RECENT PROGRESS ON RATIONALITY PROBLEMS

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The subject of this talk is a venerable problem, often called the Lüroth problem, which has seen great advances in the last three years. The story begins in 1875 with a two pages paper by A. Lüroth. He starts with a curve $C \subset \mathbb{C}^2$, defined by a polynomial equation $P(x,y) = 0$, which can be parametrized by rational functions $(x(t), y(t))$ (so that $P(x(t), y(t)) = 0$) – we say now that C is *unirational*. Then Lüroth proves that one can find another such parametrization such that a general point of C corresponds to exactly one value of the parameter – we say that C is *rational*.

20 years later, Castelnuovo proves the analogous results in dimension 2: a unirational surface is rational. In contrast with the Lüroth theorem, which is not difficult, this is a rather subtle result, one of the cornerstone of the classification of algebraic surfaces.

The Italian algebraic geometers became soon convinced that the result would not hold in dimension ≥ 3. There is a long story of attempted counter-examples, unfortunately with incomplete proofs; it is only in 1971, with the powerful tools developed by modern algebraic geometry, that three different (correct) counter-examples appeared. Each of them gives a method to check that a given variety is not rational. These methods are quite efficient in dimension 3, where they show that many unirational varieties are not rational; they are far less successful in higher dimension, where only very particular examples were known till 3 years ago. At that time the situation changed spectacularly with a new idea of Claire Voisin, which allows (roughly) to get non-rationality results for a general variety in a family, when some special member of the family, with mild singularities, is known to be non-rational.

Voisin's idea has inspired a number of remarkable results, leading to considerable progress on rationality questions. In the talk I will first survey the colorful history of the problem, describe briefly the new methods introduced in the 70's, then explain Voisin's idea and the new results which have followed. I'll conclude with an overview of the case of cubic hypersurfaces, still mysterious despite some important recent progress.

REFERENCES

For a survey of various aspects of rationality problems :

A. Beauville : *The Lüroth problem*. Rationality Problems in Algebraic Geometry, p. 1-27. Springer Lecture Notes **2172** (2016). (The other contributions of the volume are also relevant.)

Some of the recent breakthroughs that I will discuss :

C. Voisin : *Unirational threefolds with no universal codimension 2 cycle*. Invent. math. **201** (2015), no. 1, 207–237.

B. Hassett, A. Pirutka, Y. Tschinkel : *Intersections of three quadrics in* $\mathbb{P}^7$. Preprint `arXiv:1706.01371`.

M. Kontsevich, Y. Tschinkel : *Specialization of birational types* . Preprint `arXiv:1708.05699`.

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