The Rank of the Normal Function of the Ceresa Cycle

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Introduction

- Motivated by a question of Shou-Wu Zhang — will state his applications later on.
- The Ceresa cycle is a motivic incarnation of the Johnson homomorphism
  \[ H_1(T_g) \to (\Lambda^3 H_1(C))/H_1(C) \]
- Two way traffic:
  mapping class groups \( \leftrightarrow \) Arakelov and algebraic geometry of \( \mathcal{M}_g \).
The Ceresa cycle

Throughout $C$ will be a smooth projective curve (over $\mathbb{C}$) of genus $g \geq 2$. For each $x \in C$ the Abel–Jacobi map

$$\alpha_x : C \to \text{Jac } C$$

takes $y$ to the divisor class of $y - x$. Denote its image by $C_x$. This is an algebraic 1-cycle in $\text{Jac } C$. Let $\iota$ be the involution $u \mapsto -u$ of $\text{Jac } C$. Set

$$C_x^{-} = \iota_* C_x.$$

Since $\iota$ acts as $-1$ on $H^1(\text{Jac } C)$, it acts as $(-1)^k$ on $H^k(\text{Jac } C)$. So the algebraic 1-cycle

$$C_x - C_x^{-}$$

on $\text{Jac } C$ is homologous to 0. This is the Ceresa cycle of $(C, x)$. 
Detecting homologically trivial cycles via Hodge theory

Suppose that $Z = \sum_j n_j Z_j$ is an algebraic $d$-cycle on a smooth projective variety $X$. When $[Z] = 0$ in $H_{2d}(X)$, there is an extension

$$0 \to H_{2d+1}(X)(-d) \to E_Z \to \mathbb{Z}(0) \to 0$$

Pull back the LES of $(X, |Z|)$ along $cl_Z : \mathbb{Z} \to H_{2d}(|Z|)$:

$$\begin{array}{c}
0 & \to & H_{2d+1}(X) & \to & H_{2d+1}(X, |Z|) & \to & H_{2d}(|Z|) & \to & H_{2d}(X) \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
0 & \to & H_{2d+1}(X) & \to & E_Z(d) & \to & \mathbb{Z}(d) & \to & 0
\end{array}$$

The extension $E_Z$ is generated by $H_{2d}(X)$ and $\Gamma$, where $\partial \Gamma = Z$. 
The group of 1-extensions

Suppose that $V$ is a Hodge structure of negative weight, then

$$\text{Ext}^1_{\text{MHS}}(\mathbb{Z}, V) \cong J(V) := V_{\mathbb{C}}/(V_{\mathbb{Z}} + F^0 V).$$

If $V$ has weight $-1$, then $V_{\mathbb{C}} = F^0 V \oplus \overline{F^0 V}$, which implies that

$$V_{\mathbb{R}} \to V_{\mathbb{C}}/F^0 V$$

is an $\mathbb{R}$-linear isomorphism. It induces an isomorphism

$$J(V_{\mathbb{R}}) := V_{\mathbb{R}}/V_{\mathbb{Z}} \to J(V)$$

of tori. In particular, $J(V)$ is compact (but typically not algebraic).
The Griffiths invariant

A homologically trivial $d$-cycle on $X$ thus determines a point\(^1\)

$$\nu_Z \in J(H_{2d+1}(X)).$$

So the Ceresa cycle $C_x - C_x^-$ determines

$$\nu_{C,x} \in J(H_3(\text{Jac } C)).$$

It is

$$\int_{\Gamma} \in \text{Hom}(F^2H^3(\text{Jac } C), \mathbb{C})/H_3(\text{Jac } C; \mathbb{Z}) \cong J(H_3(\text{Jac } C))$$

where $\partial \Gamma = C_x - C_x^-$. This intermediate jacobian is not algebraic.

