Mixed Motives Associated to Classical Modular Forms

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Overview

This talk concerns mixed motives associated to classical modular forms.

Goals

These include making progress on:

1. new cases of Beilinson's conjectures on special values of $L$-functions of modular forms,
2. Morita's conjecture on the Galois action on the unipotent fundamental group of a smooth projective curve of arbitrary genus,
3. a variant (all genera) of the unipotent-de Rham version of the Grothendieck-Teichmüller conjecture.

Collaborators: Francis Brown, Makoto Matsumoto
Beilinson proposed that there is a $\mathbb{Q}$-linear tannakian category $\mathbf{MM}(X)$ of mixed motives associated a smooth scheme $X$ over $\mathbb{Z}$ (say) with the correct Ext groups:

$$
\text{Ext}^i_{\mathbf{MM}(X)}(\mathbb{Q}, \mathbb{Q}(n)) = H^i_{\text{mot}}(X, \mathbb{Q}(n)) := K_{2n-j}(X)^{(n)}.
$$

The dimension of these groups should, in most cases, be the dimension of the real Deligne cohomology group

$$
H^i_D(X^{\text{an}}, \mathbb{R}(n))^F
$$

or, equivalently, the order of vanishing of a certain $L$-function of $X$ at the appropriate point.
If \( X = \text{Spec} \mathcal{O}_{F,S} \), where \( F \) is a number field, then \( H^j(X, \mathbb{Q}(n)) \) vanishes when \( j > 1 \) or \( n < 0 \). Have \( H^0(X, \mathbb{Q}(0)) = \mathbb{Q} \) and

\[
H^1_{\text{mot}}(X, \mathbb{Q}(n)) = K_{2n-1}(\mathcal{O}_K,S) \otimes \mathbb{Q} \cong \begin{cases} 
\mathbb{Q}^{r_1+r_2+|S|-1} & n = 1, \\
\mathbb{Q}^{r_1+r_2} & n > 1 \text{ odd}, \\
\mathbb{Q}^{r_2} & n > 0 \text{ even}.
\end{cases}
\]

All other groups vanish. The ranks are given by the order of vanishing of the Dedekind zeta function of \( \mathcal{O}_K,S \) at negative integers.
Voevodsky (also Levine & Hanamura) has constructed a triangulated tensor category of motives associated to schemes over a perfect field $k$ with the correct Ext groups. It is not tannakian and it is not known whether $H^i_{\text{mot}}(X, \mathbb{Q}(n))$ vanishes when $j < 0$ (Beilinson-Soulé vanishing).

NB: Since $H^i_{\text{mot}}(X, \mathbb{Q}(n))$ vanishes when $j > 2n$, the vanishing conjecture implies vanishing when $n < 0$. 
There is one case where things work well, almost as well as we want. Levine and Deligne-Goncharov have constructed (from Voevodsky’s motives) a $\mathbb{Q}$-linear tannakian category $\text{MTM}(\mathcal{O}_K, S)$ of mixed Tate motives over a number field $K$, unramified over $S$, with the correct ext groups:

$$H^i_{\text{mot}}(\text{Spec } \mathcal{O}_K, S, \mathbb{Q}(n)) = \text{Ext}^i_{\text{MTM}(\mathcal{O}_K, S)}(\mathbb{Q}, \mathbb{Q}(n)).$$

Mixed Tate motives have weight filtrations. Their Hodge realizations are mixed Hodge structures whose weight graded quotients are sums of Tate Hodge structures $\mathbb{Q}(r)$. 
The fundamental group of MTM($\mathcal{O}_K,S$) is an extension of $\mathbb{G}_m$ by a free pronipotent group. When $\mathcal{O}_K,S = \mathbb{Z}$, the Lie algebra of the kernel is

$$\mathfrak{k} = \mathbb{L}(\mathbb{Z}_3, \mathbb{Z}_5, \mathbb{Z}_7, \mathbb{Z}_9, \ldots)^\wedge$$

where $\mathbb{G}_m$ acts on $\mathbb{Z}_{2m+1}$ with weight $2m + 1$.

