Polylogs: prehistory and future directions

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Outline

I: Prehistory: 1842 to 1990

II: Goncharov's Hodge Correlators

III: The Goldman–Turaev Lie Bialgebra

The Dirichlet Unit Theorem (1842)

Suppose that F is a number field of degree $r_1 + 2r_2$, where:

- r_1 is the number of embeddings $\nu : F \hookrightarrow \mathbb{R}$
- r_2 is the number of complex conjugate pairs of complex (non-real) embeddings $\nu : F \hookrightarrow \mathbb{C}$.

Theorem (Dirichlet)

The group of units $\mathcal{O}_F^\times$ in the ring of integers $\mathcal{O}_F$ is finitely generated of rank $r_1 + r_2 - 1$.

Dedekind's formula (1863) + Hecke & Landau, 20th C

- For F a number field, define the *regulator mapping*

$$\mathrm{reg} : \mathcal{O}_F^\times \to [\mathbb{R}^{\mathrm{Hom}(F,\mathbb{C})}]^{\mathrm{Gal}(\mathbb{C}/\mathbb{R})} \text{ by } u \mapsto (\log|\nu(u)|)_\nu.$$

  Its image lies in the hyperplane $\sum x_\nu = 0$ (of dimension $r_1 + r_2 - 1$) as each unit has norm 1.

- One has the Dedekind zeta function $\zeta_F(s)$. It has a pole of order 1 at $s = 1$.
- Dedekind's theorem says that the kernel of reg is torsion and its image is a lattice in this hyperplane of covolume

$$R_F = \frac{w_F\sqrt{|d_F|}}{2^{r_1}(2\pi)^{r_2}h_F}\,\mathrm{Res}_{s=1}\,\zeta_F(s)$$

  where d_F is the discriminant of F, h_F the order of the class group and w_F the order of the torsion in $\mathcal{O}_F^\times$. This is the *regulator* of F.

Quillen (1972)

Algebraic K-theory:

- $K_0(R)$ is the Grothendieck group of finitely generated projective R-modules.
- For a commutative ring R (or an affine scheme $\mathrm{Spec}\, R$):

$$K_m(R) := \pi_m(B\mathrm{GL}(R)^+) \quad \text{when } m > 0,$$

  where $\mathrm{GL}(R) = \varinjlim_N \mathrm{GL}_N(R)$. The plus construction $B\mathrm{GL}(R) \to B\mathrm{GL}(R)^+$ abelianizes π_1, and induces an isomorphism on homology.
- The determinant $\det : \mathrm{GL}(R) \to R^\times$ induces a surjection

$$K_1(R) \to R^\times$$

  It is an isomorphism when R is a field.

- Quillen showed that the K-groups of the ring of integers $\mathcal{O}_F$ in a number field F are finitely generated.

$$K_0(\mathcal{O}_F) = \mathbb{Z} \oplus \text{(class group)}\,, \quad K_1(\mathcal{O}_F) = \mathcal{O}_F^\times.$$

- He also showed that $K_m(\mathcal{O}_F) \otimes \mathbb{Q} \to K_m(F) \otimes \mathbb{Q}$ is an isomorphism when $m > 1$.
- Standard topology implies that

$$K_\bullet(R) \otimes \mathbb{Q} \to H_\bullet(\mathrm{GL}(R); \mathbb{Q})$$

is injective. So, to understand the “rational K-groups” of R we have to understand group homology of $\mathrm{GL}(R)$ — equivalently, the stable rational homology of $\mathrm{GL}_N(R)$.

Borel (1974, 1977)

- Borel computed the stable rational homology of $\mathrm{SL}_n(\mathcal{O}_K)$ and thus $K_\bullet(\mathcal{O}_K) \otimes \mathbb{Q}$. It vanishes in even positive degrees.
- When $m > 1$, $K_{2m-1}(\mathcal{O}_F)$ has rank

$$d_m = \begin{cases} r_1 + r_2 & m \text{ odd} \\ r_2 & m \text{ even} \end{cases}$$

- He constructed a class β_m (the “Borel element”) in $H^{2m-1}(\mathrm{SL}(\mathbb{C}); \mathbb{R})$. It gives a map $K_{2m-1}(\mathbb{C}) \to \mathbb{R}$. It gives *higher regulator mappings*

$$\mathrm{reg}_m : K_{2m-1}(\mathcal{O}_F) \to \mathbb{R}^{d_m} \subset \mathbb{R}^{\mathrm{Hom}(F,\mathbb{C})}$$

  The kernel is finite; the image is a lattice (except when $m = 1$, when it lies in hyperplane).

Theorem (Borel)

When $m > 1$, the covolume $R_{F,m}$ of the regulator mapping

$$\mathrm{reg}_m : K_{2m-1}(\mathcal{O}_F) \to \mathbb{R}^{d_m} \subset \mathbb{R}^{\mathrm{Hom}(F,\mathbb{C})}$$

satisfies

$$\zeta_F(m) \sim_{\mathbb{Q}^\times} \frac{\pi^{m\, d_{m+1}}}{\sqrt{|d_F|}} R_{F,m}$$

where d_F is the discriminant of F.

Bloch (1978–)

- Bloch contributed important ideas and tools for studying codimension 2 cycles. One of them was the dilogarithm.
- In particular he (and Wigner) introduced the single valued dilogarithm $D_2 : \mathbb{P}^1(\mathbb{C}) \to \mathbb{R}$:

$$D_2(z) = -\operatorname{Im} \int_0^z \log(1-z)\frac{dz}{z} + \log|z| \operatorname{Arg}(1-z).$$

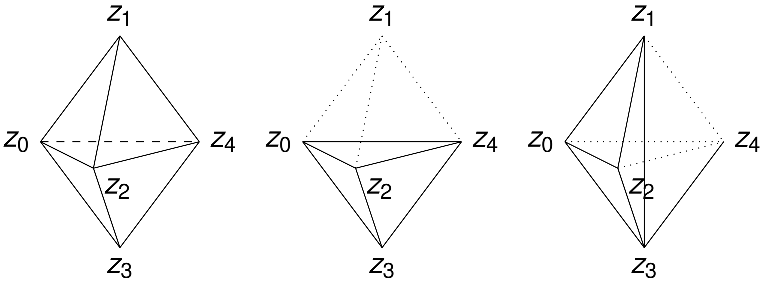
  It is the volume of the ideal hyperbolic tetrahedron with vertices $0, 1, z, \infty \in \mathbb{P}^1(\mathbb{C})$, boundary of $\mathscr{H}_3$.
- It satisfies Abel's functional equation

$$\sum_{j=0}^{4} (-1)^j D_2([z_0 : \cdots : \widehat{z_j} : \cdots : z_4]) = 0.$$

- It is a 3-cocycle on $\mathrm{GL}_2(\mathbb{C})$.

Abel's equation for Bloch–Wigner dilogarithm

$$\langle z_0, z_1, z_2, z_3, z_4\rangle = \langle z_0, \widehat{z_1}, z_2, z_3, z_4\rangle + \langle z_0, z_1, z_2, \widehat{z_3}, z_4\rangle$$
$$= \langle \widehat{z_0}, z_1, z_2, z_3, z_4\rangle + \langle z_0, z_1, \widehat{z_2}, z_3, z_4\rangle + \langle z_0, z_1, z_2, z_3, \widehat{z_4}\rangle$$



Beilinson 1984

- Beilinson (and Gillet, independently) constructed Chern classes on K-theory, into Deligne–Beilinson cohomology:

$$c_m : K_j(X) \to H^{2m-j}_{\mathscr{D}}(X, \mathbb{Z}(m)) \to H^{2m-j}_{\mathscr{D}}(X, \mathbb{R}(m))$$

- When X is defined over a number field F

$$H^{\bullet}_{\mathscr{D}}(X, \mathbb{R}(m)) = \Big[\bigoplus_{\nu : F \hookrightarrow \mathbb{C}} H^{\bullet}_{\mathscr{D}}(X_\nu(\mathbb{C}), \mathbb{R}(m)) \Big]^{\mathrm{Gal}(\mathbb{C}/\mathbb{R})}$$

In particular, when $X = \operatorname{Spec} F$:

$$H^1_{\mathscr{D}}(\operatorname{Spec} F, \mathbb{R}(m)) \cong \Big[\bigoplus_{\nu : F \hookrightarrow \mathbb{C}} \mathbb{C}/i^m\mathbb{R} \Big]^{\mathrm{Gal}(\mathbb{C}/\mathbb{R})} \cong \mathbb{R}^{d_m}.$$

- He showed that (up to a constant), Borel's regulators are Chern classes.