\(^1\)From now on I will suppress the Tate twist — always twist so that the odd weight Hodge structure $V$ in $J(V)$ has weight $-1$. 
Eliminating the base point

Set $H = H_1(C)$ and

$$\theta = \sum_{j=1}^{g} a_j \wedge b_j \in \Lambda^2 H$$

where $a_1, \ldots, a_g, b_1, \ldots, b_g$ is a symplectic basis of $H$. The inclusion

$$H \overset{\wedge \theta}{\longrightarrow} \Lambda^3 H \cong H_3(\text{Jac } C).$$

induces an inclusion

$$\text{Jac } C = J(H) \hookrightarrow J(\Lambda^3 H).$$

The primitive part of $H_3(\text{Jac } C)$ is the quotient

$$\Lambda^3_0 H = (\Lambda^3 H)/(\theta \cdot H).$$

Its intermediate jacobian is

$$J(\Lambda^3_0 H) = J(\Lambda^3 H)/\text{Jac } C.$$
Proposition (Pulte)

If $x, y \in C$, then

$$\nu_{C,x} - \nu_{C,y} = \text{the image of } 2([x] - [y]) \in \text{Jac } C \subset J(\Lambda^3 H)$$

Consequently, the image of $\nu_{C,x}$ in $J(\Lambda^3_0 H)$ does not depend on $x \in C$. It vanishes when $C$ is hyperelliptic.

**Notation:** Denote the image of $\nu_{C,x}$ in $J(\Lambda^3_0 H)$ by $\nu_C$.

One can study $\nu_C$ by letting $C$ vary and use variational methods.
Families of homologically trivial cycles

Suppose that $f: X \to S$ is a smooth projective morphism and that $Z$ an algebraic cycle on $X$ whose restriction to each fiber is homologically trivial and has codimension $e$ and dimension $d$:

$$Z_s \leftarrow Z \rightarrow X$$

$$\downarrow \quad \downarrow \quad \downarrow^f$$

$$\{s\} \rightarrow S \rightarrow S$$

Set

$$\mathbb{V} = R^{2e-1}f_*\mathbb{Z}_X(e)$$

This has fiber $H_{2d+1}(X_s)(-d)$ over $s \in S$ and weight $-1$. We have the family

$$J(\mathbb{V}) \to S$$

of intermediate jacobians.
The normal function of a family of cycles

These data give rise to an extension over $S$ of VMHS

$$0 \rightarrow \mathcal{V} \rightarrow \mathcal{E} \rightarrow \mathbb{Z}_S \rightarrow 0$$

(†)

It corresponds to the section

$$\nu_Z : s \mapsto \nu_{Z_s} \in J(V_s)$$

of $J(\mathcal{V})$. It is holomorphic and satisfies Griffiths infinitesimal period relation:

$$\nabla \tilde{\nu} \in F^{-1}\mathcal{V} \otimes \Omega^1_S$$

where $(\mathcal{V}, \nabla)$ is the associated flat bundle $\mathcal{V} \otimes \mathcal{O}_S$ and $\tilde{\nu}$ is a local lift of $\nu$ to a section of $\mathcal{V}$.

It also satisfies strong conditions “at infinity” which correspond to the existence of a limit MHS on $\mathcal{E}$ at points of $\overline{S} - S$. 
The relative Ceresa cycle: setup

Here $S = \mathcal{M}_{g,1}$, the moduli space of smooth pointed genus curves $(C, x)$, and $X$ is the universal jacobian $\mathcal{J}$ over it. Let

$\mathcal{C} \xrightarrow{\mu_x} \mathcal{J}$

be the universal curve over $\mathcal{M}_{g,1}$ with tautological section $x$. We have the diagram

$\xymatrix{ \mathcal{C} \ar[rr]^{\mu_x} \ar@{->}[d]_{f} & & \mathcal{J} \ar@{<->}[d] \ar@{<->}[l]_{\sigma} \ar@{<->}[r]_{\iota} \ar@{<->}[d] & \mathcal{M}_{g,1} \ar@{<->}[d] \ar@{<->}[l] \ar@{<->}[r] & \mathcal{M}_{g,1} }$

where $\mu_x$ is the relative Abel–Jacobi map.
The relative Ceresa cycle and its normal function