In general,

$$\mathfrak{k}_{K,S} = \mathbb{L}\left(\bigoplus_{n>0} K_{2n-1}(\mathcal{O}_K,S)^*\right)^\wedge,$$

where $\mathbb{G}_m$ acts on $K_{2n-1}(\mathcal{O}_K,S)^*$ with weight $n$. 
The *unipotent completion* $\Gamma_{\text{un}}^F$ of a discrete group $\Gamma$ over a field $F$ of characteristic zero is the tannakian fundamental group of the category of unipotent representations of $\Gamma$ on finite dimensional $F$ vector spaces.

Every unipotent representation of $\Gamma$ over $F$ factors through $\Gamma_{\text{un}}^F$:

$$\Gamma \longrightarrow \Gamma_{\text{un}}^F(F) \longrightarrow \text{Aut } V$$

The coordinate ring (equivalently, the Lie algebra) of the unipotent fundamental group $\pi_1^{\text{un}}(X, b)$ of a complex algebraic variety has a natural MHS. Here $b$ may be a tangential base point.
Deligne and Goncharov showed that $\pi^\text{un}_1(\mathbb{P}^1 - \{0, 1, \infty\}, \vec{v})$ is a (pro-) object of $\text{MTM}(\mathbb{Z})$, where $\vec{v} = \partial / \partial x \in T_0 \mathbb{P}^1$. Its periods are multi-zeta values (MZVs):

$$\zeta(n_1, \ldots, n_r) = \sum_{0 < k_1 < \cdots < k_r} \frac{1}{k_1^{n_1} k_2^{n_2} \cdots k_r^{n_r}} \quad n_r > 1$$

$$= \int_0^1 \omega_1 \odot \omega_0 \odot \omega_0 \omega_1 \odot \omega_0 \odot \omega_0 \cdots \omega_1 \odot \omega_0 \odot \omega_0$$

where $\omega_0 = dx / x$ and $\omega_1 = dx / (1 - x)$.

**Question:** Do the MZV span the periods of objects of $\text{MTM}(\mathbb{Z})$?
Brown’s Theorem

Theorem (Brown)

\[ \pi_1(\text{MTM}(\mathbb{Z})) \text{ acts faithfully on } \pi_1^{\text{un}}(\mathbb{P}^1 - \{0, 1, \infty\}, \vec{v}). \]

Consequently, the periods of all objects of \( \text{MTM}(\mathbb{Z}) \) are MZVs.

Corollary

\( \mathcal{O}(\pi_1^{\text{un}}(\mathbb{P}^1 - \{0, 1, \infty\}, \vec{v})) \) generates \( \text{MTM}(\mathbb{Z}) \) as a tannakian category and \( \text{MTM}(\mathbb{Z}) \) is isomorphic to the sub tannakian category of \( \text{MHS}_\mathbb{Q} \) generated by it.

So one could define \( \text{MTM}(\mathbb{Z}) \) to be the full subcategory of \( \text{MHS}_\mathbb{Q} \) generated by \( \mathcal{O}(\pi_1^{\text{un}}(\mathbb{P}^1 - \{0, 1, \infty\}, \vec{v})) \).
The theory of mixed Tate motives appears to be very much a “genus 0 story”, or perhaps a “hyperplane complement story”.

Is there a “higher genus story”? 

If so (by Harer connectivity), genus 1 moduli spaces should be the fundamental building block. (Cf. general Grothendieck-Teichmüller story.)

The elliptic case relates to genus 0 by specialization to the nodal cubic and to higher genus by degeneration to trees of elliptic curves.
Motives Associated to Genus 1 Moduli Spaces

1. The stack $\mathcal{M}_{1,1}$ is defined over $\mathbb{Z}$ and has everywhere good reduction. So its cohomology groups should be motives unramified over $\mathbb{Z}$.

2. $\mathcal{H}$ is the local system $R^1 f_* \mathbb{Q}$ associated to the universal elliptic curve $f : \mathcal{E} \to \mathcal{M}_{1,1}$. It is a PVHS of weight 1.

