# The Rank of the Normal Function of the Ceresa Cycle

Richard Hain

Duke University

IMAG
Université de Montpellier
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# Introduction

- Motivated by a question of Shou-Wu Zhang — will state his applications later on.
- The Ceresa cycle is a motivic incarnation of the Johnson homomorphism

$$H_1(T_g) \to (\Lambda^3 H_1(C))/H_1(C)$$

- Two way traffic:

$$\text{mapping class groups } \leftrightarrow$$
$$\text{Arakelov and algebraic geometry of } \mathcal{M}_g.$$

# The Ceresa cycle

Throughout $C$ will be a smooth projective curve (over $\mathbb{C}$) of genus $g \geq 2$. For each $x \in C$ the Abel–Jacobi map

$$\alpha_x : C \to \operatorname{Jac} C$$

takes $y$ to the divisor class of $y - x$. Denote its image by $C_x$. This is an algebraic 1-cycle in $\operatorname{Jac} C$. Let $\iota$ be the involution $u \mapsto -u$ of $\operatorname{Jac} C$. Set

$$C_x^- = \iota_* C_x.$$

Since $\iota$ acts as $-1$ on $H^1(\operatorname{Jac} C)$, it acts as $(-1)^k$ on $H^k(\operatorname{Jac} C)$. So the algebraic 1-cycle

$$C_x - C_x^-$$

on $\operatorname{Jac} C$ is homologous to 0. This is the Ceresa cycle of $(C, x)$.

# Detecting homologically trivial cycles via Hodge theory

Suppose that $Z = \sum_j n_j Z_j$ is an algebraic $d$-cycle on a smooth projective variety $X$. When $[Z] = 0$ in $H_{2d}(X)$, there is an extension

$$0 \longrightarrow H_{2d+1}(X)(-d) \longrightarrow E_Z \longrightarrow \mathbb{Z}(0) \longrightarrow 0$$

Pull back the LES of $(X, |Z|)$ along $cl_Z : \mathbb{Z} \to H_{2d}(|Z|)$:

$$\begin{tikzcd}
0 \arrow[r] & H_{2d+1}(X) \arrow[r] & H_{2d+1}(X,|Z|) \arrow[r] & H_{2d}(|Z|) \arrow[r] & H_{2d}(X) \\
0 \arrow[r] & H_{2d+1}(X) \arrow[r] \arrow[u, equal] & E_Z(d) \arrow[r] \arrow[u, hook] & \mathbb{Z}(d) \arrow[r] \arrow[u, "cl_X"'] & 0 \arrow[u, dotted]
\end{tikzcd}$$

The extension $E_Z$ is generated by $H_{2d}(X)$ and $\Gamma$, where $\partial\Gamma = Z$.

# The group of 1-extensions

Suppose that $V$ is a Hodge structure of negative weight, then

$$\mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Z}, V) \cong J(V) := V_{\mathbb{C}}/(V_{\mathbb{Z}} + F^0 V).$$

If $V$ has weight $-1$, then $V_{\mathbb{C}} = F^0 V \oplus \overline{F^0 V}$, which implies that

$$V_{\mathbb{R}} \to V_{\mathbb{C}}/F^0 V$$

is an $\mathbb{R}$-linear isomorphism. It induces an isomorphism

$$J(V_{\mathbb{R}}) := V_{\mathbb{R}}/V_{\mathbb{Z}} \to J(V)$$

of tori. In particular, $J(V)$ is compact (but typically not algebraic).

# The Griffiths invariant

A homologically trivial $d$-cycle on $X$ thus determines a point[1]

$$\nu_Z \in J(H_{2d+1}(X)).$$

So the Ceresa cycle $C_x - C_x^-$ determines

$$\nu_{C,x} \in J(H_3(\mathrm{Jac}\, C)).$$

It is

$$\int_\Gamma \in \mathrm{Hom}(F^2 H^3(\mathrm{Jac}\, C), \mathbb{C})/H_3(\mathrm{Jac}\, C; \mathbb{Z}) \cong J(H_3(\mathrm{Jac}\, C))$$

where $\partial\Gamma = C_x - C_x^-$. This intermediate jacobian is *not* algebraic.