Set
\[ \mathcal{H} = (R^1 f_* \mathcal{Z})^\vee \text{ and } \mathcal{V} = \Lambda^3_0 \mathcal{H}(-1). \]

The restriction of the algebraic cycle
\[ \mathcal{Z} = (\mu_x)_* \mathcal{C} - \iota_*(\mu_x)_* \mathcal{C} \subset \mathcal{J} \]
to the fiber \( \text{Jac} C \) of \( \mathcal{J} \) over \([C, x]\) is \( C_x - C_x^- \). It gives rise to the admissible normal function

\[ J(\Lambda^3_0 \mathcal{H}) \xrightarrow{\nu} \mathcal{M}_{g,1} \]

which descends to the *Ceresa* normal function

\[ J(\Lambda^3_0 \mathcal{H}) \xrightarrow{\nu} \mathcal{M}_g \]

It vanishes on the hyperelliptic locus and thus in genus 2.
Applications

Ceresa used it to prove:

**Theorem (Ceresa)**

*If* \( g \geq 3 \), *then the Ceresa cycle has infinite order mod algebraic equivalence for general* \([C]\) *in* \( \mathcal{M}_g \).

Nori used Ceresa’s result to prove:

**Theorem (Nori)**

*For the general abelian 3-fold* \( A \)

\[
\dim \left( \frac{\text{algebraic 1-cycles in } A}{\text{algebraic equivalence}} \right) \otimes \mathbb{Q} = \infty.
\]
The rank of a normal function

Suppose that $\nu$ is a normal function section of $J(\mathbb{V}) \to S$. The inclusion $\mathbb{V}_\mathbb{R} \hookrightarrow \mathbb{V}_\mathbb{C}$ induces a canonical isomorphism

$$\mathbb{V}_\mathbb{R}/\mathbb{V}_\mathbb{Z} =: J(\mathbb{V}_\mathbb{R}) \to J(\mathbb{V}).$$

So $\nu$ corresponds to a section $\nu_\mathbb{R}$ of $J(\mathbb{V}_\mathbb{R})$ over $S$. The bundle $J(\mathbb{V}_\mathbb{R})$ is a flat family of compact real tori. So locally $\nu_\mathbb{R}$ is a map

$$\tilde{\nu}_\mathbb{R} : U \to \mathbb{V}_\mathbb{R}, \quad \text{where } U \subset S \text{ is contractible.}$$

The rank of $\nu_\mathbb{R}$ at $s$ is defined to be the rank of $\nabla \tilde{\nu}_\mathbb{R}$ at $s$. Set

$$\text{rk } \nu = \frac{1}{2} \max_{s \in S} \text{rank } _s \nu_\mathbb{R}.$$ 

This is an integer. The rank of a torsion section is zero.
The main theorem

**Theorem**

*The rank of the normal function of the Ceresa cycle has maximal rank $3g - 3$ for all $g \geq 3$."

The theorem is false in genus 2 as, in that case, the Ceresa normal function is identically zero.

The proof is by induction. The base case is $g = 3$. I will discuss the proof there and sketch the inductive setup.

I understand that Ziyang Gao has also given a proof using Ax–Schanuel.
Zhang’s application

The Gross–Schoen cycle $\text{GS}_{C,\xi}$ of a curve $C$ and a point $\xi \in \text{Pic}^1 C$ is a homologically trivial algebraic 1-cycle in $C^3$. Its normal function is an integer multiple (6, I believe) of the Ceresa normal function of $(C, \xi)$. Below $\xi$ is a $(2g - 2)$nd root of $K_C$.