3. Manin–Drinfeld: as a motive (Hodge, Galois, . . .)

$$H^1(\mathcal{M}_{1,1}^{an}, S^{2n} \mathcal{H}) = \mathbb{Q}(-2n - 1) \oplus H^1_{\text{cusp}}(\mathcal{M}_{1,1}^{an}, S^{2n} \mathcal{H})$$

The copy of $\mathbb{Q}(-2n - 1)$ corresponds to the Eisenstein series of weight $2n + 2$.

4. Eichler-Shimura:

$$H^1_{\text{cusp}}(\mathcal{M}_{1,1}^{an}, S^{2n} \mathcal{H}_{\mathbb{R}}) = \bigoplus_f V_f$$

where $V_f$ is the 2-dimensional real Hodge structure associated to the normalized Hecke eigen cusp form $f$ of weight $2n + 2$. It is of type $(2n + 1, 0), (0, 2n + 1)$. 
If there were tannakian category $\text{MM}(\mathbb{Z})$ of mixed motives over $\mathbb{Z}$, then one could take the full subcategory of it generated by the $H^1(\mathcal{M}_{1,1}, S^{2n}\mathbb{H})$ and the $\mathbb{Q}(r)$. 

Brown refers to this putative category as the category of \textit{mixed modular motives} over $\mathbb{Z}$. Denote it by $\text{MMM}(\mathbb{Z})$, or just $\text{MMM}$.

With current technology, the construction of $\text{MMM}$ from Voevodsky motives seems to be far out of reach.

Brown has an end run around this problem.

\textbf{Question:} Where can one find all of the pure motives associated to $\mathcal{M}_{1,1}$ and lots of extensions between them?

\textbf{An Answer:} In the coordinate ring of the relative unipotent completion of $\pi_1(\mathcal{M}_{1,1}, \partial/\partial q)$. 

Richard Hain

Mixed Motives Associated to Classical Modular Forms
The relative unipotent completion $G^{\text{rel}}$ of $\SL_2(\mathbb{Z})$ is the fundamental group of the tannakian category whose objects are finite dimensional representations $V$ of $\Gamma$ (over $\mathbb{Q}$, say) that admit a filtration

$$V = V^0 \supset V^1 \supset V^2 \supset \cdots \supset V^N \supset V^{N+1} = 0$$

with the property that each $V^j / V^{j+1}$ is a sum of copies of modules of the form $S^m H$, where $H$ is the fundamental representation of $\SL_2$.

It is an affine group scheme (over $\mathbb{Q}$) (equivalently, a proalgebraic $\mathbb{Q}$-group) that is an extension

$$1 \to U^{\text{rel}} \to G^{\text{rel}} \to \SL_2 \to 1$$

where $U^{\text{rel}}$ is prounipotent. The natural homomorphism $\SL_2(\mathbb{Z}) \to G^{\text{rel}}(\mathbb{Q})$ is Zariski dense.
1. \( \text{SL}_2(\mathbb{Z}) \) is naturally isomorphic to \( \pi_1(\mathcal{M}^{an}_{1,1}, \vec{t}) \) where \( \vec{t} = \partial / \partial q \). The fundamental representation \( H \) of \( \text{SL}_2 \) can be viewed as \( H^1(E_{\vec{t}}) \). It is isomorphic to \( \mathbb{Q}(0) \oplus \mathbb{Q}(-1) \).

2. The coordinate ring \( \mathcal{O}(G_{\text{rel}}) \) has a natural (limit) MHS. Its periods are (regularized) iterated integrals of modular forms. These include Manin’s iterated Shimura integrals, but there are a lot more.

3. The category of Hodge representations of \( G_{\text{rel}} \) is equivalent to the category of the admissible VMHS over \( \mathcal{M}^{an}_{1,1} \) whose weight graded quotients are sums of variations of the form \( S^mH \otimes A \), where \( A \) is a Hodge structure.

4. For each prime \( \ell \), have a \( G_{\mathbb{Q}} \) action on \( \mathcal{O}(G_{\text{rel}}) \otimes \mathbb{Q}_\ell \). This is unramified at all primes (Mochizuki+Tamagawa) and crystalline at \( \ell \) (Olsson).
1. The Lie algebra $u^{\text{rel}}$ of $\mathcal{U}^{\text{rel}}$ is free. So it is (not naturally) isomorphic to $\mathbb{L}(H_1(u^{\text{rel}}))^\wedge$.

