Is the Goldman–Turaev Lie Bialgebra Motivic?

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Initial setting

- For a topological space X, define $\lambda(X) = [S^1, X]$.
- When X is path connected (as it will be from now on)

$$\lambda(X) = \text{ conjugacy classes in } \pi_1(X, x).$$

- For a commutative ring $\mathbb{k}$ (for us $\mathbb{Z}$ or a field of char 0) set

$$\mathbb{k}\lambda(X) = \text{free } \mathbb{k}\text{-module generated by } \lambda(X).$$

- There is an inclusion $\mathbb{k} \to \mathbb{k}\lambda(X)$ that takes $\mathbf{1}$ to the boundary of a disk and a projection $\mathbb{k}\lambda(X) \to \mathbb{k}$ that takes each loop to 1. This gives a natural decomposition

$$\mathbb{k}\lambda(X) = \mathbb{k}\mathbf{1} \oplus I_{\mathbb{k}}\lambda(X)$$

- The *cyclic quotient* of an associative $\mathbb{k}$-algebra A is

$$\mathscr{C}(A) = |A| := A/\operatorname{span}\{uv - vu : u, v \in A\}.$$

- For example the cyclic quotient of the free associative algebra $\mathbb{k}\langle x : x \in \mathscr{X}\rangle$ is spanned by the "cyclic words" in the elements x of the alphabet $\mathscr{X}$:

$$x_1 x_2 \dots x_m \sim x_2 \dots x_m x_1 .$$

- We have $\mathbb{k}\lambda(X) = \mathscr{C}(\mathbb{k}\pi_1(X, x))$.

The Goldman–Turaev Lie bialgebra

The *Goldman bracket* is a map

$$\{\ ,\ \} : \Bbbk\lambda(X) \otimes \Bbbk\lambda(X) \to \Bbbk\lambda(X)$$

that makes $\Bbbk\lambda(X)$ into a Lie algebra. The *Turaev cobracket* is a map

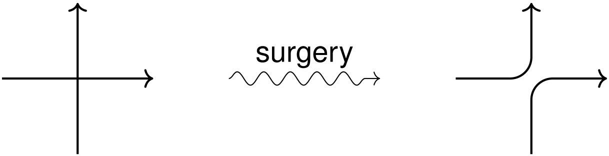
$$\delta_\xi : \Bbbk\lambda(X) \to \Bbbk\lambda(X) \otimes \Bbbk\lambda(X)$$

that depends on a framing ξ (a nowhere vanishing vector field) on X. Together they form a *Lie bialgebra*:

$$\delta_\xi\{u, v\} = u \cdot \delta_\xi(v) - v \cdot \delta_\xi(u)$$

where $w \cdot (x \otimes y) = \{w, x\} \otimes y + x \otimes \{w, y\}$.

The bracket and cobracket are defined using elementary surgery: Each element of $\lambda(X)$ can be represented by an immersed circle with simple normal crossings. (So no triple points, etc). One can perform surgery at a double point:



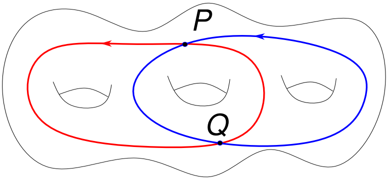
Goldman bracket

To define the Goldman bracket of $\alpha, \beta \in \lambda(X)$, represent them by oriented, transversally intersecting, immersed circles. Their Goldman bracket is

$$\{\alpha, \beta\} = \sum_{P} \epsilon_P \, \alpha \#_P \beta$$

where P ranges over the points where α intersects β, $\epsilon_P = \pm 1$ is the local intersection number at P and $\alpha \#_P \beta$ is the loop obtained by simple surgery at P.