**Theorem (S.-W. Zhang)**

*For each $g \geq 3$, there is a non-empty Zariski open subset $U$ of $\mathcal{M}_g/\mathbb{Q}$ such that:*

- **Northcott property of Bloch–Beilinson height of the Gross–Schoen cycle:** for $H, D \in \mathbb{R}_+$

  \[ \# \{ [C] \in U(\overline{\mathbb{Q}}) : \text{deg}[C] < D \text{ and } \langle \text{GS}_{C,\xi}, \text{GS}_{C,\xi} \rangle_{BB} < H \} < \infty; \]

- **for all $[C] \in U(\mathbb{C}) - U(\overline{\mathbb{Q}})$, $\text{GS}_{C,\xi}$ has infinite order in $\text{CH}^2(C^3)$.
This is a summary of work of Griffiths, Green, Nori, with a few additions. Suppose that $\mathcal{V} \to S$ is a PVHS of weight $-1$. Let

$$\mathcal{V} = \mathcal{V} \otimes \mathcal{O}_S$$

and

$$\nabla : \mathcal{V} \to \mathcal{V} \otimes \Omega^1_S$$

be the associated flat bundle and its connection. It satisfies Griffiths transversality

$$\nabla : F^p \mathcal{V} \to F^p(\mathcal{V} \otimes \Omega^1_S) = F^{p-1} \mathcal{V} \otimes \Omega^1_S.$$ 

A basic tool for studying a normal function $\nu : S \to J(\mathcal{V})$ is the complex $\mathcal{V} \otimes \Omega^\bullet_S$ and its Hodge graded quotients

$$\text{Gr}^F_p(\mathcal{V} \otimes \Omega^\bullet_S) : 0 \to \text{Gr}^F_p \mathcal{V} \to \text{Gr}^{p-1}_F \mathcal{V} \otimes \Omega^1_S \to \text{Gr}^{p-2}_F \mathcal{V} \otimes \Omega^2_S \to$$

Its differential $\overline{\nabla}$ is $\mathcal{O}_S$-linear. So this is a complex of holomorphic vector bundles.
The Green–Griffiths infinitesimal invariant

- Locally a normal function $\nu : S \to J(V)$ can be lifted to a holomorphic section $\tilde{\nu}$ of $V$. It is well defined up to a section of $F^0 V$.

- The Griffiths infinitesimal invariant $\delta(\nu)$ of $\nu$ is the image of $\nabla \tilde{\nu}$ in

  $$H^0(S, \mathcal{H}^1(F^0(V \otimes \Omega^\bullet_S))).$$

- Green’s variant — the Green–Griffiths invariant — is its image $\overline{\delta}(\nu)$ in $H^0(S, \mathcal{H}^1(\text{Gr}_F^0(V \otimes \Omega^\bullet_S)))$. 
A canonical cocycle representative of $\overline{\delta}(\nu)$

- The $(1, 0)$ component of the derivative $\nabla \nu_R$ of a real lift of $\nu$ is an element $\nabla' \nu_R$ of
  \[
  H^0(S, \text{Gr}_F^{-1} \nu \otimes \Omega^1_S)
  \]
  that provides a canonical 1-cocycle that represents $\overline{\delta}(\nu)$.
- For each $s \in S$, it can be regarded as a $\mathbb{C}$ linear map
  \[
  \nabla' \nu_R : T_s S \to \text{Gr}_F^{-1} V_s
  \]
  This has rank equal to the rank of $\nu$ at $s$. 
Every non-hyperelliptic curve $C$ of genus 3 is a plane quartic via its canonical embedding

$$C \to \mathbb{P}(H^0(\Omega^1_C))^\vee.$$ 

Collino and Pirola showed that away from the hyperelliptic locus

$$\mathcal{H}^1(Gr^0_F(\mathcal{N} \otimes \Omega^\bullet))$$

is a vector bundle with fiber $S^4H^0(\Omega^1_C) \otimes \det H^0(\Omega^1_C)^\vee$ over $[C]$.