Polylogarithms and Chern classes

- In several contexts, the first Chern class is $\log$. For example
$$c_1 : K_1(\mathbb{C}) \to H^1_{\mathscr{D}}(\operatorname{Spec}\mathbb{C}, \mathbb{R}(1)) \cong \mathbb{R}$$
is $\log|\ \ |$ as $K_1(\mathbb{C}) = \mathbb{C}^\times$.
- The formula in Cech cohomology for the first Chern class of a complex line bundle uses the logarithm and its multivaluedness.
- The Bloch–Wigner dilogarithm D_2 defines 3-cocycle on $\mathrm{GL}_2(\mathbb{C})$. It represents $c_2 : K_3(\mathbb{C}) \to H^1_{\mathscr{D}}(\operatorname{Spec}\mathbb{C}, \mathbb{R}(2)) \cong \mathbb{R}$.
- Beilinson and Deligne showed that the second Chern class
$$c_2 : K_2(X) \to H^2_{\mathcal{D}}(X; \mathbb{Z}(2)) \cong H^1(X; \mathbb{C}^\times)$$
where X is a complex curve, can be defined using the dilog and its multivaluedness.

Ideas that were floating around in the mid 1980s

- The m-logarithm should be related to various incarnations of the mth Chern class on algebraic K-theory.
- As such, it should satisfy a $(2m+1)$-term functional equation that will make (a single valued version of it) into a $(2m-1)$-cocycle on $\mathrm{GL}_m(\mathbb{C})$.
- With the correct normalizations, this should represent the Borel class.
- Nobody $(\ldots)$ could make this work for the trilog.
- This lead to the idea of Grassmann polylogs — origins in the work of Gelfand and MacPherson.

Grassmann polylogs: first steps

- Denote the "coordinate simplex" in $\mathbb{P}^N$ by Δ_N. It is the union of $N+1$ copies of $\mathbb{P}^{N-1}$. Each face intersects the other faces in its coordinate simplex.
- Let G_q^p be the subset of $G(q, \mathbb{P}^{p+q})$ consisting of those $L \subset \mathbb{P}^{p+q}$ that do not intersect the $p-1$ stratum of of Δ_{p+q}:

$$G_q^p = \{(v_0, \ldots, v_{p+q}) \in \mathbb{C}^p : \text{each } p \times p \text{ minor} \neq 0\}/\mathrm{GL}_p.$$

- The map $G_q^p \to Y_q^p$ is a trivial $(\mathbb{C}^\times)^{p+q}$ torsor, where

$$Y_q^p := \{(x_0, \ldots, x_{p+q}) \in \mathbb{P}^{p-1} : \text{each } p \text{ span } \mathbb{P}^{p-1}\}/\mathrm{PGL}_p.$$

- $G_0^p = (\mathbb{C}^\times)^p$, $G_1^2 = Y_1^2 \times (\mathbb{C}^\times)^3$, $Y_1^2 = \mathbb{C} - \{0, 1\} = \mathcal{M}_{0,4}$,
- $G_2^2 = Y_2^2 \times (\mathbb{C}^\times)^4$, where

$$Y_2^2 = (\mathbb{C} - \{0, \})^2 - \text{diagonal} = \mathcal{M}_{0,5}.$$

The Grassmann complex

- Intersecting with the $p+q+1$ coordinate hyperplanes defines "face maps" $A_j : G^p_q \to G^p_{q-1}$, $j = 0, \dots, p+q$. These lie over face maps $Y^p_q \to Y^p_{q-1}$.
- Example: The face maps $A_j : Y^2_2 \to Y^2_1$ are:

$$A_j : (x_0, x_1, x_2, x_3, x_4) \mapsto [x_0 : \cdots : \widehat{x_j} : \cdots : x_4]$$

$$A_j(y, x, 1, 0, \infty) = \begin{cases} x & j = 0 \\ y & j = 1 \\ y/x & j = 2 \\ (1-y)/(1-x) & j = 3 \\ x(1-y)/y(1-x) & j = 4 \end{cases}$$

These are the functions that occur in the functional equation of the dilogarithm.

This leads to the *Grassmann complex* $G^p_\bullet$:

$$\{G^p_q : 0 \leq q \leq p\} + \text{ face maps } A_j : G^p_q \to G^p_{q-1}.$$

Example:

$$G^2_\bullet = \left[\; G^2_2 \;\underset{A_4}{\overset{A_0}{\rightrightarrows\!\!\!\!\rightrightarrows\!\!\!\!\rightarrow}}\; G^2_1 \;\underset{A_3}{\overset{A_0}{\rightrightarrows\!\!\!\!\rightrightarrows}}\; G^2_0 \; \right]$$

Set

$$\mathrm{vol}_p := \frac{dx_1}{x_1} \wedge \frac{dx_2}{x_2} \wedge \cdots \wedge \frac{dx_p}{x_p} \in \Omega^p(G^p_0).$$

Basic fact:

$$A^*\mathrm{vol}_p := \sum_{j=0}^{p+1} (-1)^j A_j^* \mathrm{vol}_p = 0 \text{ in } \Omega^p(G^p_1).$$

Grassmann polylogs

- $W_{2m}\widetilde{\Omega}^k(X)$ consists of logarithmic p-forms on X with coefficients that are (closed) iterated integrals of length $\leq m-k$ of logarithmic 1-forms.
- $\mathrm{vol}_p \in W_{2p}\Omega^p(G_0^p)$, $\log(1-x)\,dx/x \in W_4\widetilde{\Omega}(G_1^2)$.
- Double complex $(W_{2p}\widetilde{\Omega}^\bullet(G_\bullet^p), d, A^*)$. Set $D = d \pm A^*$.
- $D\,\mathrm{vol}_p = A^*\mathrm{vol}_p = 0$. Is it exact?
- A *Grassmann p-logarithm* is an element Z_p of this complex satisfying $D\,Z_p = \mathrm{vol}_p$. Existence was established for $p \leq 3$.
- If exists, get $\mathscr{L}_p \in W_{2p}\widetilde{\mathcal{O}}(G_{p-1}^p)$ that satisfies the $(2p+1)$ term functional equation $A^*\mathscr{L}_p = 0$.
- Hope is that Z_p represents

$$c_p : K_m(X) \to H_{\mathscr{D}}^{2p-m}(X, \mathbb{Z}(p))$$

Example: the dilogarithm

The $p = 2$ Grassmann complex is:

$$G^2_\bullet = \left[\; G^2_2 \;\overset{A_0}{\underset{A_4}{\rightrightarrows}}\; G^2_1 \;\overset{A_0}{\underset{A_3}{\rightrightarrows}}\; G^2_0 \;\right]$$

The double complex is:

$$\begin{array}{ccccc}
W_4\widetilde{\mathcal{O}}(G^2_2) & \xrightarrow{d} & W_4\widetilde{\Omega}^1(G^2_2) & & \\
\uparrow{\scriptstyle A^*} & & \uparrow{\scriptstyle A^*} & & \\
\color{red}{W_4\widetilde{\mathcal{O}}(G^2_1)} & \xrightarrow{d} & W_4\widetilde{\Omega}^1(G^2_1) & \xrightarrow{d} & \Omega^2(G^2_1) \\
 & & \uparrow{\scriptstyle A^*} & & \uparrow{\scriptstyle A^*} \\
 & & \color{red}{W_4\widetilde{\Omega}^1(G^2_0)} & \xrightarrow{d} & \color{red}{\Omega^2(G^2_0)}
\end{array}$$

Zagier's conjecture (1990 - ϵ)

Suppose that $m > 1$ and that F is a number field. For a certain single valued version D_m of the classical m-logarithm, there are are elements

$$y_1, \ldots, y_{d_{m-1}} \in \mathbb{Q}[F - \{0,1\}]$$

such that

$$\zeta_F(m) \sim_{\mathbb{Q}^\times} \frac{\pi^{m\, d_{m+1}}}{\sqrt{|d_F|}} \det P$$

where P is the $d_m \times d_m$ matrix whose entries are the values of D_m at representatives of the images of the y_k under the r_2 complex places (when m is odd) or all places (when m is even). Alternatively,

$$\det P \sim_{\mathbb{Q}^\times} R_{F,m}.$$

He proved this when $m = 2$.

