \mu.$$

Example 2.2. Intuition behind Definition 2.1 is that μ should be "Fréchet-localized", in the sense that "retracting" to the tangent cone at $\bar{\mu}$ captures all of the mass. For instance, if μ is a measure on a CAT(κ) space $\mathcal{M}$ that is supported in a metric ball $B(\bar{\mu}, R_\mu)$ of radius $R_\mu < \frac{1}{2}R_\kappa = \pi/(2\sqrt{\kappa})$—this can be any measure when $\kappa = 0$—then μ is localized. Indeed, thanks to results by Kuwae [Kuw14], the Fréchet mean $\bar{\mu}$ of such a measure is unique and the Fréchet function of μ is k-uniform convex in a small ball around $\bar{\mu}$. In addition, such a measure is retractable because the cut locus (the closure of the set of points with more than shortest path to $\bar{\mu}$) has measure 0.

Definition 2.3. Fix a localized measure μ on a CAT(κ) space $\mathcal{M}$. The Fréchet function F has *directional derivative* at $\bar{\mu}$ given by

$$\begin{aligned} \nabla_{\bar{\mu}} F : T_{\bar{\mu}}\mathcal{M} &\to \mathbb{R} \\ V &\mapsto \frac{d}{dt} F(\exp_{\bar{\mu}} tV)|_{t=0} \end{aligned}$$

in which the exponential is a geodesic with constant speed and tangent V at $\bar{\mu}$.

The next three concepts are [MMT23b, Definition 2.12, 2.17, and 2.18], where explanations of their background, geometry, and motivation can be found.

Definition 2.4 (Escape cone)**.** The *escape cone* of a localized measure μ on a CAT(κ) space $\mathcal{M}$ is the set E_μ of tangent vectors along which the directional derivative of the Fréchet function vanishes at $\bar{\mu}$:

$$E_\mu = \{X \in T_{\bar{\mu}}\mathcal{M} \mid \nabla_{\bar{\mu}} F(X) = 0\}.$$

Proposition 2.5. *If μ is a measure on a* CAT(κ) *space $\mathcal{M}$, then the escape cone E_μ is a closed, path-connected, geodesically convex subcone of $(T_{\bar{\mu}}\mathcal{M}, \mathbf{d}_{\bar{\mu}})$.*

Definition 2.6 (Hull)**.** If $\mathcal{X}$ is a conical CAT(0) space, then the *hull* of any subset $\mathcal{S} \subseteq \mathcal{X}$ is the smallest geodesically convex cone $\operatorname{hull}\mathcal{S} \subseteq \mathcal{X}$ containing $\mathcal{S}$. For a localized measure μ on a CAT(κ) space $\mathcal{M}$, set

$$\operatorname{hull}\mu = \operatorname{hull}\operatorname{supp}(\widehat{\mu}),$$

the hull of the support in $\mathcal{X} = T_{\bar{\mu}}\mathcal{M}$ of the pushforward measure $\widehat{\mu} = (\log_{\bar{\mu}})_\sharp\mu$.

Definition 2.7 (Fluctuating cone)**.** The *fluctuating cone* of a localized measure μ on a CAT(κ) space $\mathcal{M}$ is the intersection

$$\begin{aligned} C_\mu &= E_\mu \cap \operatorname{hull}\mu \\ &= \{V \in \operatorname{hull}\mu \mid \nabla_{\bar{\mu}} F(V) = 0\} \end{aligned}$$

of the escape cone and hull of μ. Let $\overline{C}_\mu$ be the closure in $T_{\bar{\mu}}\mathcal{M}$ of the escape cone C_μ.

Remark 2.8. The purpose of the fluctuating cone C_μ is to encapsulate those directions in which the Fréchet mean $\bar{\mu}$ can be induced to wiggle by adding to μ a point mass in $\mathcal{M}$ along that direction (this is made precise in the main theorem of [MMT23d, Section 4]); hence the terminology. However, if the measure μ is supported on a "thin" subset of $\mathcal{M}$, then it is possible to induce fluctuations in directions that have nothing whatsoever to do with the geometry in $\mathcal{M}$ of the (support of) μ by adding a point mass outside $\operatorname{supp}\mu$; see the next Example. That is why the fluctuating cone is assumed to lie within the convex hull of the support of $\widehat{\mu}$: only fluctuations of $\bar{\mu}$ that can be realized—at least in principle—by means of samples from μ itself are relevant to the asymptotics in a central limit theorem.

Example 2.9. For a concrete example, consider a measure μ supported on the spine R of an open book $\mathcal{M}$ (see [HHL$^+$13]). The spine R is simply a vector space, where the usual central limit theorem yields convergence to a Gaussian supported on R. It is true that the Fréchet mean $\bar{\mu}$ can be induced to fluctuate off of R onto any desired page of $\mathcal{M}$ by adding a point mass on the relevant page, but that observation is irrelevant to the CLT, which only cares about fluctuations of $\bar{\mu}$ within R.

References

[BBI01] Dmitri Burago, Yuri Burago, and Sergei Ivanov, *A course in metric geometry*, volume 33, American Mathematical Soc., 2001.

[HHL+13] Thomas Hotz, Stephan Huckemann, Huiling Le, J.S. Marron, Jonathan C. Mattingly, Ezra Miller, James Nolen, Megan Owen, Vic Patrangenaru, and Sean Skwerer, *Sticky central limit theorems on open books*, Annals of Applied Probability **23** (2013), no. 6, 2238–2258.

[Kuw14] Kazuhiro Kuwae, *Jensen's inequality on convex spaces*, Calculus of Variations and Partial Differential Equations **49** (2014), no. 3, 1359–1378.

[MMT23a] Jonathan Mattingly, Ezra Miller, and Do Tran, *Shadow geometry at singular points of* CAT(κ) *spaces*, preprint, 2023. arXiv:math.MG/2311.09451

[MMT23b] Jonathan Mattingly, Ezra Miller, and Do Tran, *Geometry of measures on smoothly stratified metric spaces*, preprint, 2023. arXiv:math.MG/2311.09453

[MMT23c] Jonathan Mattingly, Ezra Miller, and Do Tran, *A central limit theorem for random tangent fields on stratified spaces*, submitted, 2023. arXiv:math.MG/2311.09454

[MMT23d] Jonathan Mattingly, Ezra Miller, and Do Tran, *Central limit theorems for Fréchet means on stratified spaces*, preprint, 2023. arXiv:math.MG/2311.09455

Departments of Mathematics and Statistical Sciences, Duke University, Durham, NC 27708
URL: `https://scholars.duke.edu/person/ezra.miller`