Mixed Motives Associated to Classical Modular Forms

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Overview

This talk concerns mixed motives associated to classical modular forms.

Goals

These include making progress on:

1. new cases of Beilinson's conjectures on special values of L-functions of modular forms,
2. Morita's conjecture on the Galois action on the unipotent fundamental group of a smooth projective curve of arbitrary genus,
3. a variant (all genera) of the unipotent-de Rham version of the Grothendieck-Teichmüller conjecture.

Collaborators: Francis Brown, Makoto Matsumoto

A Holy Grail

Beilinson proposed that there is a $\mathbb{Q}$-linear tannakian category MM(X) of mixed motives associated a smooth scheme X over $\mathbb{Z}$ (say) with the correct Ext groups:

$$Ext^j_{\mathrm{MM}(X)}(\mathbb{Q}, \mathbb{Q}(n)) = H^j_{\mathrm{mot}}(X, \mathbb{Q}(n)) := K_{2n-j}(X)^{(n)}.$$

The dimension of these groups should, in most cases, be the dimension of the real Deligne cohomology group

$$H^j_{\mathcal{D}}(X^{\mathrm{an}}, \mathbb{R}(n))^{\overline{\mathcal{F}}_\infty}$$

or, equivalently, the order of vanishing of a certain L-function of X at the appropriate point.

The Standard Example: Borel and Beilinson

If $X = \operatorname{Spec} \mathcal{O}_{F,S}$, where F is a number field, then $H^j(X, \mathbb{Q}(n))$ vanishes when $j > 1$ or $n < 0$. Have $H^0(X, \mathbb{Q}(0)) = \mathbb{Q}$ and

$$H^1_{\mathrm{mot}}(X, \mathbb{Q}(n)) = K_{2n-1}(\mathcal{O}_{K,S}) \otimes \mathbb{Q} \cong \begin{cases} \mathbb{Q}^{r_1+r_2+|S|-1} & n = 1, \\ \mathbb{Q}^{r_1+r_2} & n > 1 \text{ odd}, \\ \mathbb{Q}^{r_2} & n > 0 \text{ even}. \end{cases}$$

All other groups vanish. The ranks are given by the order of vanishing of the Dedekind zeta function of $\mathcal{O}_{K,S}$ at negative integers.

Voevodsky (also Levine & Hanamura) has constructed a triangulated tensor category of motives associated to schemes over a perfect field k with the correct Ext groups. It is not tannakian and it is not known whether $H^j_{\mathrm{mot}}(X, \mathbb{Q}(n))$ vanishes when $j < 0$ (Beilinson-Soulé vanishing).

NB: Since $H^j_{\mathrm{mot}}(X, \mathbb{Q}(n))$ vanishes when $j > 2n$, the vanishing conjecture implies vanishing when $n < 0$.

Mixed Tate Motives

There is one case where things work well, almost as well as we want. Levine and Deligne-Goncharov have constructed (from Voevodsky's motives) a $\mathbb{Q}$-linear tannakian category $\mathsf{MTM}(\mathcal{O}_{K,S})$ of mixed Tate motives over a number field K, unramified over S, with the correct ext groups:

$$H^j_{\mathrm{mot}}(\operatorname{Spec}\mathcal{O}_{K,S},\mathbb{Q}(n)) = \operatorname{Ext}^j_{\mathsf{MTM}(\mathcal{O}_{K,S})}(\mathbb{Q},\mathbb{Q}(n)).$$

Mixed Tate motives have weight filtrations. Their Hodge realizations are mixed Hodge structures whose weight graded quotients are sums of Tate Hodge structures $\mathbb{Q}(r)$.

The Fundamental Group of MTM($\mathbb{Z}$)

The fundamental group of MTM($\mathcal{O}_{K,S}$) is an extension of $\mathbb{G}_m$ by a free prounipotent group. When $\mathcal{O}_{K,S} = \mathbb{Z}$, the Lie algebra of the kernel is

$$\mathfrak{k} = \mathbb{L}(\mathbf{z}_3, \mathbf{z}_5, \mathbf{z}_7, \mathbf{z}_9, \dots)^\wedge$$

where $\mathbb{G}_m$ acts on $\mathbf{z}_{2m+1}$ with weight $2m+1$.