Goncharov and the trilogarithm (1990)

- Remarkably, Goncharov succeeded in expressing the Grassmann trilog in terms of the classical trilogarithm.
- He used this to prove Zagier's conjecture for $\zeta_F(3)$.
- There was virtually no major progress until 2018 when Goncharov and Rudenko proved Zagier's conjecture for $\zeta_F(4)$ using some work of Gangl.
- I do not claim to understand this work.

The future . . .

> *And finally, in an attempt to unify the entire subject into a coherent whole, difficulties of a different order are encountered, and some central unifying principle has still to be discovered.*
>
> Leonard Lewin, 1981.

- This comment was prescient and still applies.
- It appears that Goncharov and Rudenko introduce two new tools:
  - Cluster algebras (of which I am ignorant)
  - motivic correlators (which I am trying to understand)
- I will give an introduction to the Hodge manifestation of motivic correlators.

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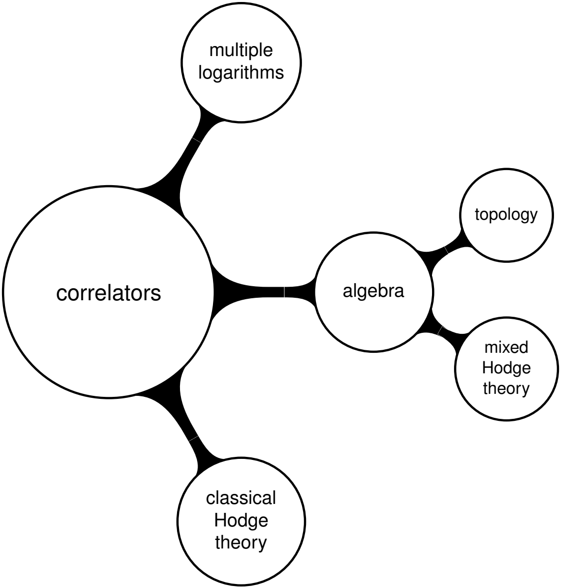
The landscape



Guided tour & plan

Goncharov's Crelle paper is 138 journal pages. Need a guide:

- currents (introduction/review)
- planar trivalent trees
- recipe for Hodge correlators
- related algebra
- selected results of Goncharov

Two more items I believe are relevant:

- topology: the Goldman (Turaev) Lie (bi)algebra
- Hodge theory

Currents

A k-current T on an n manifold M is a continuous function on the space of $n - k$ forms on M that are compactly supported in some coordinate patch. One defines bT (its boundary) by

$$\langle bT, \psi \rangle := \langle T, d\psi \rangle.$$

Every locally L^1 k-form ω on M gives a k-current $[\omega]$:

$$[\omega] : \phi \to \int_M \omega \wedge \phi.$$

When ω is smooth, $[d\omega] = d[\omega] := (-1)^{k+1} b[\omega]$. This is not true when ω is locally L^1 but not smooth.

Integration over a codimension q closed submanifold (or subvariety) Z also gives a current, denoted δ_Z:

$$\langle \delta_Z, \psi \rangle = \int_Z \psi.$$

For a k-current T on a complex manifold, we can define ∂T and $\overline{\partial} T$ by:

$$\langle \partial T, \psi \rangle := (-1)^{k+1} \langle T, \partial \psi \rangle \text{ and } \langle \overline{\partial} T, \psi \rangle := (-1)^{k+1} \langle T, \overline{\partial} \psi \rangle$$

We have $dT = \partial T + \overline{\partial} T$.

When $M = \mathbb{C}$,

$$\partial\overline{\partial}[\log|z|^2] = -2\pi i\,\delta_{[0]}.$$

That is, for all smooth, compactly supported functions h on $\mathbb{C}$

$$\langle \partial\overline{\partial}[\log|z|^2], h\rangle = -2\pi i\, h(0).$$

Note that if f is a smooth function on $\mathbb{C}$, then

$$\partial\overline{\partial} f = \frac{1}{4}\Delta f\, dz \wedge d\overline{z} = \frac{1}{2i}\Delta f\, dx \wedge dy, \quad z = x + iy.$$

So we get the classical formula of distributions

$$\Delta[\log|z|] = 2\pi\delta_0.$$

Here is a proof so that you can see how to work with currents:

$$\langle \partial\overline{\partial} \log |z|^2, h \rangle = \langle \overline{\partial} \log |z|^2, \partial h \rangle$$

h is smooth with compact support, $\overline{\partial}[\log |z|^2]$ is a 1-current

Here is a proof so that you can see how to work with currents:

$$\langle \partial\overline{\partial} \log |z|^2, h \rangle = -\langle \log |z|^2, \overline{\partial}\partial h \rangle$$

$[\log |z|^2]$ is a 0-current

Here is a proof so that you can see how to work with currents:

$$\langle \partial\overline{\partial} \log |z|^2, h\rangle = \langle \log |z|^2, \partial\overline{\partial} h\rangle$$

as $\overline{\partial}\partial = -\partial\overline{\partial}$

Here is a proof so that you can see how to work with currents:

$$\begin{aligned}\langle \partial\overline{\partial} \log |z|^2, h\rangle &= \langle \log |z|^2, \partial\overline{\partial} h\rangle \\ &= \int_{\mathbb{C}} \log |z|^2 \partial\overline{\partial} h\end{aligned}$$

the definition — the integrand is L^1

Here is a proof so that you can see how to work with currents:

$$
\begin{aligned}
\langle \partial\overline{\partial} \log |z|^2, h\rangle &= \langle \log |z|^2, \partial\overline{\partial} h\rangle \\
&= \int_{\mathbb{C}} \log |z|^2 \partial\overline{\partial} h \\
&= \lim_{\epsilon\to 0} \int_{|z|\geq\epsilon} \log |z|^2 \partial\overline{\partial} h
\end{aligned}
$$

by absolute continuity of the Lebesgue integral

Here is a proof so that you can see how to work with currents:

$$\begin{aligned}\langle \partial\bar{\partial} \log |z|^2, h\rangle &= \langle \log |z|^2, \partial\bar{\partial} h\rangle \\ &= \int_{\mathbb{C}} \log |z|^2 \partial\bar{\partial} h \\ &= \lim_{\epsilon\to 0} \int_{|z|\geq\epsilon} \log |z|^2 \partial\bar{\partial} h \\ &= -\lim_{\epsilon\to 0} \int_{|z|=\epsilon} \left(h\frac{dz}{z} + 2\log\epsilon\, \bar{\partial} h\right)\end{aligned}$$

via Stokes as $\log |z|^2 \partial\bar{\partial} h = d\big(log|z|^2 \bar{\partial} h + h\frac{dz}{z}\big)$

Here is a proof so that you can see how to work with currents:

$$\begin{aligned}
\langle \partial\bar{\partial} \log |z|^2, h\rangle &= \langle \log |z|^2, \partial\bar{\partial} h\rangle \\
&= \int_{\mathbb{C}} \log |z|^2 \partial\bar{\partial} h \\
&= \lim_{\epsilon \to 0} \int_{|z| \geq \epsilon} \log |z|^2 \partial\bar{\partial} h \\
&= -\lim_{\epsilon \to 0} \int_{|z| = \epsilon} \left(h \frac{dz}{z} + 2 \log \epsilon \, \bar{\partial} h \right) \\
&= -2\pi i \, h(0)
\end{aligned}$$

as $\epsilon^2 \log \epsilon \to 0$ and as h continuous

Here is a proof so that you can see how to work with currents:

$$
\begin{aligned}
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&= -\lim_{\epsilon\to 0} \int_{|z|=\epsilon} \left(h\frac{dz}{z} + 2\log\epsilon\,\overline{\partial} h \right) \\
&= -2\pi i\, h(0) \\
&= -2\pi i\, \langle \delta_{[0]}, h\rangle
\end{aligned}
$$

the definition of $\delta_{[0]}$

Suppose that X is a compact Riemann surface and that $\phi_1, \dots, \phi_g, \psi_1, \dots, \psi_g$ are harmonic representatives of a symplectic basis of $H^1(X)$. Choose any 2-form (or 2-current) μ with $\int_X \mu = 1$. Then, in $H^2(X \times X)$, we have the formula

$$[\Delta] = [\mu] \times 1 + 1 \times [\mu] - \sum_{j=1}^{g} ([\phi_j] \times [\psi_j] - [\psi_j] \times [\phi_j]).$$

Harmonic theory implies that there is a 0-current G_μ such that

$$\overline{\partial}\partial G_\mu = \delta_\Delta - \mu \times 1 - 1 \times \mu + \sum_{j=1}^{g} (\phi_j \times \psi_j - \psi_j \times \phi_j).$$

It is symmetric and uniquely determined, up to a constant, by μ.

