Hypersurfaces of low entropy and mean curvature flow

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In this talk, I will present joint work with Jacob Bernstein on properties of hypersurfaces of low entropy using mean curvature flow (MCF).

A MCF is a one-parameter family of hypersurfaces in $\mathbb{R}^{n+1}$, $\{\Sigma_t\}_{t\in I}$ that satisfies

$$\left(\frac{\partial \mathbf{x}}{\partial t}\right)^{\perp} = \mathbf{H}_{\Sigma_t} \text{ for } \mathbf{x} \in \Sigma_t,$$

where $\mathbf{H}_{\Sigma_t} = \Delta_{\Sigma_t}\mathbf{x} = -H_{\Sigma_t}\mathbf{n}_{\Sigma_t} = -(\mathrm{div}_{\Sigma_t}\mathbf{n}_{\Sigma_t})\mathbf{n}_{\Sigma_t}$ is the mean curvature vector, $\mathbf{n}_{\Sigma_t}$ is the unit normal, and $(\cdot)^{\perp}$ is the normal part. MCF is the negative gradient flow of the volume functional, so it decreases the volume of hypersurfaces in the steepest manner.

By an avoidance principle, one uses the self-shrinking spheres as barriers to see that any MCF starting from a closed (i.e., compact without boundary) hypersurface will develop singularities in finite time. By the Huisken monotonicity formula [13] (see also [15]) the asymptotic behavior of the flow near a singular point in spacetime is modeled by self-similar shrinking solutions of MCF (simply called shrinkers). When $n = 1$, by work of Gage-Hamilton [11] and Grayson [12], any closed simple curve in plane is evolved by its curvature vector to a convex curve and eventually becomes a round point in finite time. However, in higher dimensions, singularity formation of MCF is much more complicated, in part because there are infinitely many shrinkers – cf. [1], [18], [17] and [19] – suggesting it is virtually hopeless to classify all shrinkers.

Motivated by the generic MCF conjecture of Huisken [16], Colding and Minicozzi [8] introduced the following notion of entropy of hypersurface:

$$\lambda[\Sigma] = \sup_{\mathbf{x}_0\in\mathbb{R}^{n+1}, t_0>0} (4\pi t_0)^{-\frac{n}{2}} \int_{\Sigma} e^{-\frac{|\mathbf{x}-\mathbf{x}_0|^2}{4t_0}}.$$

Entropy is invariant under translations and dilations about $\mathbf{0}$, and is monotonically decreasing under MCF. Clearly, $\lambda[\Sigma] \geq 1$ with equality iff Σ is a hyperplane. Moreover, a computation of Stone [20] gives

$$2 > \lambda[\mathbb{S}^1] > \frac{3}{2} > \lambda[\mathbb{S}^2] > \cdots > \lambda[\mathbb{S}^n] > \lambda[\mathbb{S}^{n+1}] > \cdots \to \sqrt{2}.$$

Suggested by their dynamic approach, Colding, Ilmanen, Minicozzi and White [9] made the following two related conjectures.

Conjecture 1 *For $n \leq 6$, the entropy of any closed hypersurface $\Sigma \subset \mathbb{R}^{n+1}$ is at least as that of the round sphere with equality iff Σ is a round sphere.*

Conjecture 2 *Given $n \leq 6$, there is an $\epsilon = \epsilon(n) > 0$ so that if $\Sigma \subset \mathbb{R}^{n+1}$ is a non-flat shrinker that is not equal to the round sphere, then $\lambda[\Sigma] \geq \lambda[\mathbb{S}^n] + \epsilon$.*

Both conjectures are known to be true when $n = 1$ by combining results of [11] and [12]. The second conjecture has been verified for closed shrinkers for all dimensions by the four authors [9].

In [2], Bernstein and myself used a notion of weak mean curvature flow, the level set flow (cf. [10], [7] and [14]), to analyze singularities of the flow at its extinct time and was able to address the first conjecture. After that, J. Zhu [22] generalized the conjecture to all dimensions.

Theorem 3 *Conjecture 1 is true for all n.*

A natural follow-up question is the stability of the round sphere under small perturbations of entropy. For this purpose, it is inevitable to consider the second conjecture. In [3] and [4], Bernstein and myself obtained some restriction on topology of shrinkers with low entropy.

Theorem 4 *For $n \geq 2$ let Σ be an asymptotically conical shrinker in $\mathbb{R}^{n+1}$. If $\lambda[\Sigma] \leq \lambda[\mathbb{S}^{n-1}]$, then Σ is contractible, and the link of the asymptotic cone of Σ is a homology $(n-1)$-sphere. In particular, when $n = 2$ or 3, Σ is diffeomorphic to $\mathbb{R}^n$.*

Combining this with work of Brendle [6] and Colding-Ilmanen-Minicozzi-White [9], we get

Theorem 5 *Conjecture 2 is true for $n = 2$. Indeed, there is a $\delta > 0$ so that if Σ is a shrinker in $\mathbb{R}^3$ that is not equal to plane, round sphere, or round cylinder, then $\lambda[\Sigma] \geq \lambda[\mathbb{S}^1] + \delta$.*

Theorem 6 *If Σ is a closed surface in $\mathbb{R}^3$ and $\lambda[\Sigma] \leq \lambda[\mathbb{S}^1]$, then Σ is isotopic to $\mathbb{S}^2$.*

It would be desired to extend Theorem 6 to higher dimensions. However, due to the lack of classification of shrinkers of low entropy in general dimensions, in [4], we first used the level set flow together with a surgery procedure to prove a weaker version of Theorem 6 for shrinkers in $\mathbb{R}^4$.

Theorem 7 *If Σ is a closed hypersurface in $\mathbb{R}^4$ and $\lambda[\Sigma] \leq \lambda[\mathbb{S}^2]$, then Σ is diffeomorphic to $\mathbb{S}^3$.*

If the 4-dimensional Smooth Schoenflies Conjecture were true, the hypersurface Σ in Theorem 7 would be *isotopic* to $\mathbb{S}^3$. To prove this statement rigorously, one main obstacle is the non-uniqueness of MCF starting from a cone. In [5], we initiated an investigation on the global structure of the space of asymptotically conical self-expanders, which is an extension of work of White [21] for minimal surfaces to non-compact weighted setting.

Theorem 8 *The space of asymptotically conical self-expanders of a fixed diffeomorphism type is a smooth Banach manifold of infinite dimensions. As a corollary, for a "generic" cone C there do not exist self-expanders asymptotic to C that have non-trivial Jacobi vector fields.*

In an ongoing project, we will discuss situations in which there is a well-defined *integer* degree theory on the space of asymptotically conical self-expanders.

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