Tutorial 3: Shadows and tangential collapse

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joint with Jonathan Mattingly (Duke)
Do Tran (Deutsche Bank (was: Göttingen))

http://arxiv.org/abs/2311.09455
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Interactions of Statistics and Geometry (ISAG) II

National University of Singapore

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Outline

1. Shadows
2. Fréchet means and log maps
3. Radial transport
4. Tangential collapse

Shadows

Def. $S_{\bar{\mu}}M$ = unit sphere in $T_{\bar{\mu}}M$ has metric $\mathbf{d}_s$. Vectors $U, V \in S_{\bar{\mu}}M$ have

- angle $\angle(U, V) = \begin{cases} \mathbf{d}_s(U, V) & \text{if } < \pi \\ \pi & \text{otherwise} \end{cases}$
- angular pairing $\langle U, V \rangle_{\bar{\mu}} = \|U\|\|V\| \cos\big(\angle(U, V)\big)$.

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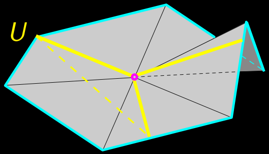

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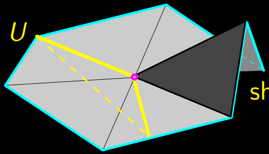

$\text{shadow}(U) = \{V \in S_{\bar{\mu}}M \mid \angle(U,V) \geq \pi\}$

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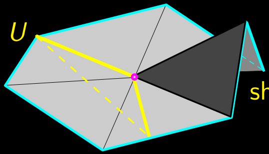

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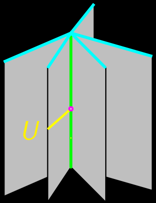

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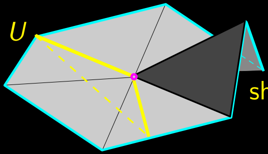

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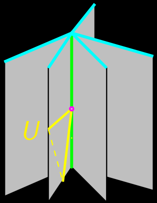

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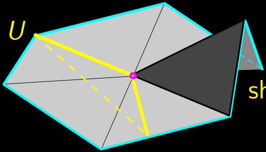

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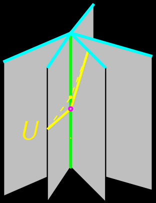

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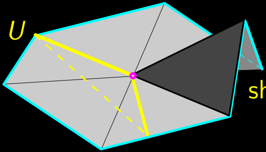

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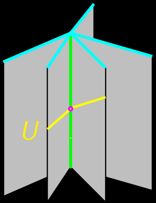

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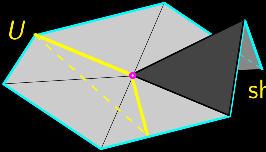

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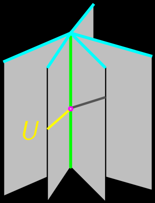

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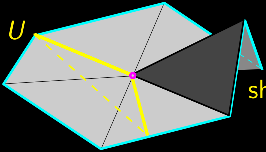

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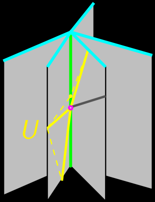

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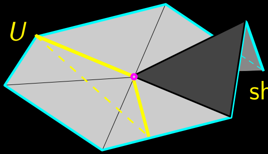

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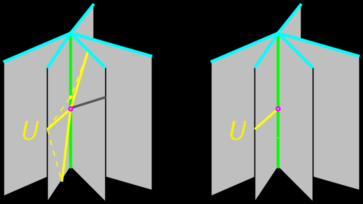

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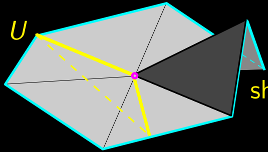

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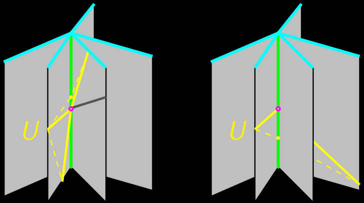

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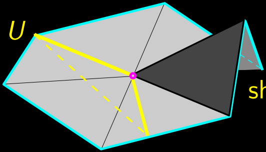

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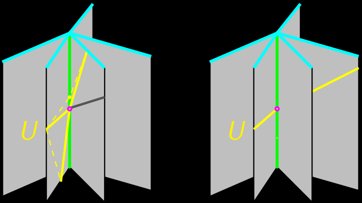

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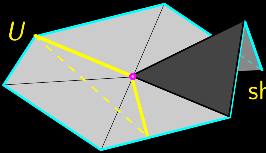