**Theorem (Collino–Pirola, 1995)**

*If $C$ is a non-hyperelliptic curve of genus 3, the Green–Griffiths invariant $\nabla \nu$ at $C$ is a defining equation of the canonical image of $C$.***
Linear algebra

- Suppose that \( C \) is a genus 3 curve. Set

\[
A = F^0 H_1(C) \quad \text{and} \quad B = \text{Gr}^{-1}_F H_1(C) = H^0(\Omega^1_C).
\]

The intersection pairing induces an isomorphism \( A \cong B^\vee \).

- If \( C \) is non-hyperelliptic, there are natural isomorphisms

\[
T^\vee_{[C]} M_3 \cong H^0(\Omega^2_C) \cong S^2 H^0(\Omega^2_C) \cong T^\vee_{[\text{Jac} C]} A_3 \cong S^2 B.
\]

- The fiber over \([ C ]\) of the complex \( \text{Gr}^0 F(\mathcal{V} \otimes \Omega^\bullet_{M_3}) \) is

\[
\begin{array}{c}
A \otimes \Lambda^2 B \\
\mathcal{B}
\end{array} \rightarrow \begin{array}{c}
\Lambda^2 A \otimes B \\
\mathcal{A}
\end{array} \otimes S^2 B \rightarrow \Lambda^3 A \otimes \Lambda^2 S^2 B.
\]

It is a complex of \( GL(B) \) modules.

- The differential \( \nabla \) is induced by \( B \rightarrow A \otimes S^2 B \) adjoint to \( B^\otimes 2 \rightarrow S^2 B \).
A little representation theory shows that the group of 1-cocycles is

\[ S^2 S^2 B \otimes \det A = S^4 B \otimes \det A + S^2 A \otimes \det B. \]

and the group of 1-coboundaries is \( S^2 A \otimes \det B \).

This gives the Collino–Pirola computation

\[ H^1(\text{Gr}_F^0(V \otimes \Omega^\bullet)) = S^4 B \otimes \det A. \]

The computation implies that, away from the hyperelliptic locus, \( \nabla' \nu_{\mathbb{R}} \) is a symmetric bilinear form

\[ S^2 A \otimes S^2 A \rightarrow \det A. \]

Its rank is the rank of \( \nu \) at \( C \).
The part coming from $S^4 B \otimes \det A$ is

$$S^2 A \otimes S^2 A \rightarrow \mathbb{C}, \quad u \otimes v \mapsto f(uv), \quad u, v \in S^2 A$$

where $f \in S^4 B$ is a quartic defining equation of $C$. This and its rank are easily computed from $f$.

**Proposition**

*If $C$ is the Klein quartic, then the coboundary component of $\nabla' \nu_\mathbb{R}$ vanishes and the other part has rank 6.*

**Corollary**

*The genus 3 Ceresa normal function has maximal rank on a dense open subset of $\mathcal{M}_3$.***
Conjecture

If $C$ is not hyperelliptic, then the component of $\nabla' \nu_R$ in $S^2 A \otimes \det B$ vanishes.

- I believe I have a proof. It uses recent work of Cléry, Faber and van der Geer. If this component were not zero, it would have to be a Teichmüller modular form of type $(0, 2, -1)$, of which there are none.

- The component of $\nabla' \nu_R$ with values in $S^4 B \otimes \det A$ is a non-zero multiple of the Teichmüller modular form $\chi_{4,0,-1}$ that plays a significant role in their paper. This should also yield a new proof of the Collino–Pirola theorem.

- If the conjecture is true, one can explicitly compute the rank of $\nu$ at all non-hyperelliptic curves.
References

- R. Hain: *The Rank of the normal function of the Ceresa cycle*, notes (February, 2024) and manuscript (in preparation, 2024).
- S.-W. Zhang: *A Northcott property for Gross–Schoen cycles and Ceresa cycles*, manuscript, April, 2024.