In general,

$$\mathfrak{k}_{K,S} = \mathbb{L}\Bigg(\bigoplus_{n>0} K_{2n-1}(\mathcal{O}_{K,S})^*\Bigg)^\wedge .$$

where $\mathbb{G}_m$ acts on $K_{2n-1}(\mathcal{O}_{K,S})^*$ with weight n.

Unipotent Fundamental Groups

The *unipotent completion* $\Gamma^{\mathrm{un}}_{/F}$ of a discrete group Γ over a field F of characteristic zero is the tannakian fundamental group of the category of unipotent representations of Γ on finite dimensional F vector spaces.

Every unipotent representation of Γ over F factors through $\Gamma^{\mathrm{un}}_{/F}$:

$$\begin{tikzcd} \Gamma \arrow[r] \arrow[rr, bend left] & \Gamma^{\mathrm{un}}(F) \arrow[r, dotted] & \operatorname{Aut} V \end{tikzcd}$$

The coordinate ring (equivalently, the Lie algebra) of the unipotent fundamental group $\pi_1^{\mathrm{un}}(X, b)$ of a complex algebraic variety has a natural MHS. Here b may be a tangential base point.

Examples of Mixed Tate Motives

Deligne and Goncharov showed that $\pi_1^{\mathrm{un}}(\mathbb{P}^1 - \{0, 1, \infty\}, \vec{v})$ is a (pro-) object of $\mathrm{MTM}(\mathbb{Z})$, where $\vec{v} = \partial/\partial x \in T_0\mathbb{P}^1$. Its periods are multi-zeta values (MZVs):

$$\zeta(n_1, \dots, n_r) = \sum_{0<k_1<\cdots<k_r} \frac{1}{k_1^{n_1} k_2^{n_2} \dots k_r^{n_r}} \qquad n_r > 1$$

$$= \int_0^1 \omega_1 \overbrace{\omega_0 \dots \omega_0}^{n_1-1} \omega_1 \overbrace{\omega_0 \dots \omega_0}^{n_2-1} \dots \omega_1 \overbrace{\omega_0 \dots \omega_0}^{n_r-1}$$

where $\omega_0 = dx/x$ and $\omega_1 = dx/(1-x)$.

Question: Do the MZV span the periods of objects of $\mathrm{MTM}(\mathbb{Z})$?

Brown's Theorem

Theorem (Brown)

$\pi_1(\mathrm{MTM}(\mathbb{Z}))$ *acts faithfully on* $\pi_1^{\mathrm{un}}(\mathbb{P}^1 - \{0, 1, \infty\}, \vec{v})$.
Consequently, the periods of all objects of $\mathrm{MTM}(\mathbb{Z})$ *are MZVs.*

Corollary

$\mathcal{O}\big(\pi_1^{\mathrm{un}}(\mathbb{P}^1 - \{0, 1, \infty\}, \vec{v})\big)$ *generates* $\mathrm{MTM}(\mathbb{Z})$ *as a tannakian category and* $\mathrm{MTM}(\mathbb{Z})$ *is isomorphic to the sub tannakian category of* $\mathrm{MHS}_{\mathbb{Q}}$ *generated by it.*

So one could define $\mathrm{MTM}(\mathbb{Z})$ to be the full subcategory of $\mathrm{MHS}_{\mathbb{Q}}$ generated by $\mathcal{O}(\pi_1^{\mathrm{un}}(\mathbb{P}^1 - \{0, 1, \infty\}, \vec{v}))$.