[1] From now on I will suppress the Tate twist — always twist so that the odd weight Hodge structure $V$ in $J(V)$ has weight $-1$.

# Eliminating the base point

Set $H = H_1(C)$ and

$$\theta = \sum_{j=1}^{g} a_j \wedge b_j \in \Lambda^2 H$$

where $a_1, \ldots, a_g, b_1, \ldots, b_g$ is a symplectic basis of $H$. The inclusion

$$H \xrightarrow{\wedge\theta} \Lambda^3 H \cong H_3(\operatorname{Jac} C).$$

induces an inclusion

$$\operatorname{Jac} C = J(H) \hookrightarrow J(\Lambda^3 H).$$

The primitive part of $H_3(\operatorname{Jac} C)$ is the quotient

$$\Lambda_0^3 H = (\Lambda^3 H)/(\theta \cdot H).$$

Its intermediate jacobian is

$$J(\Lambda_0^3 H) = J(\Lambda^3 H)/\operatorname{Jac} C.$$

### Proposition (Pulte)

*If $x, y \in C$, then*

$$\nu_{C,x} - \nu_{C,y} = \textit{the image of } 2([x] - [y]) \in \operatorname{Jac} C \subset J(\Lambda^3 H)$$

*Consequently, the image of $\nu_{C,x}$ in $J(\Lambda^3_0 H)$ does not depend on $x \in C$. It vanishes when $C$ is hyperelliptic.*

**Notation:** Denote the image of $\nu_{C,x}$ in $J(\Lambda^3_0 H)$ by $\nu_C$.

One can study $\nu_C$ by letting $C$ vary and use variational methods.

# Families of homologically trivial cycles

Suppose that $f : X \to S$ is a smooth projective morphism and that $Z$ an algebraic cycle on $X$ whose restriction to each fiber is homologically trivial and has codimension $e$ and dimension $d$:

$$\begin{tikzcd} Z_s \arrow[r, hook] \arrow[d] & Z \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \{s\} \arrow[r] & S \arrow[r, equal] & S \end{tikzcd}$$

Set

$$\mathbb{V} = R^{2e-1} f_* \mathbb{Z}_X(e)$$

This has fiber $H_{2d+1}(X_s)(-d)$ over $s \in S$ and weight $-1$. We have the family

$$J(\mathbb{V}) \to S$$

of intermediate jacobians.

# The normal function of a family of cycles

These data give rise to an extension over $S$ of VMHS

$$0 \to \mathbb{V} \to \mathbb{E} \to \mathbb{Z}_S \to 0 \qquad (\dagger)$$

It corresponds to the section

$$\nu_Z : s \mapsto \nu_{Z_s} \in J(V_s)$$

of $J(\mathbb{V})$. It is holomorphic and satisfies Griffiths infinitesimal period relation:

$$\nabla\tilde{\nu} \in F^{-1}\mathcal{V} \otimes \Omega^1_S$$

where $(\mathcal{V}, \nabla)$ is the associated flat bundle $\mathbb{V} \otimes \mathcal{O}_S$ and $\tilde{\nu}$ is a local lift of $\nu$ to a section of $\mathcal{V}$.

It also satisfies strong conditions “at infinity” which correspond to the existence of a limit MHS on $\mathbb{E}$ at points of $\overline{S} - S$.

# The relative Ceresa cycle: setup

Here $S = \mathcal{M}_{g,1}$, the moduli space of smooth pointed genus curves $(C, x)$, and $X$ is the universal jacobian $\mathcal{J}$ over it. Let

$$\begin{tikzcd} \mathscr{C} \arrow[r, "f"'] & \mathcal{M}_{g,1} \arrow[l, bend right, "x"'] \end{tikzcd}$$

be the universal curve over $\mathcal{M}_{g,1}$ with tautological section $x$. We have the diagram

$$\begin{tikzcd} \mathscr{C} \arrow[r, "\mu_x"] \arrow[d, shift left, "f"] & \mathcal{J} \arrow[d] \arrow[loop right, "\iota"] \\ \mathcal{M}_{g,1} \arrow[u, bend left, "x"] \arrow[r, equal] & \mathcal{M}_{g,1} \end{tikzcd}$$

where $\mu_x$ is the relative Abel–Jacobi map.