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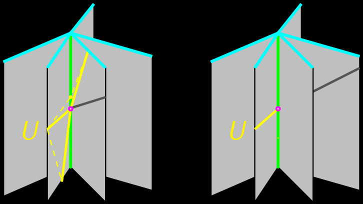

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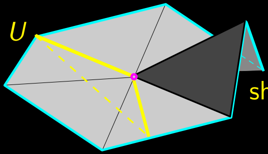

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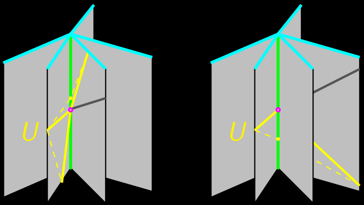

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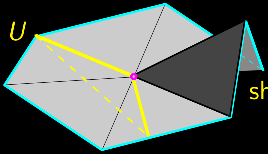

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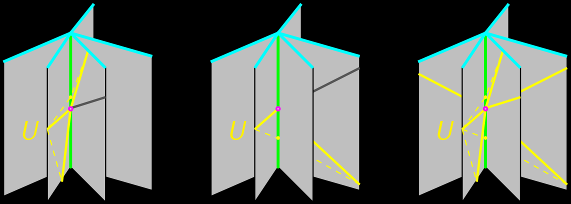

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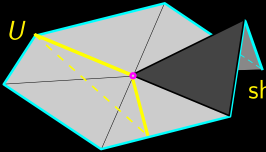

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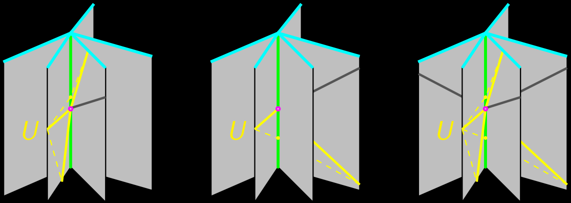

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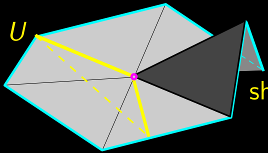

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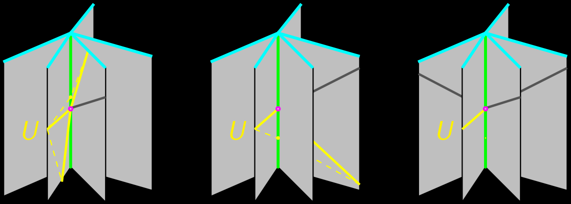

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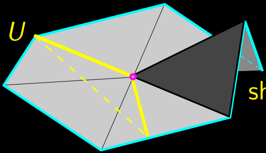

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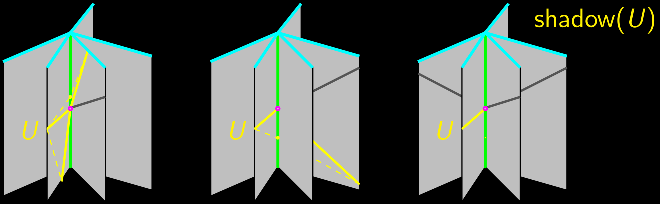

Fréchet means

Def. Probability distribution μ on any metric space M has Fréchet function

$$F_\mu(y) = \frac{1}{2}\int_M \underset{\substack{\uparrow \\ \text{square distance}}}{d(x,y)^2}\,\underset{\substack{\uparrow \\ \text{measure induced by } \mu}}{\mu(dx)}$$

and Fréchet mean $\bar{\mu} = \underset{y \in M}{\operatorname{argmin}}\, F_\mu(y)$.

Prop. M is CAT(κ)
$\Rightarrow$ M has tangent spaces (cones)

Def. The logarithm map is

$$\log_{\bar{\mu}} : M \to T_{\bar{\mu}}M$$
$$x \mapsto d(\bar{\mu}, x)V,$$

where V = unit tangent to geodesic from $\bar{\mu}$ to x.

Note. M singular at $\bar{\mu} \Leftrightarrow T_{\bar{\mu}}M \not\cong \mathbb{R}^d$

Prop. M smoothly stratified
$\Rightarrow$ $T_{\bar{\mu}}M$ is a smoothly stratified CAT(0) cone.