Example

When $\mathbb{P}^1$ and $\mu = \delta_{[\infty]}$

$$G_{[\infty]}(x, y) = \frac{1}{2\pi i} \log |x - y|^2 \qquad (x, y) \in \mathbb{C}^2$$

as

$$\overline{\partial}\partial \log |x - y|^2 = 2\pi i(\delta_{\Delta_{\mathbb{P}^1}} - \delta_{[\infty]} \times 1 - 1 \times \delta_{[\infty]}).$$

In general, $G_\mu(x, y) - \log |x - y|^2 / 2\pi i$ is smooth near the diagonal.

Choices of μ and normalization

Three natural choices of μ are:

- a current $\delta_{[a]}$ for some $a \in X$ — works for all $g \geq 0$;
- the *Arakelov volume form* — pulled back from the flat metric on $\operatorname{Jac} X$ along $X \to \operatorname{Jac} X$ — works for all $g \geq 1$;
- the volume form of the hyperbolic metric on X — works for $g \geq 2$.

To fix G_μ, Goncharov chooses a point $x_o \in X$ (not a) and a non-zero tangent vector $\vec{v} \in T_{x_o}X$. Then take a holomorphic arc $t : (\mathbb{D}, 0) \to (X, x_o)$ with $\partial/\partial t = \vec{v}$. One insists that the restriction of $2\pi i\, G - \log|t|^2$ to $t \mapsto (x(t), x_o)$ is smooth.

Example: $G_{[\infty]}$ above satisfies this when $X = \mathbb{P}^1$, $x_o \in \mathbb{C}$ and $\vec{v} = \partial/\partial z$.

Set up

For the rest of this talk, X is a compact Riemann surface of genus $g \geq 0$ and $S = \{s_0, s_1, \ldots, s_n\}$ is a finite subset. Set

$$X' = X - S \text{ and } S_0 := S - \{s_0\} = \{s_1, \ldots, s_n\}.$$

We will assume that X' is *hyperbolic*:

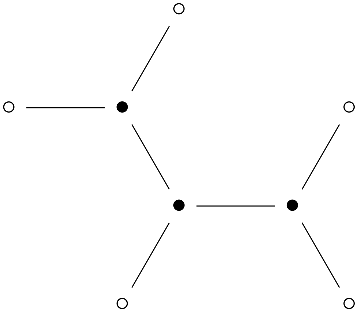
$$\chi(X') = 2 - 2g - n - 1 < 0.$$

The space of complex valued harmonic forms on X is

$$\mathscr{H} = \Omega^1(X) \oplus \overline{\Omega^1(X)}.$$

Planar trivalent trees

- m internal vertices
- $m+2$ leaves (external vertices)
- $2m+1$ edges



$m = 3$

Note that the exterior vertices are cyclically ordered.

Cyclic words

Our alphabet is $\mathscr{H} \cup S_0$. A word of length r in this alphabet is an expression

$$v_1 \ldots v_r, \qquad v_j \in \mathscr{H} \cup S_0.$$

Let $\sim$ be the equivalence relation on these words generated by

$$v_1 \ldots v_r \sim v_r v_1 \ldots v_{r-1}.$$

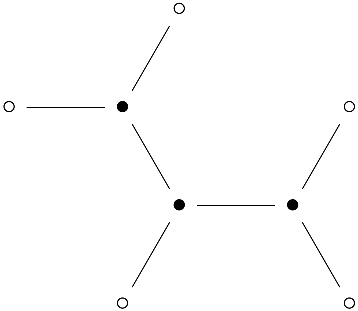
A cyclic word of length r is an equivalence class. We'll denote it by

$$c(v_1 v_2 \ldots v_r).$$

Example: $\mathbf{w} = c(\omega' s_3\, s_1\, \omega' s_2)$, where ω' and $\omega'' \in \mathscr{H}$.

Decorated planar trivalent trees

A trivalent planar tree T with m internal vertices can be labelled by a cyclic word $\mathbf{w}$ of length $\ell(\mathbf{w}) = m + 2$.



$$\mathbf{w} = c(\omega' s_3\, s_1\, \omega' s_2)$$

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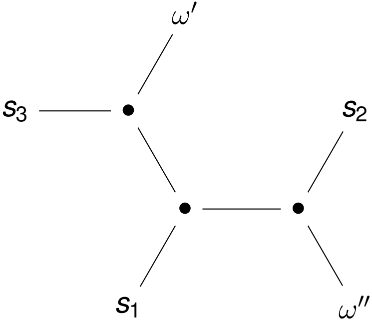
A trivalent planar tree T with m internal vertices can be labelled by a cyclic word $\mathbf{w}$ of length $\ell(\mathbf{w}) = m + 2$.



$$\mathbf{w} = c(\omega' s_3\, s_1\, \omega' s_2)$$

One more definition

For 0-currents $G_0, \ldots, G_r$ on M define $\varphi_{r+1}(G_0, \ldots, G_r)$ by

$$\frac{1}{(r+1)!} \sum_{k=0}^{r} (-1)^k \sum_{\sigma \in \mathbb{S}_{r+1}} \operatorname{sgn}(\sigma)$$
$$G_{\sigma(0)} \partial G_{\sigma(1)} \wedge \cdots \wedge \partial G_{\sigma(k)} \wedge \bar{\partial} G_{\sigma(k+1)} \wedge \cdots \wedge \bar{\partial} G_{\sigma(r)}.$$

It is a current of degree r on M and alternating in its arguments.

Examples:

$$\varphi_1(G_0) = G_0$$

and

$$\varphi_2(G_0, G_1) = \frac{1}{2}\left(G_0 \bar{\partial} G_1 - G_0 \partial G_1 - G_1 \bar{\partial} G_0 + G_1 \partial G_0\right)$$

Useful formulas

If f is a rational function on X (not just a curve), then

$$d\left[\frac{df}{f}\right] = \bar{\partial}\left[\frac{df}{f}\right] = 2\pi i\,\delta_{[\mathrm{div}\,f]}, \quad \partial\left[\frac{df}{f}\right] = 0.$$

$$\partial\bar{\partial}[\log|f|^2] = -2\pi i\,\delta_{[\mathrm{div}\,f]}$$

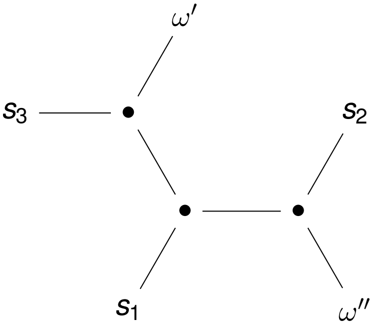
$$\partial[\log|f|^2] = \left[\frac{df}{f}\right], \quad \bar{\partial}[\log|f|^2] = \left[\frac{d\bar{f}}{\bar{f}}\right]$$