# The relative Ceresa cycle and its normal function

Set

$$\mathbb{H} = (R^1 f_* \mathbb{Z})^\vee \text{ and } \mathbb{V} = \Lambda^3_0 \mathbb{H}(-1).$$

The restriction of the algebraic cycle

$$Z = (\mu_x)_* \mathscr{C} - \iota_* (\mu_x)_* \mathscr{C} \subset \mathcal{J}$$

to the fiber $\mathrm{Jac}\, C$ of $\mathcal{J}$ over $[C, x]$ is $C_x - C_x^-$. It gives rise to the admissible normal function

$$\begin{tikzcd} J(\Lambda^3 \mathbb{H}) \arrow[r] & \mathcal{M}_{g,1} \arrow[l, bend right, "\nu_x"'] \end{tikzcd}$$

which descends to the *Ceresa* normal function

$$\begin{tikzcd} J(\Lambda^3_0 \mathbb{H}) \arrow[r] & \mathcal{M}_{g} \arrow[l, bend right, "\nu"'] \end{tikzcd}$$

**It vanishes on the hyperelliptic locus and thus in genus 2.**

# Applications

Ceresa used it to prove:

## Theorem (Ceresa)

*If $g \geq 3$, then the Ceresa cycle has infinite order mod algebraic equivalence for general $[C]$ in $\mathcal{M}_g$.*

Nori used Ceresa's result to prove:

## Theorem (Nori)

*For the general abelian 3-fold A*

$$\dim\left(\frac{\textit{algebraic 1-cycles in A}}{\textit{algebraic equivalence}}\right) \otimes \mathbb{Q} = \infty.$$

# The rank of a normal function

Suppose that $\nu$ is a normal function section of $J(\mathbb{V}) \to S$. The inclusion $\mathbb{V}_\mathbb{R} \hookrightarrow \mathbb{V}_\mathbb{C}$ induces a canonical isomorphism

$$\mathbb{V}_\mathbb{R}/\mathbb{V}_\mathbb{Z} =: J(\mathbb{V}_\mathbb{R}) \to J(\mathbb{V}).$$

So $\nu$ corresponds to a section $\nu_\mathbb{R}$ of $J(\mathbb{V}_\mathbb{R})$ over $S$. The bundle $J(\mathbb{V}_\mathbb{R})$ is a flat family of compact real tori. So locally $\nu_\mathbb{R}$ is a map

$$\tilde{\nu}_\mathbb{R} : U \to V_\mathbb{R}, \qquad \text{where } U \subset S \text{ is contractible.}$$

The *rank* of $\nu_\mathbb{R}$ at $s$ is defined to be the rank of $\nabla \tilde{\nu}_\mathbb{R}$ at $s$. Set

$$\operatorname{rk} \nu = \tfrac{1}{2} \max_{s \in S} \operatorname{rk}_s \nu_\mathbb{R}.$$

This is an integer. The rank of a torsion section is zero.

# The main theorem

**Theorem**

*The rank of the normal function of the Ceresa cycle has maximal rank $3g-3$ for all $g \geq 3$.*

The theorem is false in genus 2 as, in that case, the Ceresa normal function is identically zero.

The proof is by induction. The base case is $g=3$. I will discuss the proof there and sketch the inductive setup.

I understand that Ziyang Gao has also given a proof using Ax–Schanuel.

# Zhang's application

The Gross–Schoen cycle $\mathrm{GS}_{C,\xi}$ of a curve $C$ and a point $\xi \in \mathrm{Pic}^1 C$ is a homologically trivial algebraic 1-cycle in $C^3$. Its normal function is an integer multiple (6, I believe) of the Ceresa normal function of $(C, \xi)$. Below $\xi$ is a $(2g-2)$nd root of $K_C$.