An example

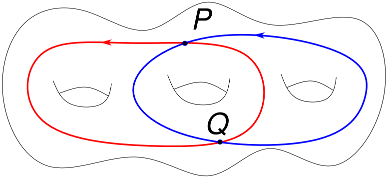


$\epsilon_P = 1$

$\epsilon_Q = -1$

$$\{\alpha, \beta\} = \epsilon_P\, \alpha \#_P \beta + \epsilon_Q\, \alpha \#_Q \beta$$

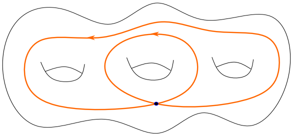
An example



$\epsilon_P = 1$

$\epsilon_Q = -1$

$\alpha \#_P \beta$



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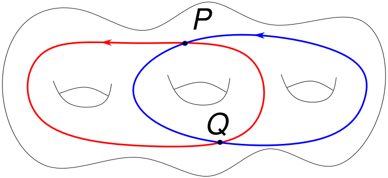
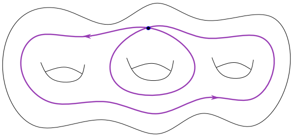


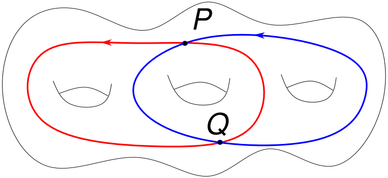
$\epsilon_P = 1$

$\epsilon_Q = -1$

$\alpha \#_Q \beta$



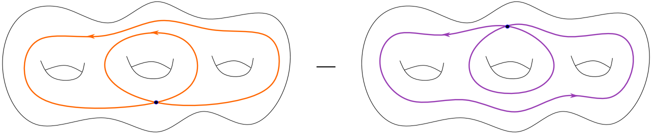
An example



$\epsilon_P = 1$

$\epsilon_Q = -1$

$$\{\alpha, \beta\} = \epsilon_P\, \alpha \#_P \beta + \epsilon_Q\, \alpha \#_Q \beta = \alpha \#_P \beta - \alpha \#_Q \beta$$

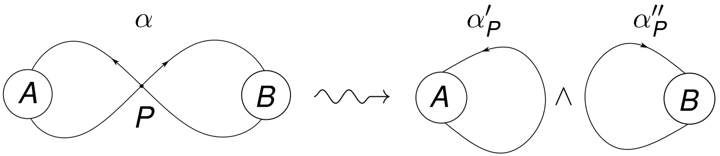


The Turaev cobracket

For convenience, we denote the element $v \otimes w - w \otimes v$ of $V^{\otimes 2}$ by $v \wedge w$. Suppose that α is an immersed circle with simple normal crossings. The first step in defining the cobracket is to define for each double point P of α

$$\delta_P(\alpha) = \alpha'_P \wedge \alpha''_P$$

where



To define $\delta_\xi(\alpha)$ represent α by an immersed loop with simple normal crossings **and trivial winding number** with respect to the framing:

$$\mathrm{rot}_\xi\, \alpha = 0.$$

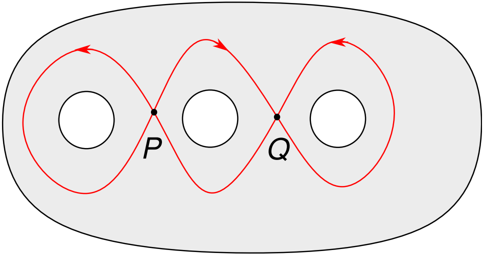
(Add some "backflips" as necessary.) The cobracket is defined by

$$\delta_\xi(\alpha) = \sum_{\text{double points } P} \epsilon_P\, \delta_P(\alpha)$$

where $\epsilon_P = \pm 1$ is the local intersection number of the initial arcs of α'_P and α''_P (in that order).