Example: if $f_0, f_1 \in \mathbb{C}(X)^\times$, then

$$\varphi_2(\log|f_0|^2, \log|f_1|^2) = \left(\log|f_0|\frac{d\bar{f}_1}{\bar{f}_1} - \log|f_0|\frac{df_1}{f_1} - \log|f_1|\frac{d\bar{f}_0}{\bar{f}_0} + \log|f_1|\frac{df_0}{f_0}\right)$$

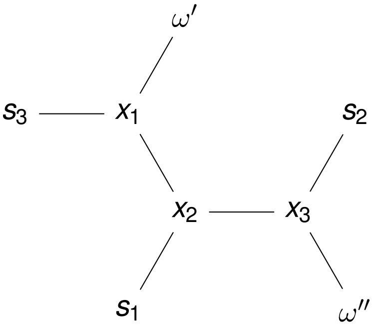
The recipe I

Consider the **w** decorated planar tree



The recipe I

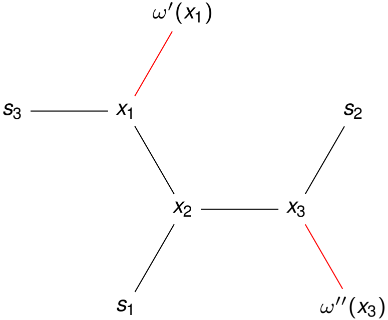
Take a copy of X for each internal vertex:



$$(x_1, x_2, x_3) \in X^3$$

The recipe II

Associate $G(x_j, x_k)$ to the edge joining x_j and x_k, and $G(x_j, s_k)$ the edge that joins x_j to s_k:



Define

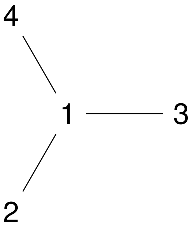
$$\Omega_T(\mathbf{w}) = \pm\varphi_5(G(s_3, x_1), G(x_1, x_2), G(x_2, x_3), G(x_2, s_1), G(x_3, s_2)) \wedge \omega'(x_1) \wedge \omega''(x_3).$$

Terms correspond to an ordered list of edges.

How to compute the sign

In fact

$$\Omega_T(\mathbf{w}) = -\varphi_5(G(s_3, x_1), G(x_1, x_2), G(x_2, x_3), G(x_2, s_1), G(x_3, s_2)) \wedge \omega'(x_1) \wedge \omega''(x_3).$$

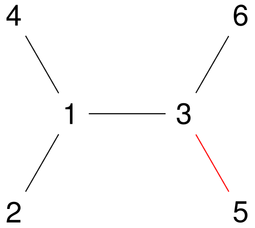


$$e_{12} \wedge e_{13} \wedge e_{14}$$

How to compute the sign

In fact

$$\Omega_T(\mathbf{w}) = -\varphi_5(G(s_3, x_1), G(x_1, x_2), G(x_2, x_3), G(x_2, s_1), G(x_3, s_2)) \wedge \omega'(x_1) \wedge \omega''(x_3).$$

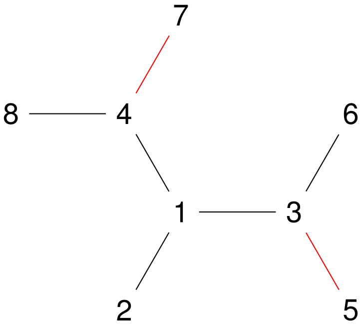


$$e_{12} \wedge e_{13} \wedge e_{14} \wedge e_{35} \wedge e_{36}$$

How to compute the sign

In fact

$$\Omega_T(\mathbf{w}) = -\varphi_5(G(s_3, x_1), G(x_1, x_2), G(x_2, x_3), G(x_2, s_1), G(x_3, s_2)) \wedge \omega'(x_1) \wedge \omega''(x_3).$$



$$\mathrm{or}_T = e_{12} \wedge e_{13} \wedge e_{14} \wedge e_{35} \wedge e_{36} \wedge e_{47} \wedge e_{48}$$
$$= -e_{48} \wedge e_{14} \wedge e_{13} \wedge e_{12} \wedge e_{36} \wedge e_{47} \wedge e_{35}$$

The recipe III

Here $\Omega_T(\mathbf{w})$ is a 6-current on X^3 that depends on the "variables" $(s_1, \ldots, s_n)$. One obtains a function of (s_1, s_2, s_3) by integrating it over X^3.

In general, for each $\mathbf{w}$ decorated (planar trivalent) graph T a current $\Omega_T(\mathbf{w})$ on X^m, where m is $\ell(\mathbf{w}) - 2$. Define the *correlator* associated to $\mathbf{w}$ (and μ) by

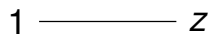
$$\mathrm{Cor}_\mu(\mathbf{w}) := \sum_{T \vdash \mathbf{w}} \int_{X^m} \Omega_T(\mathbf{w})$$

where the sum ranges over all trivalent planar trees T decorated by $\mathbf{w}$. It is a complex number or a function of $(s_1, \ldots, s_n)$ depending on your point of view.

Examples

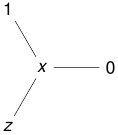
Take $X = \mathbb{P}^1$, $\mu = \delta_{[\infty]}$. Then $2\pi i\, G_\mu(x, y) = \log|x - y|^2$.

The logarithm: $S = \{0, z, \infty\}$, $\mathbf{w} = c(1\, z)$ and the $\mathbf{w}$ decorated T is



Then $2\pi i\, \Omega_T(\mathbf{w}) = \log|z|^2$.

The dilogarithm: $S = \{\infty, 0, 1, z\}$, $\mathbf{w} = c(0\, 1\, z)$ and the $\mathbf{w}$ decorated T is



$$3!(2\pi i)^3 \int_{x \in \mathbb{P}^1} \Omega_T(\mathbf{w}) = \varphi_3(\log|x|^2, \log|x - 1|^2, \log|x - z|^2)$$
$$= (\text{coefficient}) D_2(z).$$

Algebra: preparation

Recall $S = \{s_0, \dots, s_n\}$, $X' = X - S$ and $S_0 = \{s_1, \dots, s_n\}$. We have an exact sequence

$$0 \to H_2(X) \to H_0(S) \to H_1(X') \to H_1(X) \to 0$$

Denote the class of a small (positive) loop about s_j by $\mathbf{e}_j$. Then

$$H_0(S)/H_2(S) = \bigoplus_{k=0}^{n} \Bbbk \mathbf{e}_k / \Bbbk(\mathbf{e}_0 + \cdots + \mathbf{e}_n), \quad \Bbbk = \mathbb{Z}, \mathbb{Q}, \mathbb{R}.$$

Set

$$E_0 = \bigoplus_{k=1}^{n} \Bbbk \mathbf{e}_k$$

Then $E_0 \xrightarrow{\sim} H_0(S)/H_2(S)$ is an iso. Define a symmetric bilinear form on E_0 by declaring $\mathbf{e}_1, \dots, \mathbf{e}_n$ to be orthonormal. The intersection pairing defines a symplectic form on $H_1(X)$.

Algebra: setup

Suppose that $V = H \oplus E$ is a $\Bbbk$ vector space ($\Bbbk = \mathbb{Q}, \mathbb{R}$), where H has a symplectic inner product and E has a non-degenerate symmetric inner product. Give H weight -1 and E weight -2.

Example: $V = \mathrm{Gr}^W_\bullet H_1(X') = H_1(X) \oplus E_0$.

Fix an orthonormal basis $\mathbf{e}_1, \dots, \mathbf{e}_n$ of E. Denote the dual space by $V^\vee = H^\vee \oplus E^\vee$. Denote the dual basis of $E^\vee$ by $s_1, \dots, s_n$.

Example: $V^\vee = \mathrm{Gr}^W_\bullet H^1(X') = H^1(X) \oplus E_0^\vee$. The residue map gives an isomorphism

$$\mathrm{Res} : \mathrm{Gr}^W_2 H^1(X') \xrightarrow{\simeq} \widetilde{H}_0(S) = \left\{ \textstyle\sum_{k=0}^n a_k s_k : \sum_k a_k = 0 \right\}.$$

The dual orthonormal basis on $E_0^\vee$ is $\{s_1, \dots, s_n\}$.