## Theorem (S.-W. Zhang)

*For each $g \geq 3$, there is a non-empty Zariski open subset $U$ of $\mathcal{M}_g/\mathbb{Q}$ such that:*

- *Northcott property of Bloch–Beilinson height of the Gross–Schoen cycle: for $H, D \in \mathbb{R}_+$*

  $$\#\{[C] \in U(\overline{\mathbb{Q}}) : \deg[C] < D \text{ and } \langle \mathrm{GS}_{C,\xi}, \mathrm{GS}_{C,\xi} \rangle_{BB} < H\} < \infty;$$

- *for all $[C] \in U(\mathbb{C}) - U(\overline{\mathbb{Q}})$, $\mathrm{GS}_{C,\xi}$ has infinite order in $\mathrm{CH}^2(C^3)$.*

# Technical tools

This is a summary of work of Griffiths, Green, Nori, with a few additions. Suppose that $\mathbb{V} \to S$ is a PVHS of weight $-1$. Let

$$\mathcal{V} = \mathbb{V} \otimes \mathcal{O}_S \text{ and } \nabla : \mathcal{V} \to \mathcal{V} \otimes \Omega^1_S$$

be the associated flat bundle and its connection. It satisfies Griffiths transversality

$$\nabla : F^p\mathcal{V} \to F^p(\mathcal{V} \otimes \Omega^1_S) = F^{p-1}\mathcal{V} \otimes \Omega^1_S.$$

A basic tool for studying a normal function $\nu : S \to J(\mathbb{V})$ is the complex $\mathcal{V} \otimes \Omega^\bullet_S$ and its Hodge graded quotients

$$\mathrm{Gr}^p_F(\mathcal{V} \otimes \Omega^\bullet_S) : 0 \to \mathrm{Gr}^p_F\,\mathcal{V} \to \mathrm{Gr}^{p-1}_F\,\mathcal{V} \otimes \Omega^1_S \to \mathrm{Gr}^{p-2}_F\,\mathcal{V} \otimes \Omega^2_S \to$$

Its differential $\overline{\nabla}$ is $\mathcal{O}_S$-linear. So this is a complex of holomorphic vector bundles.

# The Green–Griffiths infinitesimal invariant

- Locally a normal function $\nu : S \to J(\mathbb{V})$ can be lifted to a holomorphic section $\tilde{\nu}$ of $\mathcal{V}$. It is well defined up to a section of $F^0\mathcal{V}$.
- The Griffiths infinitesimal invariant $\delta(\nu)$ of $\nu$ is the image of $\nabla\tilde{\nu}$ in
$$H^0(S, \mathcal{H}^1(F^0(\mathcal{V} \otimes \Omega_S^\bullet))).$$
- Green's variant — the *Green–Griffiths invariant* — is its image $\overline{\delta}(\nu)$ in $H^0(S, \mathcal{H}^1(\mathrm{Gr}_F^0(\mathcal{V} \otimes \Omega_S^\bullet)))$.

# A canonical cocycle representative of $\overline{\delta}(\nu)$

- The $(1,0)$ component of the derivative $\nabla\nu_{\mathbb{R}}$ of a *real* lift of $\nu$ is an element $\nabla'\nu_{\mathbb{R}}$ of

$$H^0(S, \mathrm{Gr}_F^{-1}\mathcal{V} \otimes \Omega_S^1)$$

  that provides a *canonical* 1-cocycle that represents $\overline{\delta}(\nu)$.
- For each $s \in S$, it can be regarded as a $\mathbb{C}$ linear map

$$\nabla'\nu_{\mathbb{R}} : T_sS \to \mathrm{Gr}_F^{-1} V_s$$

  This has rank equal to the rank of $\nu$ at $s$.