Sample cobracket

To compute the cobracket of

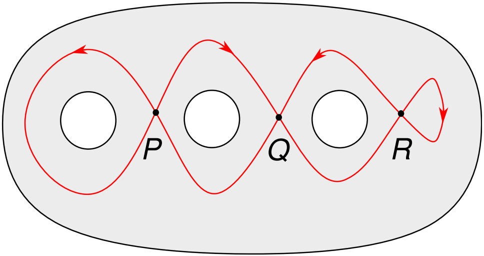


$\xi = \partial/\partial x$

$\mathrm{rot}_\xi\, \alpha = 1$

Sample cobracket

represent it by



$\xi = \partial/\partial x$

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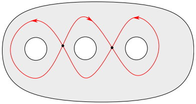
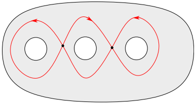
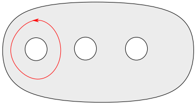
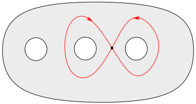
Sample cobracket

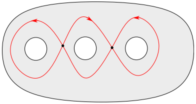
to see that δ_ξ takes  to

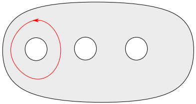
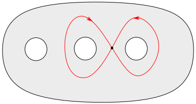
Sample cobracket

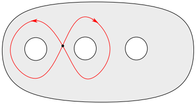
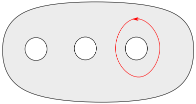
to see that δ_ξ takes  to

$-$  $\wedge$ 

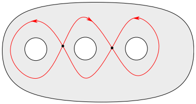
Sample cobracket

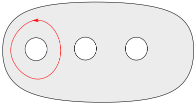
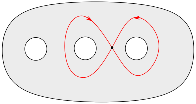
to see that δ_ξ takes  to

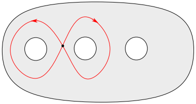
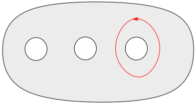
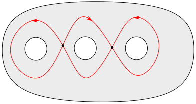
$-$  $\wedge$ 

$+$  $\wedge$ 

Sample cobracket

to see that δ_ξ takes  to

$-$  $\wedge$ 

$+$  $\wedge$  $-$  $\wedge 1$

- The Goldman–Turaev Lie bialgebra is *involutive*. That is

$$\mathbb{k}\lambda(X) \xrightarrow{\delta_\xi} \mathbb{k}\lambda(X) \otimes \mathbb{k}\lambda(X) \xrightarrow{\{\ ,\ \}} \mathbb{k}\lambda(X)$$

  is zero.
- The cobracket δ_ξ induces a map

$$\overline{\delta} : \mathbb{k}\lambda(X)/\mathbb{k}\mathbf{1} \to (\mathbb{k}\lambda(X)/\mathbb{k}\mathbf{1})^{\otimes 2}$$

  It does not depend on the framing ξ. This is called the *reduced cobracket*.

The Kawazumi–Kuno action and Turaev coaction

- Let $\vec{v}$ be a tangential base point — equivalently, a base point in the boundary of X.
- Kawazumi and Kuno extended the constructions of Goldman and Turaev to define a Lie algebra homomorphism

$$\kappa_{\vec{v}} : \mathbb{k}\lambda(X) \to \mathrm{Der}\, \mathbb{k}\pi_1(X, \vec{v}).$$

Turaev defined a coaction

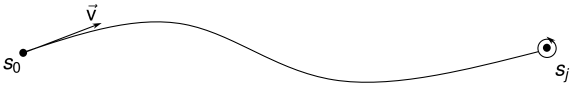
$$\mathbb{k}\pi_1(X; \vec{v}) \to \mathbb{k}\lambda(X) \otimes \mathbb{k}\pi_1(X; \vec{v}).$$

Special derivations

A derivation D of $\mathbb{k}\pi_1(X, \vec{v})$ is *special* if there are $\mu_1, \ldots, \mu_n \in \mathbb{k}\pi_1(X, \vec{v})$ (resp., its completion) such that $D(\gamma_0) = 0$ and

$$D(\gamma_j) = [\gamma_j, \mu_j] := \gamma_j\mu_j - \mu_j\gamma_j \text{ when } j > 0.$$

Here γ_j is any path of the form



Loops act as special derivations, so

$$\kappa_{\vec{v}} : \mathbb{k}\lambda(X) \to \mathrm{SDer}\, \mathbb{k}\pi_1(X, \vec{v}).$$