Remark: Our alphabet is $\mathscr{H} \cup S_0 \cong H^1(X) \cup S_0$.

Special derivations

Let TV be the tensor algebra on V. It is the universal enveloping algebra of the free Lie algebra $\mathbb{L}(V)$. Both are graded by weight. There are canonical graded isomorphisms

$$\operatorname{Gr}^W_\bullet \Bbbk\pi_1(X', \vec{v})^\wedge \cong T(H_1(X) \oplus E_0) = TV.$$

Define $\mathbf{e}_0 \in \operatorname{Gr}^W_{-2} TV$ by

$$\mathbf{e}_0 + \mathbf{e}_1 + \cdots + \mathbf{e}_n + \sum_{j=1}^{g} [\mathbf{p}_j, \mathbf{q}_j] = 0$$

where $\mathbf{p}_1, \dots, \mathbf{p}_g, \mathbf{q}_1, \dots, \mathbf{q}_g$ is a symplectic basis of $H_1(X)$.

A derivation δ of TV is called *special* if $\delta(\mathbf{e}_0) = 0$ and there are $\mathbf{u}_k \in TV$ such that $\delta(\mathbf{e}_k) = [\mathbf{e}_k, \mathbf{u}_k]$ when $k \neq 0$. A derivation δ of $\mathbb{L}(V)$ is *special* if each $\mathbf{u}_j \in \mathbb{L}(V)$.

Cyclic words

The *cyclic quotient* of an associative $\Bbbk$-algebra A is

$$\mathscr{C}(A) = A/\langle uv - vu : u, v \in A \rangle.$$

Elements of $\mathscr{C}(TV)$ are cyclic words in the alphabet $\{\mathbf{p}_j, \mathbf{q}_j, \mathbf{e}_k : 1 \leq j \leq g, 1 \leq k \leq n\}$. It is a Lie algebra with graded bracket (after a shift by 2):

$$\{\ ,\ \}_0 : \mathrm{Gr}^W_{2-j}\mathscr{C}(TV) \otimes \mathrm{Gr}^W_{2-k}\mathscr{C}(TV) \to \mathrm{Gr}^W_{2-j-k}\mathscr{C}(TV)$$

There is also a *surjective* Lie algebra homomorphism

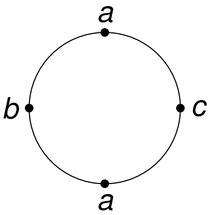
$$\Phi_0 : \mathscr{C}(TV) \to \mathrm{SDer}\, TV.$$

Its kernel is spanned by $\mathbf{e}_j^m$ where $j \neq 0$ and $m \geq 0$.

Formula for the action

Suppose $A = \Bbbk\langle a_1, \ldots, a_m \rangle$. We have operators $\frac{\partial}{\partial a_j} : \mathscr{C}(A) \to A$ of weight $+2$. For example:

$\frac{\partial}{\partial a}$:



$\mapsto \quad bac + cab$

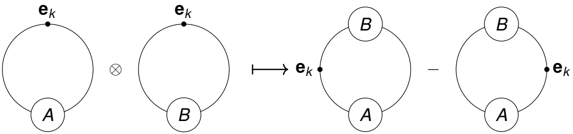
For $F \in \mathscr{C}(TV)$, $\Phi_0(F) \in \mathrm{SDer}(TV)$ is defined by

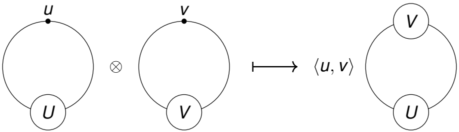
$$\Phi_0(F) : \begin{cases} \mathbf{p}_j \mapsto -\partial F / \partial \mathbf{q}_j, \\ \mathbf{q}_j \mapsto \quad \partial F / \partial \mathbf{p}_j, \\ \mathbf{e}_k \mapsto [\mathbf{e}_k, \partial F / \partial \mathbf{e}_k] \quad k \neq 0. \end{cases}$$

Formula for the bracket

For $F, G \in \mathscr{C}(TV)$

$$\{F,G\}_0 = c\left(\sum_{k\neq 0}\left[\frac{\partial F}{\partial \mathbf{e}_k}, \frac{\partial G}{\partial \mathbf{e}_k}\right]\mathbf{e}_k + \sum_{j=1}^{g}\left(\frac{\partial F}{\partial \mathbf{p}_j}\frac{\partial G}{\partial \mathbf{q}_j} - \frac{\partial G}{\partial \mathbf{p}_j}\frac{\partial F}{\partial \mathbf{q}_j}\right)\right)$$

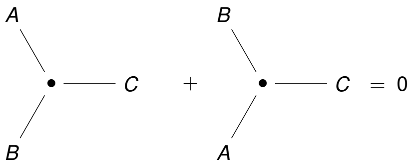




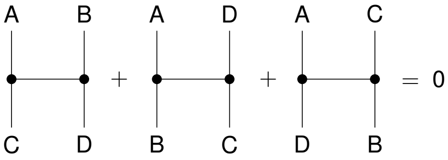
Here $k \neq 0$, $u, v \in H$ and $A, B, U, V \in TV$.

The Lie algebra $\mathscr{C}(\mathbb{L}(V))$

The Lie algebra $\mathscr{C}(\mathbb{L}(V))$ is defined to be the Lie algebra of V-decorated trivalent planar graphs modulo the AS-relation



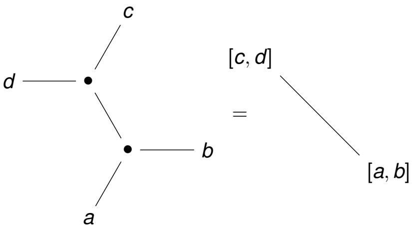
and the IHX-relation



The homomorphism $\mathscr{C}(\mathbb{L}(V)) \to \mathscr{C}(TV)$

Expanding V-labelled planar trivalent trees defines an injective Lie algebra homomorphism

$$\mathscr{C}(\mathbb{L}(V)) \to \mathscr{C}(TV)$$



$$\mapsto \quad c\big((ab - ba)(cd - dc)\big)$$

The PBW Theorem gives a coalgebra isomorphism:

$$TV = U\mathbb{L}(V) = \bigoplus_{m\geq 0} \operatorname{Sym}^m \mathbb{L}(V).$$

"Cutting" an edge of a decorated tree defines a well-defined map $\mathscr{C}(\mathbb{L}(V)) \to |\operatorname{Sym}^2 \mathbb{L}(V)|$. It has an obvious inverse, so we have a Lie algebra isomorphism

$$\mathscr{C}(\mathbb{L}(V)) \cong |\operatorname{Sym}^2 \mathbb{L}(V)|.$$

The restriction of $\mathscr{C}(TV) \to \operatorname{SDer} \mathbb{L}(V)$ to $\mathscr{C}(\mathbb{L}(V))$ is surjective and has kernel

$$\operatorname{span}\{\mathbf{e}_1^2, \ldots, \mathbf{e}_n^2\}.$$

Correlators revisited

Recall that, after fixing a “volume form” μ and **w** a cyclic word in $\mathscr{H} \cup S_0$, we defined

$$\mathrm{Cor}_\mu(\mathbf{w}) = \sum_{T \vdash \mathbf{w}} \int_{X^{\ell(\mathbf{w})-2}} \Omega_T(\mathbf{w}) \in \mathbb{C}.$$

The cyclic words **w** are actually elements of $\mathscr{C}(TV)^\vee$. So

$$\mathrm{Cor}_\mu(\mathbf{w}) \in \mathscr{C}(TV).$$

Summing over all cyclic words **w** in the alphabet $\{\phi_j, \psi_j, s_k\}$ gives

$$\mathrm{Cor}_\mu \in \mathscr{C}(TV)$$

and therefore a special derivation $\delta_{X',\vec{v}} := \Phi_0(\mathrm{Cor}_\mu) \in \mathrm{SDer}\, TV$.

The correlator Cor_μ is purely imaginary and lies in $F^{-1} \cap \overline{F}^{-1} \mathscr{C}(TV)$. That is $\mathrm{Cor}_\mu \in i\mathscr{C}(\mathbb{L}(V))_{\mathbb{R}}$.