Interlude

Questions and Comments

1. The theory of mixed Tate motives appears to be very much a “genus 0 story”, or perhaps a “hyperplane complement story”.
2. Is there a “higher genus story”?
3. If so (by Harer connectivity), genus 1 moduli spaces should be the fundamental building block. (Cf. general Grothendieck-Teichmüller story.)
4. The elliptic case relates to genus 0 by specialization to the nodal cubic and to higher genus by degeneration to trees of elliptic curves.

Motives Associated to Genus 1 Moduli Spaces

1. The stack $\mathcal{M}_{1,1}$ is defined over $\mathbb{Z}$ and has everywhere good reduction. So its cohomology groups should be motives unramified over $\mathbb{Z}$.
2. $\mathbb{H}$ is the local system $R^1 f_* \mathbb{Q}$ associated to the universal elliptic curve $f : \mathcal{E} \to \mathcal{M}_{1,1}$. It is a PVHS of weight 1.
3. Manin–Drinfeld: as a motive (Hodge, Galois, ...)
$$H^1(\mathcal{M}_{1,1}^{\mathrm{an}}, S^{2n}\mathbb{H}) = \mathbb{Q}(-2n-1) \oplus H^1_{\mathrm{cusp}}(\mathcal{M}_{1,1}^{\mathrm{an}}, S^{2n}\mathbb{H})$$
The copy of $\mathbb{Q}(-2n-1)$ corresponds to the Eisenstein series of weight $2n+2$.
4. Eichler-Shimura:
$$H^1_{\mathrm{cusp}}(\mathcal{M}_{1,1}^{\mathrm{an}}, S^{2n}\mathbb{H}_{\mathbb{R}}) = \bigoplus_f V_f$$
where V_f is the 2-dimensional real Hodge structure associated to the normalized Hecke eigen cusp form f of weight $2n+2$. It is of type $(2n+1, 0)$, $(0, 2n+1)$.

Mixed Modular Motives

1. If there were tannakian category $\mathrm{MM}(\mathbb{Z})$ of mixed motives over $\mathbb{Z}$, then one could take the full subcategory of it generated by the $H^1(\mathcal{M}_{1,1}, S^{2n}\mathbb{H})$ and the $\mathbb{Q}(r)$.
2. Brown refers to this putative category as the category of *mixed modular motives* over $\mathbb{Z}$. Denote it by $\mathrm{MMM}(\mathbb{Z})$, or just MMM.
3. With current technology, the construction of MMM from Voevodsky motives seems to be far out of reach.
4. Brown has an end run around this problem.

Question: Where can one find all of the pure motives associated to $\mathcal{M}_{1,1}$ and lots of extensions between them?

An Answer: In the coordinate ring of the relative unipotent completion of $\pi_1(\mathcal{M}_{1,1}, \partial/\partial q)$.

Relative Unipotent Completion of $\mathrm{SL}_2(\mathbb{Z})$

The *relative unipotent completion* $\mathcal{G}^{\mathrm{rel}}$ of $\mathrm{SL}_2(\mathbb{Z})$ is the fundamental group of the tannakian category whose objects are finite dimensional representations V of Γ (over $\mathbb{Q}$, say) that admit a filtration

$$V = V^0 \supset V^1 \supset V^2 \supset \cdots \supset V^N \supset V^{N+1} = 0$$

with the property that each V^j/V^{j+1} is a sum of copies of modules of the form $S^m H$, where H is the fundamental representation of SL_2.

It is an affine group scheme (over $\mathbb{Q}$) (equivalently, a proalgebraic $\mathbb{Q}$-group) that is an extension

$$1 \to \mathcal{U}^{\mathrm{rel}} \to \mathcal{G}^{\mathrm{rel}} \to \mathrm{SL}_2 \to 1$$

where $\mathcal{U}^{\mathrm{rel}}$ is prounipotent. The natural homomorphism $\mathrm{SL}_2(\mathbb{Z}) \to \mathcal{G}^{\mathrm{rel}}(\mathbb{Q})$ is Zariski dense.