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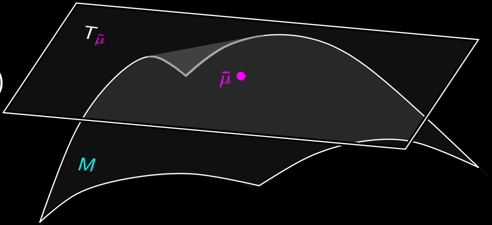

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($d(x,y)^2$ ↑ square distance; $\mu(dx)$ ↑ measure induced by μ)

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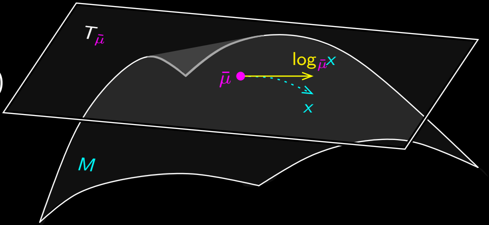

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Fréchet means

Def. Probability distribution μ on any metric space M has Fréchet function

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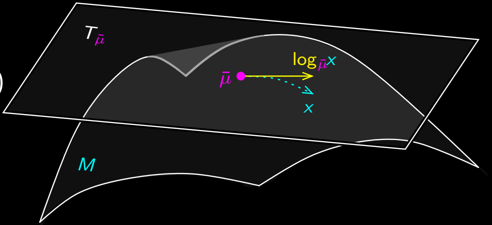

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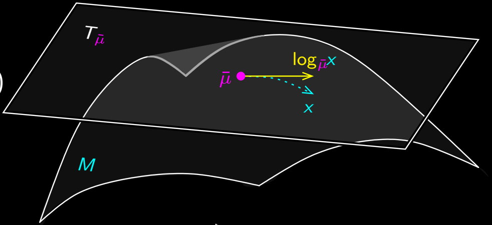

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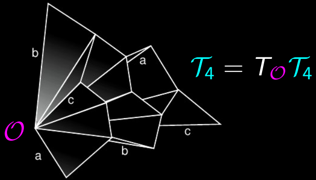

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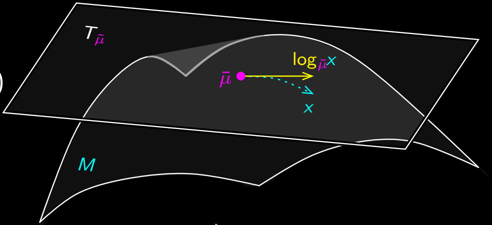

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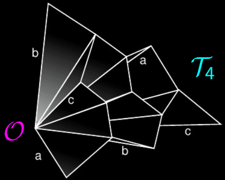

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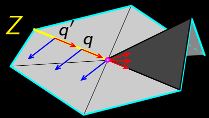

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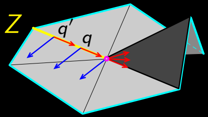

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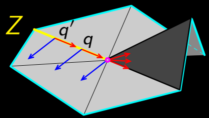

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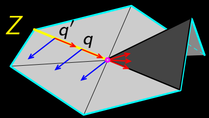

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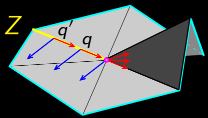

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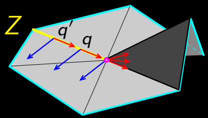

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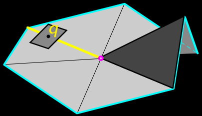

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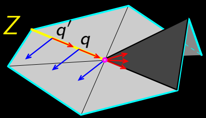

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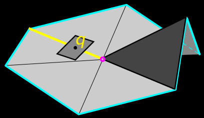

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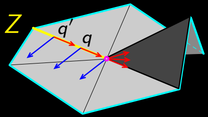

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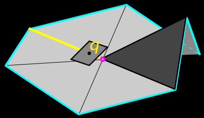

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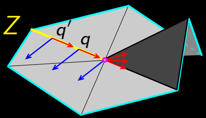

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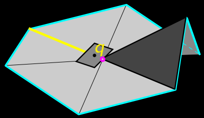

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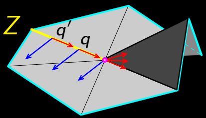

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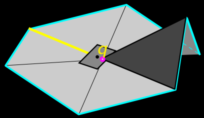

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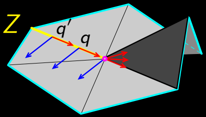

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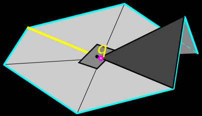

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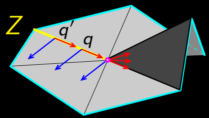
Iterate to get $T_{\bar{\mu}}M \to \mathbb{R}^m$ = tangent space to some smooth stratum

- choose resolving vectors Z appropriately
- to ensure μ pushes forward appropriately, assume μ is localized: unique $\bar{\mu}$, locally convex F_μ, and $\mu(\text{cut locus}) = 0$

Radial transport

Prop. Set $\mathcal{X} = T_{\bar\mu} M$ with apex $\mathcal{O}$. Fix

- $Z = \log_{\mathcal{O}} z \in T_{\mathcal{O}}\mathcal{X}$
- $q \in [\mathcal{O}, z]$
- $q' \in (\mathcal{O}, z]$



Then radial transport $T_q\mathcal{X} \to T_{q'}\mathcal{X}$ is isometry if $q \neq \mathcal{O}$.