Theorem (Goncharov)

1. $\mathrm{Cor}_\mu \in \mathscr{C}(\mathbb{L}(V))$, *so that* $\delta_{X',\vec{v}} \in i\,\mathrm{SDer}\,\mathbb{L}(V)_{\mathbb{R}}$.
2. *The derivation* $\delta_{X',\vec{v}}$ *determines a MHS on the completed group algebra of* $\pi_1(X',\vec{v})$ *via the map*

$$\exp \delta_{X',\vec{v}} : \prod_{m \geq 0} \mathrm{Gr}^W_{-m} \mathbb{R}\pi_1(X',\vec{v}) \to \prod_{m \geq 0} \mathrm{Gr}^W_{-m} \mathbb{C}\pi_1(X',\vec{v}).$$

Apparently, this is the canonical MHS.

Outline

I: Prehistory: 1842 to 1990

II: Goncharov's Hodge Correlators

III: The Goldman–Turaev Lie Bialgebra

Enter topology

These cyclic constructions in correlators comes from topology — specifically from the *Goldman Lie algebra* and the *Kawazumi–Kuno action* of it on $\Bbbk\pi_1(X', \vec{v})$. In topology, there is additional structure — the *Turaev cobracket* which does not (yet) appear in correlators.

- For a connected, oriented surface Y, set

$$\lambda(Y) = [S^1, Y] = \{\text{conjugacy classes in } \pi_1(Y, y)\}.$$

- For a commutative ring $\Bbbk$ (e.g., $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$), set

$$\Bbbk\lambda(Y) = \text{free } \Bbbk\text{-module generated by } \lambda(Y).$$

- It is the cyclic quotient of the group algebra:

$$\Bbbk\lambda(Y) = \mathscr{C}(\Bbbk\pi_1(Y, x))$$

The Goldman–Turaev Lie bialgebra

The *Goldman bracket* is a map

$$\{\ ,\ \} : \mathbb{k}\lambda(Y) \otimes \mathbb{k}\lambda(Y) \to \mathbb{k}\lambda(Y)$$

that makes $\mathbb{k}\lambda(Y)$ into a Lie algebra. The *Turaev cobracket* is a map

$$\delta_\xi : \mathbb{k}\lambda(Y) \to \mathbb{k}\lambda(Y) \otimes \mathbb{k}\lambda(Y)$$

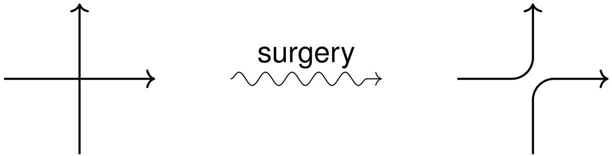
that depends on a framing ξ (a nowhere vanishing vector field) on X. Together they form a *Lie bialgebra*:

$$\delta_\xi\{u, v\} = u \cdot \delta_\xi(v) + \delta_\xi(u) \cdot v$$

where $u \cdot (x \otimes y) = \{u, x\} \otimes y$ and $(x \otimes y) \cdot v = x \otimes \{y, v\}$.

An elementary surgery

The bracket and cobracket are defined using elementary surgery: Each element of $\lambda(Y)$ can be represented by an immersed circle with simple normal crossings. (So no triple points, etc). One can perform surgery at a double point:



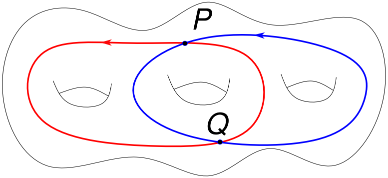
The Goldman bracket

To define the Goldman bracket of $\alpha, \beta \in \lambda(Y)$, represent them by oriented, transversally intersecting, immersed circles. Their Goldman bracket is

$$\{\alpha, \beta\} = \sum_{P} \epsilon_P \, \alpha \#_P \beta$$

where P ranges over the points where α intersects β, $\epsilon_P = \pm 1$ is the local intersection number at P and $\alpha \#_P \beta$ is the loop obtained by simple surgery at P.

An example

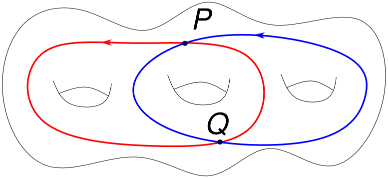


$\epsilon_P = 1$

$\epsilon_Q = -1$

$$\{\alpha, \beta\} = \epsilon_P \, \alpha \#_P \beta + \epsilon_Q \, \alpha \#_Q \beta$$

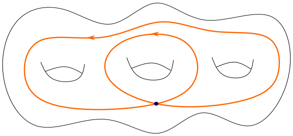
An example



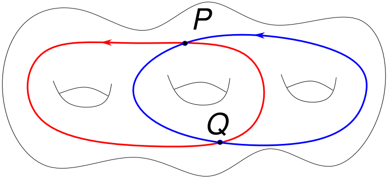
$\epsilon_P = 1$

$\epsilon_Q = -1$

$\alpha \#_P \beta$



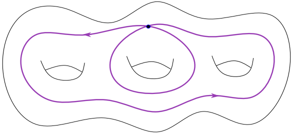
An example



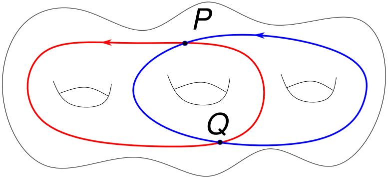
$\epsilon_P = 1$

$\epsilon_Q = -1$

$\alpha \#_Q \beta$



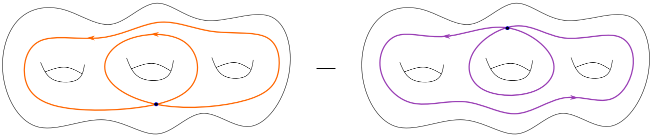
An example



$\epsilon_P = 1$

$\epsilon_Q = -1$

$$\{\alpha, \beta\} = \epsilon_P\, \alpha \#_P \beta + \epsilon_Q\, \alpha \#_Q \beta = \alpha \#_P \beta - \alpha \#_Q \beta$$



The Kawazumi–Kuno action

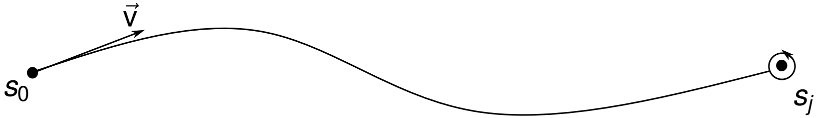
We will take $Y = X' = X - S$. There is a similarly defined Lie algebra homomorphism

$$\kappa_{\vec{v}} : \Bbbk\lambda(X', \vec{v}) \to \mathrm{SDer}\, \Bbbk\pi_1(X', \vec{v})$$

where here a derivation δ of $\Bbbk\pi_1(X', \vec{v})$ is *special* if there are $\mu_1, \ldots, \mu_n \in \Bbbk\pi_1(X', \vec{v})$ such that $\delta(\gamma_0) = 0$ and

$$\delta(\gamma_j) = [\gamma_j, \mu_j] := \gamma_j\mu_j - \mu_j\gamma_j \text{ when } j > 0.$$

Here γ_j is any path of the form



Completions

Now suppose that $\mathbb{k}$ is a field of characteristic 0.

- We can complete the group algebra $\mathbb{k}\pi_1(X', \vec{v})$ in the standard way:

$$\mathbb{k}\pi_1(X', \vec{v})^\wedge := \varprojlim_m \mathbb{k}\pi_1(X', \vec{v})/I^m$$

  where I is the kernel of the augmentation $\mathbb{k}\pi_1(X', \vec{v}) \to \mathbb{k}$. This has a natural topology — the I-adic topology.

- The corresponding completion of $\mathbb{k}\lambda(X')$ is

$$\mathbb{k}\lambda(X')^\wedge := \mathscr{C}(\mathbb{k}\pi_1(X', \vec{v})^\wedge).$$

  Give this the quotient topology — also called the I-adic topology.