Structure and Properties of $\mathcal{G}^{\mathrm{rel}}$

1. $\mathrm{SL}_2(\mathbb{Z})$ is naturally isomorphic to $\pi_1(\mathcal{M}_{1,1}^{\mathrm{an}}, \vec{\mathrm{t}})$ where $\vec{\mathrm{t}} = \partial/\partial q$. The fundamental representation H of SL_2 can be viewed as $H^1(E_{\vec{\mathrm{t}}})$. It is isomorphic to $\mathbb{Q}(0) \oplus \mathbb{Q}(-1)$.
2. The coordinate ring $\mathcal{O}(\mathcal{G}^{\mathrm{rel}})$ has a natural (limit) MHS. Its periods are (regularized) iterated integrals of modular forms. These include Manin's iterated Shimura integrals, but there are a lot more.
3. The category of Hodge representations of $\mathcal{G}^{\mathrm{rel}}$ is equivalent to the category of the admissible VMHS over $\mathcal{M}_{1,1}^{\mathrm{an}}$ whose weight graded quotients are sums of variations of the form $S^m\mathbb{H} \otimes A$, where A is a Hodge structure.
4. For each prime ℓ, have a $G_{\mathbb{Q}}$ action on $\mathcal{O}(\mathcal{G}^{\mathrm{rel}}) \otimes \mathbb{Q}_\ell$. This is unramified at all primes (Mochizuki+Tamagawa) and crystalline at ℓ (Olsson).

Structure and Properties of $\mathcal{G}^{\mathrm{rel}}$ (ctd)

1. The Lie algebra $\mathfrak{u}^{\mathrm{rel}}$ of $\mathcal{U}^{\mathrm{rel}}$ is free. So it is (not naturally) isomorphic to $\mathbb{L}(H_1(\mathfrak{u}^{\mathrm{rel}}))^\wedge$.
2. As an $\mathrm{SL}(H)$-module and as a MHS
$$H_1(\mathfrak{u}^{\mathrm{rel}}) \cong \prod_{n>0} H^1(\mathcal{M}_{1,1}^{\mathrm{an}}, S^{2n}\mathbb{H}) \otimes S^{2n}H(2n+1).$$
3. This implies that all of the Hodge structures
$$V_{f_1} \otimes \cdots \otimes V_{f_m}(r)$$
appear in $\mathrm{Gr}^W_\bullet \mathcal{O}(\mathcal{G}^{\mathrm{rel}})$. So the coordinate ring of $\mathcal{G}^{\mathrm{rel}}$ contains "compatible families of extensions" (cf. Deligne)
4. Which extensions does one get?

Brown's End Run

Since $(\mathcal{M}_{1,1}, \vec{\mathrm{t}})$ is defined over $\mathbb{Z}$ and has everywhere good reduction, $\mathcal{G}^{\mathrm{rel}}$ should be an object of MMM.

Brown's candidate

Define MMM to be the full tannakian subcategory of $MHS_{\mathbb{Q}}$ generated by the coordinate ring of $\mathcal{G}^{\mathrm{rel}}$.

It contains MTM($\mathbb{Z}$) and (after tensoring with $\mathbb{R}$) all simple factors of

$$V_{f_1} \otimes \cdots \otimes V_{f_m}(r).$$

where the f_j are eigen forms, and thus extensions between them. If true, will exhibit Hodge and Galois realizations of all elements of

$$\mathrm{Ext}^1_{\mathrm{MM}(\mathbb{Z})}(\mathbb{Q}, V_{f_1} \otimes \cdots \otimes V_{f_m}(r))$$

in subquotients of $\mathcal{O}(\mathcal{G}^{\mathrm{rel}})$.

Universal Mixed Elliptic Motives

A *universal MEM* is a mixed Tate motive $(V, M_\bullet)$ with an $\mathrm{SL}_2(\mathbb{Z})$ symmetry. The $\mathrm{SL}_2(\mathbb{Z})$ action is required to factor through an action of $\mathcal{G}^{\mathrm{rel}}$. The monodromy coaction

$$V \to V \otimes \mathcal{O}(\mathcal{G}^{\mathrm{rel}})$$

is required to be a morphism of MHS.