Idea. Z points out of stratum containing $\mathcal{O}$
$\Rightarrow q \in (\mathcal{O}, z]$ is strictly less singular than $\mathcal{O}$
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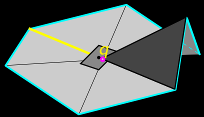

Def [Mattingly, M–, Tran & Barden, Le]. The limit tangent cone along Z is

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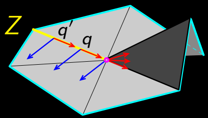

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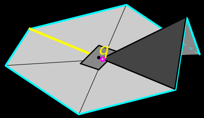

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Tangential collapse

Def. Localized μ on smoothly stratified M has fluctuating cone

$$C_\mu = \{X \in T_{\bar\mu} M \mid \nabla_{\bar\mu} F(X) = 0 \text{ and } X \in \text{convex cone generated by } \operatorname{supp}(\mu \circ \log_{\bar\mu}^{-1})\}$$

Lemma. Adding mass to μ can only cause $\bar\mu$ to move into C_μ

Thm [Mattingly, M–, Tran 2023]**.** M smoothly stratified $\Rightarrow$ some sequence of limit log maps, followed by convex projection to the relevant smooth stratum, is a tangential collapse: a continuous map $\mathcal{L} : T_{\bar\mu} M \to \mathbb{R}^m$ that is

- injective on C_μ and
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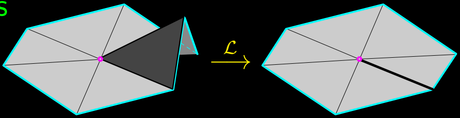
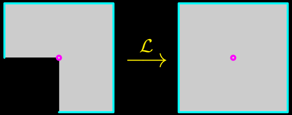
Examples

- kale: 

- nonconvex quadrants: 

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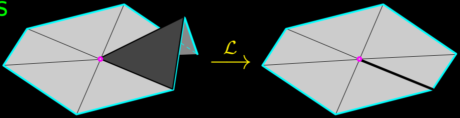
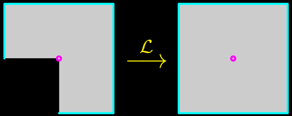
- kale: 

- nonconvex quadrants: 

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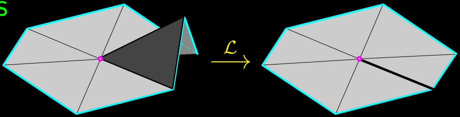
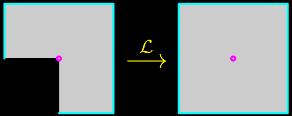
- kale: 

- nonconvex quadrants: 

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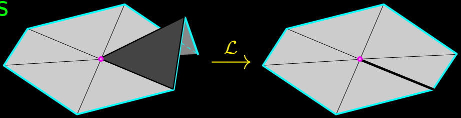
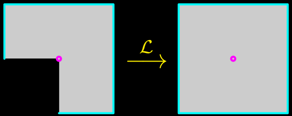
- kale: 

- nonconvex quadrants: 

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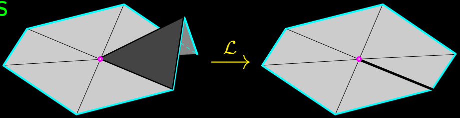
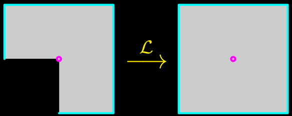
- kale: 

- nonconvex quadrants: 

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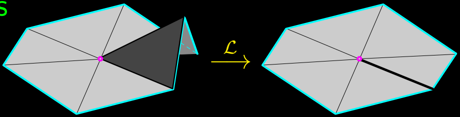
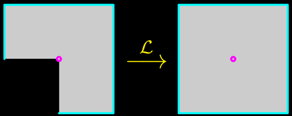
- kale: 

- nonconvex quadrants: 

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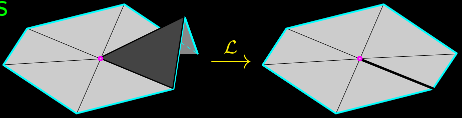

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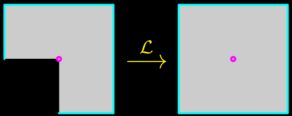
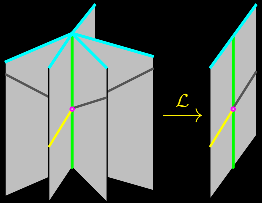


- open book:



Next lecture: convergence to Gaussian objects