The completed Goldman Lie algebra

Kawazumi–Kuno: the bracket and the KK-action are continuous and so induce continuous mappings

$$\{\ ,\ \} : \mathbb{k}\lambda(X')^\wedge \otimes \mathbb{k}\lambda(X')^\wedge \to \mathbb{k}\lambda(X')^\wedge$$

$$\kappa_{\vec{v}} : \mathbb{k}\lambda(X')^\wedge \to \mathrm{SDer}\,\mathbb{k}\pi_1(X', \vec{v})^\wedge.$$

Theorem

- *$\mathbb{Q}\lambda(X')^\wedge$ has a canonical mixed Hodge structure (MHS).*
- *It is a quotient of the canonical MHS on $\mathbb{Q}\pi_1(X', \vec{v})^\wedge$ and does not depend on the choice of $s_0 \in S$ or $\vec{v} \in TX_{s_0}$.*
- *The Tate twist $\mathbb{Q}\lambda(X')^\wedge(-1)$ is a Lie algebra in the category of pro-MHS.*
- *The action $\mathbb{Q}\lambda(X')^\wedge(-1) \to \mathrm{SDer}\,\mathbb{Q}\pi_1(X', \vec{v})^\wedge$ is a morphism of pro-MHS.*

Hodge theory and splittings

Hodge theory gives natural isomorphisms of a MHS V with its associated weight graded $\mathrm{Gr}^W_\bullet V$. There is a canonical isom

$$\mathrm{Gr}^W_\bullet \mathbb{Q}\pi_1(X', \vec{\mathrm{v}})^\wedge \cong T(\mathrm{Gr}^W_\bullet H_1(X')) \cong T(H_1(X) \oplus E_0) = TV.$$

Theorem

- *The graded Lie algebra* $\mathrm{Gr}^W_\bullet \mathbb{Q}\lambda(X')^\wedge$ *is canonically isomorphic to* $\mathscr{C}(T(H \oplus S_0), \{\ ,\ \}_0)$.
- *The diagram*

$$\begin{array}{ccc} \mathrm{Gr}^W_\bullet \mathbb{Q}\lambda(X')^\wedge & \xrightarrow{\kappa_{\vec{\mathrm{v}}}} & \mathrm{SDer}\, \mathrm{Gr}^W_\bullet \mathbb{Q}\pi_1(X', \vec{\mathrm{v}})^\wedge \\ \cong \downarrow & & \downarrow \cong \\ \mathscr{C}(TV) & \xrightarrow{\Phi_0} & \mathrm{SDer}\, TV \end{array}$$

commutes.

- S. Bloch: *Applications of the dilogarithm function in algebraic K-theory and algebraic geometry*. Proceedings of the International Symposium on Algebraic Geometry (Kyoto Univ., Kyoto, 1977), pp. 103–114.
- A. Borel: *Stable real cohomology of arithmetic groups*, Ann. Sci. École Norm. Sup. 7 (1974), 235–272.
- A. Borel: *Cohomologie de* SL_n *et valeurs de fonctions zeta aux points entiers*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. 4 (1977), 613–636.
- B. Bost, H. Gillet, C. Soulé: *Heights of projective varieties and positive Green forms*, J. Amer. Math. Soc. 7 (1994), 903–1027.
- H. Gillet, C. Soulé: *Arithmetic intersection theory*, Inst. Hautes Études Sci. Publ. Math. No. 72 (1990), 93–174 (1991).

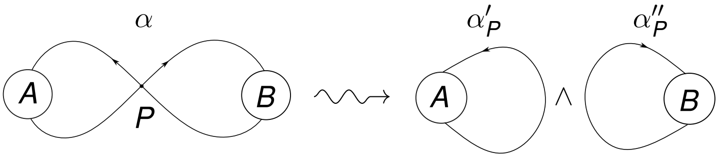
- I. Gelfand, R. MacPherson: *Geometry in Grassmannians and a generalization of the dilogarithm*, Adv. in Math. 44 (1982), 279–312.
- A. Goncharov: *The classical trilogarithm, algebraic K-theory of fields and Dedekind zeta functions*, Bull. Amer. Math. Soc. 24 (1991), 155–162.
- A. Goncharov: *Hodge correlators*, J. Reine Angew. Math. 748 (2019), 1–138.
- A. Goncharov, D. Rudenko: *Motivic correlators, cluster varieties and Zagier's conjecture on $\zeta_F(4)$*, [arXiv:1803.08585]
- R. Hain, R. MacPherson: *Higher logarithms*, Illinois J. Math. 34 (1990), 392–475.
- D. Quillen: *Higher algebraic K-theory, I*, in Algebraic K-theory, I: Higher K-theories (Proc. Conf., Battelle Inst., 1972), pp. 85–147. Springer LNM. 341, 1973.

The Turaev cobracket

For convenience, we denote the element $v \otimes w - w \otimes v$ of $V^{\otimes 2}$ by $v \wedge w$. Suppose that α is an immersed circle with simple normal crossings. The first step in defining the cobracket is to define for each double point P of α

$$\delta_P(\alpha) = \alpha'_P \wedge \alpha''_P$$

where



To define $\delta_\xi(\alpha)$ represent α by an immersed loop with simple normal crossings **and trivial winding number** with respect to the framing:

$$\mathrm{rot}_\xi\,\alpha = 0.$$

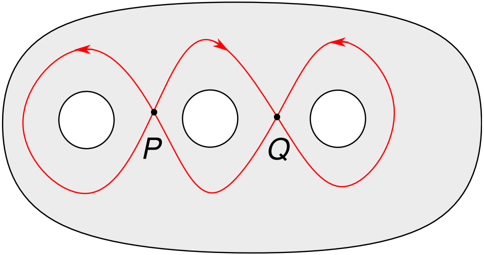
(Add some "back flips" as necessary.) The cobracket is defined by

$$\delta_\xi(\alpha) = \sum_{\text{double points } P} \epsilon_P\,\delta_P(\alpha)$$

where $\epsilon_P = \pm 1$ is the local intersection number of the initial arcs of α'_P and α''_P (in that order).

Sample cobracket

To compute the cobracket of

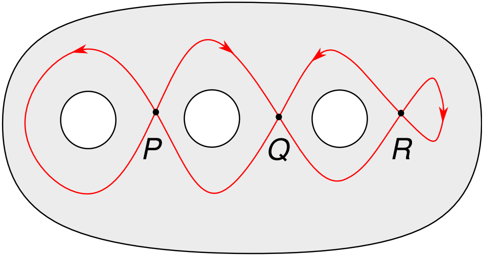


$\xi = \partial/\partial x$

$\text{rot}_\xi\, \alpha = 1$

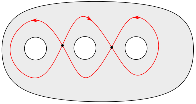
Sample cobracket

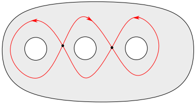
repersent it by

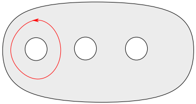
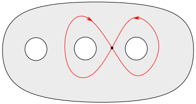


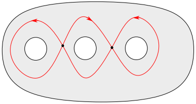
$\xi = \partial/\partial x$

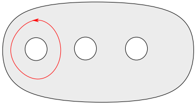
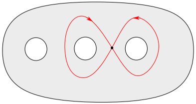
$\text{rot}_\xi\, \alpha = 0$

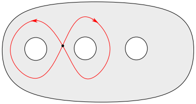
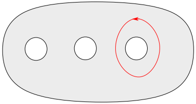
Then δ_ξ takes  to

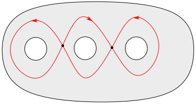
Then δ_ξ takes  to

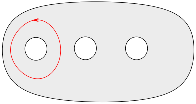
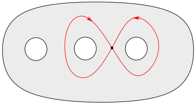
$-$  $\wedge$ 

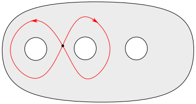
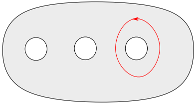
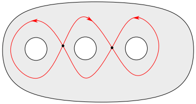
Then δ_ξ takes  to

$-$  $\wedge$ 

$+$  $\wedge$ 

Then δ_ξ takes  to

$-$  $\wedge$ 

$+$  $\wedge$  $-$  $\wedge 1$