The corresponding local system $\mathbb{V}$ over the modular curve $\mathcal{M}_{1,1}^{\mathrm{an}}$ is an admissible variation of MHS whose weight graded quotients are sums of pure variations of the form $S^m\mathbb{H}(r)$. Its fiber over $\partial/\partial q$ is the Hodge realization of $(V, M_\bullet)$.

Examples of Universal MEM

1. $S^m\mathbb{H}(r)$ for all $m \geq 0$ and all $r \in \mathbb{Z}$. These are the simple objects of MEM.
2. All objects of $\mathsf{MTM}(\mathbb{Z})$ are *geometrically constant* objects of MEM.
3. The Lie algebra of $\pi_1^{\mathrm{un}}(E'_{\vec{t}}, \vec{w})$ is a pro-object of MEM.

Since V is mixed Tate, this action factors through *the maximal Tate quotient* $\mathcal{G}^{\mathrm{eis}}$ of $\mathcal{G}^{\mathrm{rel}}$. Call this the *Eisenstein quotient* of $\mathcal{G}^{\mathrm{rel}}$. Denote the Lie algebra of its prounipotent radical by $\mathfrak{u}^{\mathrm{eis}}$. The fundamental group of MEM is

$$\pi_1(\mathsf{MEM}) \cong \pi_1(\mathsf{MTM}(\mathbb{Z})) \ltimes \mathcal{G}^{\mathrm{eis}}.$$

First Steps Towards a Presentation of $\mathfrak{u}^{\mathrm{eis}}$

The Lie algebra $\mathfrak{u}^{\mathrm{eis}}$ is a quotient of the free Lie algebra

$$\mathfrak{f} := \mathbb{L}\big(\bigoplus_{n>0} S^{2n}H\big) = \mathbb{L}(\mathbf{e}_0^j \cdot \mathbf{e}_{2n+2} : n > 0,\ 0 \le j \le 2n)^\wedge$$

on which $\mathfrak{sl}_2$ acts. The generator $\mathbf{e}_{2n+2}$ is a highest weight vector of $S^{2n}H$ dual to the Eisenstein series G_{2n}, when $n > 0$, and $\mathbf{e}_0$ is the nilpotent of weight -2 in $\mathfrak{sl}_2$.

To give a presentation of $\mathfrak{u}^{\mathrm{eis}}$, we need only give a basis of the $\mathfrak{sl}_2$ highest weight vectors in the relations. The highest weight vector of $\mathfrak{sl}_2$ weight $2n$ and degree d that lies in $[S^{2a}H, S^{2b}H]$ is

$$\mathbf{w}_{a,b}^d := \sum_{\substack{i+j=d-2 \\ i\ge 0, j\ge 0}} (-1)^i \binom{d-2}{i} (2a-i)!(2b-j)![\mathbf{e}_0^i \cdot \mathbf{e}_{2a+2}, \mathbf{e}_0^j \cdot \mathbf{e}_{2b+2}]$$

Monodromy and Pollack Relations

- The Lie algebra of $\pi_1^{\mathrm{un}}(E'_{\vec{\mathrm{t}}}, \vec{\mathrm{v}})$ is isomorphic to $\mathbb{L}(H)$. This is a pro-object of MTM $=$ MTM$(\mathbb{Z})$.
- The action of $\mathrm{SL}_2(\mathbb{Z})$ on $\pi_1^{\mathrm{un}}(E'_{\vec{\mathrm{t}}}, \vec{\mathrm{v}})$ induces a monodromy homomorphism
$$\mathfrak{u}^{\mathrm{rel}} \to \mathrm{Der}\, \mathbb{L}(H).$$
- Its image is generated by certain derivations ϵ_{2n}, $n \geq 0$. These are dual to the Eisenstein series G_{2n} when $n > 0$.
- Matsumoto and the speaker naively predicted that each cusp form should determine relations between the ϵ_{2n}'s of each degree $d \geq 2$.
- Pollack (in his undergraduate thesis) found such relations between the ϵ_{2n}'s when $d = 2$ and found relations that hold mod a certain filtration for all $d \geq 3$.
- The quadratic relations imply the Ihara-Takao relations.