Completions

- From now on, $\mathbb{k}$ is a field of characteristic zero.
- Denote the augmentation idea of $\mathbb{k}\pi_1(X, \vec{v})$ by I.
- The I-adic completion of $\mathbb{k}\pi_1(X, \vec{v})$ is

$$\mathbb{k}\pi_1(X, \vec{v})^\wedge := \varprojlim_m \mathbb{k}\pi_1(X, \vec{v})/I^m.$$

- Give $\mathbb{k}\lambda(X)$ the quotient topology via $\mathbb{k}\pi_1(X, \vec{v}) \to \mathbb{k}\lambda(X)$. Its I-adic completion is

$$\mathbb{k}\lambda(X)^\wedge = \mathscr{C}(\mathbb{k}\pi_1(X, \vec{v})^\wedge).$$

The completed GT Lie bialgebra

- Kawazumi and Kuno showed that the Goldman bracket and Turaev cobracket are continuous in the I-adic topology and thus induce maps

$$\{\ ,\ \} : \mathbb{k}\lambda(X)^{\wedge} \otimes \mathbb{k}\lambda(X)^{\wedge} \to \mathbb{k}\lambda(X)^{\wedge}$$

  and

$$\delta_{\xi} : \mathbb{k}\lambda(X)^{\wedge} \to \mathbb{k}\lambda(X)^{\wedge} \widehat{\otimes}\, \mathbb{k}\lambda(X)^{\wedge}$$

  This is the *completed GT Lie bialgebra*.
- They also showed that their action is continuous, so that there is a continuous Lie algebra homomorphism

$$\kappa_{\vec{v}} : \mathbb{k}\lambda(X)^{\wedge} \to \mathrm{SDer}\, \mathbb{k}\pi_1(X, \vec{v})^{\wedge}$$

- When $(X, \vec{v})$ is a surface of type $(g, \vec{1})$, $\kappa_{\vec{v}}$ induces an isomorphism

$$\mathbb{Q}\lambda(X)^{\wedge}/\mathbb{Q}\mathbf{1} \xrightarrow{\simeq} \mathrm{SDer}\, \mathbb{k}\pi_1(X, \vec{v})^{\wedge}$$

Hodge theory

- Suppose that $X = \overline{X} - S$ where $\overline{X}$ is a compact Riemann surface, $S = \{s_0, \dots, s_n\}$ with $n \geq 0$ and $\vec{v} \in T_{s_0}\overline{X}$, $\vec{v} \neq 0$. (So $(\overline{X}, S, \vec{v})$ is a topological surface of type $(g, n + \vec{1})$.)
- When needed, ξ is an *algebraic framing* of X. That is, a meromorphic vector field on $\overline{X}$ that is nowhere vanishing and holomorphic on X.
- There is a canonical pro-mixed Hodge structure (MHS) on $\mathbb{Q}\pi_1(X, \vec{v})^\wedge$. It induces a canonical pro-MHS on $\mathbb{Q}\lambda(X)^\wedge$.
- The MHS on $\mathbb{Q}\lambda(X)^\wedge$ does not depend on $\vec{v}$, only on X.

Theorem (H: 2020, 2021)

The completed Goldman bracket

$$\{\ ,\ \} : \Bbbk\lambda(X)^{\wedge} \otimes \mathbb{Q}(-1) \otimes \Bbbk\lambda(X)^{\wedge} \otimes \mathbb{Q}(-1) \\ \to \Bbbk\lambda(X)^{\wedge} \otimes \mathbb{Q}(-1),$$

the completed Turaev cobracket

$$\delta_{\xi} : \Bbbk\lambda(X)^{\wedge} \otimes \mathbb{Q}(1) \to \left[\Bbbk\lambda(X)^{\wedge} \otimes \mathbb{Q}(1)\right]^{\widehat{\otimes}2}$$

and the Kawazumi–Kuno action

$$\kappa_{\vec{v}} : \Bbbk\lambda(X)^{\wedge} \otimes \mathbb{Q}(-1) \to \mathrm{SDer}\, \Bbbk\pi_1(X, \vec{v})^{\wedge}$$

are all morphisms of pro-MHS.