2. As an $\text{SL}(H)$-module and as a MHS

\[ H_1(u^{\text{rel}}) \cong \prod_{n>0} H^1(\mathcal{M}_{1,1}^{\text{an}}, S^{2n}\mathbb{H}) \otimes S^{2n}H(2n + 1). \]

3. This implies that all of the Hodge structures $V_{f_1} \otimes \cdots \otimes V_{f_m}(r)$ appear in $\text{Gr}^W_\bullet \mathcal{O}(G^{\text{rel}})$. So the coordinate ring of $G^{\text{rel}}$ contains “compatible families of extensions” (cf. Deligne)

4. Which extensions does one get?
Since \((M_{1,1}, \hat{t})\) is defined over \(\mathbb{Z}\) and has everywhere good reduction, \(G^{rel}\) should be an object of MMM.

**Brown’s candidate**

Define MMM to be the full tannakian subcategory of \(MHS_{\mathbb{Q}}\) generated by the coordinate ring of \(G^{rel}\).

It contains \(MTM(\mathbb{Z})\) and (after tensoring with \(\mathbb{R}\)) all simple factors of

\[V_{f_1} \otimes \cdots \otimes V_{f_m}(r)\]

where the \(f_j\) are eigen forms, and thus extensions between them. If true, will exhibit Hodge and Galois realizations of all elements of

\[\text{Ext}^1_{MM(\mathbb{Z})}(\mathbb{Q}, V_{f_1} \otimes \cdots \otimes V_{f_m}(r))\]

in subquotients of \(O(G^{rel})\).
A universal MEM is a mixed Tate motive \((V, M)\) with an \(SL_2(\mathbb{Z})\) symmetry. The \(SL_2(\mathbb{Z})\) action is required to factor through an action of \(G^{rel}\). The monodromy coaction

\[ V \rightarrow V \otimes \mathcal{O}(G^{rel}) \]

is required to be a morphism of MHS.

The corresponding local system \(V\) over the modular curve \(M^{an}_{1,1}\) is an admissible variation of MHS whose weight graded quotients are sums of pure variations of the form \(S^m \mathbb{H}(r)\). Its fiber over \(\partial/\partial q\) is the Hodge realization of \((V, M)\).
Examples of Universal MEM

1. $S^m \mathbb{H}(r)$ for all $m \geq 0$ and all $r \in \mathbb{Z}$. These are the simple objects of MEM.

2. All objects of $\text{MTM}(\mathbb{Z})$ are geometrically constant objects of MEM.

3. The Lie algebra of $\pi_1^{\text{un}}(E'_t, \vec{w})$ is a pro-object of MEM.

Since $V$ is mixed Tate, this action factors through the maximal Tate quotient $G^{\text{eis}}$ of $G^{\text{rel}}$. Call this the Eisenstein quotient of $G^{\text{rel}}$. Denote the Lie algebra of its prounipotent radical by $u^{\text{eis}}$. The fundamental group of MEM is

$$\pi_1(\text{MEM}) \cong \pi_1(\text{MTM}(\mathbb{Z})) \rtimes G^{\text{eis}}.$$
The Lie algebra $u^{\text{eis}}$ is a quotient of the free Lie algebra

$$\mathfrak{f} := \mathbb{L}(\bigoplus_{n>0} S^{2n}H) = \mathbb{L}(e_0^j \cdot e_{2n+2} : n > 0, 0 \leq j \leq 2n)^\wedge$$

on which $\mathfrak{sl}_2$ acts. The generator $e_{2n+2}$ is a highest weight vector of $S^{2n}H$ dual to the Eisenstein series $G_{2n}$, when $n > 0$, and $e_0$ is the nilpotent of weight $-2$ in $\mathfrak{sl}_2$.