Pollack Relations Lift

Theorem (Brown, Hain, Matsumoto)

For each cusp form f of $\mathrm{SL}_2(\mathbb{Z})$ of weight $2n+2$ and each $d \geq 2$, there is a degree d element

$$\mathbf{r}_{f,d} = \sum c_a \mathbf{w}^d_{a,b} + \textit{ higher order terms}$$

of $\ker\{\mathfrak{f} \to \mathfrak{u}^{\mathrm{eis}}\} \otimes \mathbb{C}$, *where*

$$\mathrm{r}^{\epsilon}_f(x,y) = \sum c_a x^{2a-d} y^{2n-2a-d}$$

is the modular symbol of f. For each n and d as above,

$$\{\mathbf{r}_{f,d} : f \textit{ a normalized eigen cusp form of weight } 2n+2\}$$

projects to a linearly independent subset of

$$H^2\big(\mathfrak{u}^{\mathrm{eis}}, S^{2n}H(2n+d)\big)^{\mathrm{GL}(H)} \otimes \mathbb{C}.$$

Arithmetic Relations

If standard conjectures in number theory are true, these *geometric relations* and their Galois (or Hodge) conjugates will generate all relations in $\mathfrak{u}^{\mathrm{eis}}$.

The remaining task is to determine the "infinitesimal Galois action", i.e., the action of $\mathfrak{k}$ on $\mathfrak{u}^{\mathrm{eis}}$. That is, we need to determine the *arithmetic relations*

$$[\mathbf{z}_{2m+1}, \mathbf{e}_{2n}] \in \mathfrak{f}.$$

Brown and the speaker are trying to determine the quadratic terms of the RHS using his period computations. One application of this will be to Morita's Conjecture.

Morita's Conjecture

- Suppose that C is a smooth projective curve over $\mathbb{Q}$ of genus $g \geq 2$ and that $x \in C(\mathbb{Q})$. Here set $H = H_1(C^{\mathrm{an}}, \mathbb{Q}_\ell)$.
- The Lie algebra $\mathfrak{p}$ of $\pi_1^{\mathrm{un}}(C^{\mathrm{an}}, x) \otimes \mathbb{Q}_\ell$ is isomorphic to

$$\mathbb{L}(\mathbf{a}_j, \mathbf{b}_j : j = 1, \dots, g)^\wedge / (\textstyle\sum_j [\mathbf{a}_j, \mathbf{b}_j])$$

- Denote the relative completion of $\pi_1(\mathcal{M}_{g,1/\overline{\mathbb{Q}}}, [C, x])$ by $\mathcal{G}_{g,1}$. There is a monodromy action $\mathcal{G}_{g,1} \to \operatorname{Aut} \mathfrak{p}$. It induces a Lie algebra homomorphism $\mathfrak{g}_{g,1} \to \operatorname{Der} \mathfrak{p}$.
- There is a well defined map

$$\mathfrak{k} \to \operatorname{Der} \mathfrak{p} / \operatorname{im} \mathfrak{g}_{g,1}$$

  induced by the $G_{\mathbb{Q}}$ action on $\mathfrak{p}$. It is injective by Brown's Theorem plus the solution of Oda's Conjecture by Takao and others. The image is independent of (C, x).
- Morita has made an explicit conjecture about the image.

Sample References

1. F. Brown: *Multiple modular values for* $\mathrm{SL}_2(\mathbb{Z})$, [arXiv:1407.5167]
2. R. Hain: *The Hodge-de Rham theory of modular groups*, [arXiv:1403.6443]
3. R. Hain, M. Matsumoto: *Universal mixed elliptic motives*, arXiv.org soon!
4. Y. Manin: *Iterated Shimura integrals*, Mosc. Math. J. 5 (2005), 869–881.

and references therein.