Comments and Questions

- I believe that when X is defined over a number field K, then for all ℓ, the bracket and cobracket on $\mathbb{Q}_\ell\lambda(X)^\wedge$ (after a suitable Tate twists) are $\mathrm{Gal}(\overline{\mathbb{Q}}/K)$ equivariant. Similarly for the Kawazumi–Kuno action.
- I have a sketch of an indirect proof. Can this be proved directly by ‘elementary’ arguments?
- The Hodge and Galois equivariance suggests that the Goldman–Turaev Lie bialgebra is motivic. If so, what does it have to do with cycles and motives?
- It appears that there is a link to Ceresa cycle when $g \geq 3$.
- It also appears to be related to Goncharov’s Hodge correlators.

Mapping class groups

- Denote the mapping class group of $(\overline{X}; S, \vec{v})$ by $\Gamma_{X,\vec{v}}$:

$$\Gamma_{X,\vec{v}} := \pi_0 \operatorname{Diff}^+(\overline{X}, S, \vec{v}) \cong \pi_1(\mathcal{M}_{g,n+\vec{1}}, [(X, \vec{v})]).$$

  It is a mapping class group of type $(g, n + \vec{1})$.
- Assume that X is hyperbolic: $2g - 2 + n + 1 > 0$.
- Its Torelli subgroup $T_{X,\vec{v}}$ is the kernel of the homomorphism $\Gamma_{X,\vec{v}} \to \mathrm{Sp}(H_{\mathbb{k}})$, where $H = H_1(\overline{X}; \mathbb{k})$.
- We have the extension

$$1 \to T_{X,\vec{v}} \to \Gamma_{X,\vec{v}} \to \mathrm{Sp}(H_{\mathbb{Z}}) \to 1.$$

  and the natural representation $\Gamma_{X,\vec{v}} \to \operatorname{Aut} \pi_1(X, \vec{v})$.

Relative completion of mapping class groups

The relative completion of $\Gamma_{X,\vec{v}}$ consists of an affine (aka proalgebraic) group $\mathcal{G}_{X,\vec{v}}$ defined over $\mathbb{Q}$ and a homomorphism

$$\rho : \Gamma_{X,\vec{v}} \to \mathcal{G}_{X,\vec{v}}(\mathbb{Q}).$$

This group is an extension

$$1 \to \mathcal{U}_{X,\vec{v}} \to \mathcal{G}_{X,\vec{v}} \to \mathrm{Sp}(H_{\mathbb{Q}}) \to 1$$

where $\mathcal{U}_{X,\vec{v}}$ is prounipotent. The composite

$$\Gamma_{X,\vec{v}} \to \mathcal{G}_{X,\vec{v}}(\mathbb{Q}) \to \mathrm{Sp}(H_{\mathbb{Q}})$$

is the canonical homomorphism. Such extensions form a category. The relative completion is the initial object of this category.

The unipotent completion of $\pi_1(X, \vec{v})^\wedge$

- $\mathbb{Q}\pi_1(X, \vec{v})$ is a Hopf algebra; its completion $\mathbb{Q}\pi_1(X, \vec{v})^\wedge$ is a *complete* Hopf algebra.
- The set of primitive elements of $\mathbb{Q}\pi_1(X, \vec{v})^\wedge$ is the Lie algebra $\mathfrak{p}(X, \vec{v})$ of the unipotent (aka, Malcev) completion of $\pi_1(X, \vec{v})$.
- If X is affine, $\mathbb{Q}\pi_1(X, \vec{v})^\wedge$ is (un-naturally) isomorphic to the completed tensor algebra

$$T(H_1(X; \Bbbk))^\wedge$$

with the coproduct $\Delta u = 1 \otimes u + u \otimes 1$, $u \in H_1(X)$. And $\mathfrak{p}(X, \vec{v})$ is isomorphic to $\mathbb{L}(H_1(X))^\wedge$.