To give a presentation of $u^{\text{eis}}$, we need only give a basis of the $\mathfrak{sl}_2$ highest weight vectors in the relations. The highest weight vector of $\mathfrak{sl}_2$ weight $2n$ and degree $d$ that lies in $[S^{2a}H, S^{2b}H]$ is

$$w_{a,b}^d := \sum_{i+j=d-2 \atop i \geq 0, j \geq 0} (-1)^i \binom{d-2}{i} (2a-i)! (2b-j)! [e_0^i \cdot e_{2a+2}, e_0^j \cdot e_{2b+2}]$$
The Lie algebra of $\pi_{1}^{un}(E'_t, \vec{v})$ is isomorphic to $\mathbb{L}(H)$. This is a pro-object of $\text{MTM} = \text{MTM}(\mathbb{Z})$.

The action of $\text{SL}_2(\mathbb{Z})$ on $\pi_{1}^{un}(E'_t, \vec{v})$ induces a monodromy homomorphism

$$u^{\text{rel}} \to \text{Der} \mathbb{L}(H).$$

Its image is generated by certain derivations $\epsilon_{2n}, n \geq 0$. These are dual to the Eisenstein series $G_{2n}$ when $n > 0$.

Matsumoto and the speaker naively predicted that each cusp form should determine relations between the $\epsilon_{2n}$’s of each degree $d \geq 2$.

Pollack (in his undergraduate thesis) found such relations between the $\epsilon_{2n}$’s when $d = 2$ and found relations that hold mod a certain filtration for all $d \geq 3$.

The quadratic relations imply the Ihara-Takao relations.
Pollack Relations Lift

Theorem (Brown, Hain, Matsumoto)

For each cusp form $f$ of $\text{SL}_2(\mathbb{Z})$ of weight $2n+2$ and each $d \geq 2$, there is a degree $d$ element

$$r_{f,d} = \sum c_a w_{a,b}^d + \text{higher order terms}$$

of $\ker \{ f \rightarrow \text{ueis} \} \otimes \mathbb{C}$, where

$$r_f^e(x, y) = \sum c_a x^{2a-d} y^{2n-2a-d}$$

is the modular symbol of $f$. For each $n$ and $d$ as above,

$$\{ r_{f,d} : f \text{ a normalized eigen cusp form of weight } 2n + 2 \}$$

projects to a linearly independent subset of

$$H^2 \left( \text{ueis}, S^{2n} H(2n + d) \right)^{GL(H)} \otimes \mathbb{C}.$$
If standard conjectures in number theory are true, these \textit{geometric relations} and their Galois (or Hodge) conjugates will generate all relations in $u^{\text{eis}}$.

The remaining task is to determine the “infinitesimal Galois action”, i.e., the action of $\mathfrak{g}$ on $u^{\text{eis}}$. That is, we need to determine the \textit{arithmetic relations}

$$[z_{2m+1}, e_{2n}] \in \mathfrak{g}.$$ 

Brown and the speaker are trying to determine the quadratic terms of the RHS using his period computations. One application of this will be to Morita’s Conjecture.
Morita’s Conjecture

Suppose that $C$ is a smooth projective curve over $\mathbb{Q}$ of genus $g \geq 2$ and that $x \in C(\mathbb{Q})$. Here set $H = H_1(C^{\text{an}}, \mathbb{Q}_\ell)$.

The Lie algebra $\mathfrak{p}$ of $\pi^\text{un}_1(C^{\text{an}}, x) \otimes \mathbb{Q}_\ell$ is isomorphic to

$$\mathbb{L}(a_j, b_j : j = 1, \ldots, g)^{\wedge}/(\sum_j [a_j, b_j])$$

Denote the relative completion of $\pi_1(\mathcal{M}_{g, 1}/\overline{\mathbb{Q}}, [C, x])$ by $\mathcal{G}_{g, 1}$. There is a monodromy action $\mathcal{G}_{g, 1} \to \text{Aut } \mathfrak{p}$. It induces a Lie algebra homomorphism $\mathfrak{g}_{g, 1} \to \text{Der } \mathfrak{p}$.

There is a well defined map

$$\mathfrak{k} \to \text{Der } \mathfrak{p}/\text{im } \mathfrak{g}_{g, 1}$$

induced by the $G_\mathbb{Q}$ action on $\mathfrak{p}$. It is injective by Brown’s Theorem plus the solution of Oda’s Conjecture by Takao and others. The image is independent of $(C, x)$.

Morita has made an explicit conjecture about the image.
Sample References


and references therein.