# Genus 3: introduction

Every non-hyperelliptic curve $C$ of genus 3 is a plane quartic via its canonical embedding

$$C \to \mathbb{P}(H^0(\Omega^1_C))^\vee.$$

Collino and Pirola showed that away from the hyperelliptic locus

$$\mathcal{H}^1(\mathrm{Gr}^0_F(\mathcal{V} \otimes \Omega^\bullet))$$

is a vector bundle with fiber $S^4 H^0(\Omega^1_C) \otimes \det H^0(\Omega^1_C)^\vee$ over $[C]$.

**Theorem (Collino–Pirola, 1995)**

*If $C$ is a non-hyperelliptic curve of genus 3, the Green–Griffiths invariant $\overline{\nabla}\nu$ at $C$ is a defining equation of the canonical image of $C$.*

# Linear algebra

- Suppose that $C$ is a genus 3 curve. Set
$$A = F^0 H_1(C) \text{ and } B = \mathrm{Gr}_F^{-1} H_1(C) = H^0(\Omega^1_C).$$
The intersection pairing induces an isomorphism $A \cong B^\vee$.
- If $C$ is non-hyperelliptic, there are natural isomorphisms
$$T^\vee_{[C]}\mathcal{M}_3 \cong H^0(\Omega_C^{\otimes 2}) \cong S^2 H^0(\Omega^2_C) \cong T^\vee_{[\mathrm{Jac}\, C]}\mathcal{A}_3 \cong S^2 B.$$
- The fiber over $[C]$ of the complex $\mathrm{Gr}_F^0(\mathcal{V} \otimes \Omega^\bullet_{\mathcal{M}_3})$ is
$$\frac{A \otimes \Lambda^2 B}{B} \to \frac{\Lambda^2 A \otimes B}{A} \otimes S^2 B \to \Lambda^3 A \otimes \Lambda^2 S^2 B.$$
It is a complex of $\mathrm{GL}(B)$ modules.
- The differential $\overline{\nabla}$ is induced by $B \to A \otimes S^2 B$ adjoint to $B^{\otimes 2} \to S^2 B$.

- A little representation theory shows that the group of 1-cocycles is

$$S^2 S^2 B \otimes \det A = S^4 B \otimes \det A + S^2 A \otimes \det B.$$

  and the group of 1-coboundaries is $S^2 A \otimes \det B$.
- This gives the Collino–Pirola computation $H^1(\mathrm{Gr}_F^0(V \otimes \Omega^\bullet)) = S^4 B \otimes \det A$.
- The computation implies that, away from the hyperelliptic locus, $\nabla' \nu_{\mathbb{R}}$ is a symmetric bilinear form

$$S^2 A \otimes S^2 A \to \det A.$$

  Its rank is the rank of $\nu$ at $C$.

- The part coming from $S^4 B \otimes \det A$ is

$$S^2 A \otimes S^2 A \to \mathbb{C}, \quad u \otimes v \mapsto f(uv), \quad u, v \in S^2 A$$

where $f \in S^4 B$ is a quartic defining equation of $C$. This and its rank are easily computed from $f$.

## Proposition

*If $C$ is the Klein quartic, then the coboundary component of $\nabla' \nu_{\mathbb{R}}$ vanishes and the other part has rank 6.*

## Corollary

*The genus 3 Ceresa normal function has maximal rank on a dense open subset of $\mathcal{M}_3$.*

## Conjecture

*If $C$ is not hyperelliptic, then the component of $\nabla' \nu_{\mathbb{R}}$ in $S^2 A \otimes \det B$ vanishes.*

- I believe I have a proof. It uses recent work of Cléry, Faber and van der Geer. If this component were not zero, it would have to be a Teichmüllar modular form of type $(0, 2, -1)$, of which there are none.
- The component of $\nabla' \nu_{\mathbb{R}}$ with values in $S^4 B \otimes \det A$ is a non-zero multiple of the Teichmüller modular form $\chi_{4,0,-1}$ that plays a significant role in their paper. This should also yield a new proof of the Collino–Pirola theorem.
- If the conjecture is true, one can explicitly compute the rank of $\nu$ at all non-hyperelliptic curves.

# References


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