The Johnson homomorphism

- Since unipotent completion is functorial, the action of $\Gamma_{X,\vec{v}}$ on $\pi_1(X,\vec{v})$ induces a homomorphism

$$\Gamma_{X,\vec{v}} \to \operatorname{Aut} \mathfrak{p}(X,\vec{v})$$

- The universal mapping property of relative completion implies that it induces a homomorphism $\mathcal{G}_{X,\vec{v}} \to \operatorname{Aut} \mathfrak{p}(X,\vec{v})$ such that the diagram

$$\begin{array}{ccccc} T_{X,\vec{v}} & \hookrightarrow & \Gamma_{X,\vec{v}} & \hookrightarrow & \operatorname{Aut} \pi_1(X,\vec{v}) \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{U}_{X,\vec{v}}(\mathbb{Q}) & \hookrightarrow & \mathcal{G}_{X,\vec{v}}(\mathbb{Q}) & \longrightarrow & \operatorname{Aut} \mathfrak{p}(X,\vec{v}) \end{array}$$

commutes.

- Denote the Lie algebras of $\mathcal{G}_{X,\vec{v}}$ and $\mathcal{U}_{X,\vec{v}}$ by $\mathfrak{g}_{X,\vec{v}}$ and $\mathfrak{u}_{X,\vec{v}}$.
- The homomorphism $\mathcal{G}_{X,\vec{v}} \to \operatorname{Aut}\mathfrak{p}(X,\vec{v})$ induces a Lie algebra homomorphism

$$\mathfrak{g}_{X,\vec{v}} \to \operatorname{SDer}\mathfrak{p}(X,\vec{v}) \qquad (*)$$

- For each $(X,\vec{v})$, there is a canonical MHS on $\mathfrak{g}_{X,\vec{v}}$ and $(*)$ is a morphism of MHS.
- This is (for me) the *geometric* Johnson homomorphism.

The arithmetic Johnson homomorphism

- There is also a homomorphism (for $\Bbbk = \mathbb{Q}, \mathbb{R}$).

$$\mathfrak{mhs}_{\Bbbk} \to \operatorname{Der} \mathfrak{p}(X, \vec{v})$$

  where $\mathfrak{mhs}_{\Bbbk}$ is the Lie algebra of $G_{\Bbbk} = \pi_1(\mathsf{MHS}_{\Bbbk})$.
- Since $\mathfrak{mhs}_{\Bbbk}$ acts on $\mathfrak{g}_{X,\vec{v}}$, we have

$$\mathfrak{mhs}_{\Bbbk} \ltimes \mathfrak{g}_{X,\vec{v}}$$

- Since $\mathfrak{mhs}_{\Bbbk}$ acts on $\mathfrak{p}(X, \vec{v})$, the Johnson homomorphism extends to

$$\mathfrak{mhs}_{\Bbbk} \ltimes \mathfrak{g}_{X,\vec{v}} \to \operatorname{Der} \mathfrak{p}(X, \vec{v})$$

- This is the *arithmetic* Johnson homomorphism

Arithmetic versus geometric Johnson image

- Denote the images of the geometric and arithmetic Johnson homomorphisms by $\overline{\mathfrak{g}}_{X,\vec{v}}$ and $\widehat{\mathfrak{g}}_{X,\vec{v}}$, respectively.
- Denote their pronilpotent radicals by $\overline{\mathfrak{u}}_{X,\vec{v}}$ and $\widehat{\mathfrak{u}}_{X,\vec{v}}$, respectively.
- The proof of Oda's Conjecture by Takao (+ Ihara, Matsumoto, Nakamura, ...), Hodge theory and Brown's fundamental theorem (on mixed Tate motives) give:

Theorem

The Lie algebras $\overline{\mathfrak{g}}_{X,\vec{v}}$ and $\widehat{\mathfrak{g}}_{X,\vec{v}}$ have natural MHS and the inclusion is a morphism. For $\mathbb{k} = \mathbb{Q}, \mathbb{R}$, and all $g, n \geq 0$ there is a SES

$$0 \to \overline{\mathfrak{g}}_{X,\vec{v}} \to \widehat{\mathfrak{g}}_{X,\vec{v}} \to \operatorname{Lie} \pi_1(\mathsf{MTM}(\mathbb{Z})) \to 0$$

Recall that $\operatorname{Gr}^W_\bullet \operatorname{Lie} \pi_1(\mathsf{MTM}(\mathbb{Z})) \cong \mathbb{Q}(0) \oplus \mathbb{L}(\sigma_3, \sigma_5, \sigma_7, \dots)$, where σ_m has type $(-m, -m)$.

- PBW gives an isomorphism of pro-MHS

$$\mathbb{Q}\pi_1(X,\vec{v})^\wedge \cong \prod_{m\geq 0} \operatorname{Sym}^m \mathfrak{p}(X,\vec{v}).$$

- The image of $\operatorname{Sym}^m \mathfrak{p}(X,\vec{v})$ in $\mathbb{Q}\lambda(X)^\wedge$ is a sub-MHS.
- Denote its image in $|\mathbb{Q}\pi_1(X,\vec{v})^\wedge| \cong \mathbb{Q}\lambda(X)^\wedge$ by $|\operatorname{Sym}^m \mathfrak{p}(X,\vec{v})|$.
- For simplicity, I'll now restrict to the case where $(X,\vec{v})$ is of type $(g,\vec{1})$. In this case

$$\mathbb{Q}\lambda(X)^\wedge/\mathbb{Q}\mathbf{1} \to \operatorname{SDer} \mathbb{Q}\pi_1(X,\vec{v})$$

  is an *isomorphism* by a result of Kawazumi and Kuno. It restricts to an isomorphism

$$|\operatorname{Sym}^2 \mathfrak{p}(X,\vec{v})| \xrightarrow{\simeq} \operatorname{SDer} \mathfrak{p}(X,\vec{v})$$

So we have a diagram

$$\begin{array}{ccc} & & \widehat{\mathfrak{u}}_{X,\vec{v}} \\ & \swarrow & \downarrow \\ |\operatorname{Sym}^2 \mathfrak{p}(X,\vec{v})|(-1) & \xrightarrow{\simeq} & \operatorname{SDer} \mathfrak{p}(X,\vec{v}) \\ \downarrow & & \downarrow \\ \mathbb{Q}\lambda(X)^\wedge \otimes \mathbb{Q}(-1) & \twoheadrightarrow & \operatorname{SDer} \mathbb{Q}\pi_1(X,\vec{v})^\wedge \end{array}$$

of pro-MHS, where all maps are morphisms.

The restriction of the cobracket to $|\operatorname{Sym}^2 \mathfrak{p}(X,\vec{\mathrm{v}})|$ induces a map

$$\widehat{\mathfrak{u}}_{X,\vec{\mathrm{v}}} \longrightarrow |\operatorname{Sym}^2 \mathfrak{p}(X,\vec{\mathrm{v}})|(-1) \xrightarrow{\delta_\xi} \left[\mathbb{Q}\lambda(X)^\wedge\right]^{\otimes 2}$$

It is closely related to the Enomoto–Satoh trace.

Theorem (H + Enomoto–Sato, Kawazumi–Kumo)

If $g \geq 3$ (with the "right choice" of ξ), the cobracket δ_ξ almost vanishes on $\widehat{\mathfrak{u}}_{X,\vec{\mathrm{v}}}$. More precisely, its kernel is the kernel of

$$\widehat{\mathfrak{u}}_{X,\vec{\mathrm{v}}} \to H_1(\mathfrak{k}) \cong \bigoplus_{m \text{ odd}>1} \mathbb{Q}(m) = \bigoplus_{m \text{ odd}>1} \mathbb{Q}\sigma_m,$$

where $\mathfrak{k}$ is the "motivic Lie algebra" of $\operatorname{Spec}\mathbb{Z}$.

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