The Rank of the Normal Function of the Ceresa Cycle

Richard Hain

Duke University

IMAG Université de Montpellier May 14, 2024

▲□▶ ▲□▶ ▲ 三▶ ▲ 三▶ - 三 - のへぐ

Introduction

- Motivated by a question of Shou-Wu Zhang will state his applications later on.
- The Ceresa cycle is a motivic incarnation of the Johnson homomorphism

$$H_1(T_g) \rightarrow (\Lambda^3 H_1(C))/H_1(C)$$

Two way traffic:

mapping class groups \leftrightarrow

Arakelov and algebraic geometry of \mathcal{M}_g .

< □ > < 同 > < Ξ > < Ξ > < Ξ > < Ξ < </p>

The Ceresa cycle

Throughout *C* will be a smooth projective curve (over \mathbb{C}) of genus $g \ge 2$. For each $x \in C$ the Abel–Jacobi map

$$\alpha_{\mathbf{X}}: \mathbf{C} \to \mathsf{Jac} \ \mathbf{C}$$

takes *y* to the divisor class of y - x. Denote its image by C_x . This is an algebraic 1-cycle in Jac *C*. Let ι be the involution $u \mapsto -u$ of Jac *C*. Set

$$C_{X}^{-}=\iota_{*}C_{X}.$$

Since ι acts as -1 on $H^1(\operatorname{Jac} C)$, it acts as $(-1)^k$ on $H^k(\operatorname{Jac} C)$. So the algebraic 1-cycle

$$C_x - C_x^-$$

on Jac C is homologous to 0. This is the Ceresa cycle of (C, x).

Detecting homologically trivial cycles via Hodge theory

Suppose that $Z = \sum_{j} n_{j}Z_{j}$ is an algebraic *d*-cycle on a smooth projective variety *X*. When [Z] = 0 in $H_{2d}(X)$, there is an extension

$$0 \rightarrow H_{2d+1}(X)(-d) \rightarrow E_Z \rightarrow \mathbb{Z}(0) \rightarrow 0$$

Pull back the LES of (X, |Z|) along $cl_Z : \mathbb{Z} \to H_{2d}(|Z|)$:

The extension E_Z is generated by $H_{2d}(X)$ and Γ , where $\partial \Gamma = Z$.

The group of 1-extensions

Suppose that V is a Hodge structure of negative weight, then

$$\operatorname{Ext}^1_{\operatorname{MHS}}({\mathbb Z},V)\cong J(V):=V_{\mathbb C}/(V_{\mathbb Z}+{\mathcal F}^0 V).$$

If V has weight -1, then $V_{\mathbb{C}} = F^0 V \oplus \overline{F^0 V}$, which implies that

$$V_{\mathbb{R}} o V_{\mathbb{C}}/F^0 V$$

is an \mathbb{R} -linear isomorphism. It induces an isomorphism

$$J(V_{\mathbb{R}}) := V_{\mathbb{R}}/V_{\mathbb{Z}} \to J(V)$$

of tori. In particular, J(V) is compact (but typically not algebraic).

The Griffiths invariant

A homologically trivial d-cycle on X thus determines a point¹

 $\nu_Z \in J(H_{2d+1}(X)).$

So the Ceresa cycle $C_x - C_x^-$ determines

 $\nu_{C,x} \in J(H_3(\operatorname{Jac} C)).$

lt is

 $\int_{\Gamma} \in \operatorname{Hom}(F^2H^3(\operatorname{Jac} C), \mathbb{C})/H_3(\operatorname{Jac} C; \mathbb{Z}) \cong J(H_3(\operatorname{Jac} C))$

where $\partial \Gamma = C_x - C_x^-$. This intermediate jacobian is *not* algebraic.

¹From now on I will suppress the Tate twist — always twist so that the odd weight Hodge structure V in J(V) has weight -1.

Eliminating the base point Set $H = H_1(C)$ and

$$heta = \sum_{j=1}^g a_j \wedge b_j \in \Lambda^2 H$$

where $a_1, \ldots, a_g, b_1, \ldots, b_g$ is a symplectic basis of *H*. The inclusion

$$H \xrightarrow{\wedge \theta} \Lambda^3 H \cong H_3(\operatorname{Jac} C).$$

induces an inclusion

$$\mathsf{Jac}\ C = J(H) \hookrightarrow J(\Lambda^3 H).$$

The primitive part of $H_3(\operatorname{Jac} C)$ is the quotient

$$\Lambda_0^3 H = (\Lambda^3 H) / (\theta \cdot H).$$

Its intermediate jacobian is

$$J(\Lambda_0^3 H) = J(\Lambda^3 H) / \operatorname{Jac} C.$$

Proposition (Pulte) If $x, y \in C$, then

 $u_{\mathcal{C},x} - \nu_{\mathcal{C},y} = \text{the image of } 2([x] - [y]) \in \operatorname{Jac} \mathcal{C} \subset J(\Lambda^3 H)$

Consequently, the image of $\nu_{C,x}$ in $J(\Lambda_0^3 H)$ does not depend on $x \in C$. It vanishes when C is hyperelliptic.

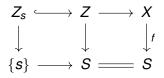
A D F A 同 F A E F A E F A Q A

Notation: Denote the image of $\nu_{C,x}$ in $J(\Lambda_0^3 H)$ by ν_C .

One can study ν_C by letting *C* vary and use variational methods.

Families of homologically trivial cycles

Suppose that $f: X \to S$ is a smooth projective morphism and that *Z* an algebraic cycle on *X* whose restriction to each fiber is homologically trivial and has codimension *e* and dimension *d*:



Set

$$\mathbb{V} = R^{2e-1} f_* \mathbb{Z}_X(e)$$

This has fiber $H_{2d+1}(X_s)(-d)$ over $s \in S$ and weight -1. We have the family

$$J(\mathbb{V}) o S$$

of intermediate jacobians.

The normal function of a family of cycles

These data give rise to an extension over S of VMHS

$$0 \to \mathbb{V} \to \mathbb{E} \to \mathbb{Z}_S \to 0$$
 (†)

It corresponds to the section

$$\nu_Z : \mathbf{s} \mapsto \nu_{Z_s} \in J(V_s)$$

of J(V). It is holomorphic and satisfies Griffiths infinitesimal period relation:

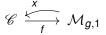
$$abla ilde{
u} \in F^{-1} \mathcal{V} \otimes \Omega^1_\mathcal{S}$$

where (\mathcal{V}, ∇) is the associated flat bundle $\mathbb{V} \otimes \mathcal{O}_S$ and $\tilde{\nu}$ is a local lift of ν to a section of \mathcal{V} .

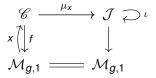
It also satisfies strong conditions "at infinity" which correspond to the existence of a limit MHS on \mathbb{E} at points of $\overline{S} - S$.

The relative Ceresa cycle: setup

Here $S = M_{g,1}$, the moduli space of smooth pointed genus curves (*C*, *x*), and *X* is the universal jacobian \mathcal{J} over it. Let



be the universal curve over $\mathcal{M}_{g,1}$ with tautological section *x*. We have the diagram



(ロ) (同) (三) (三) (三) (○) (○)

where μ_{x} is the relative Abel–Jacobi map.

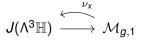
The relative Ceresa cycle and its normal function Set

$$\mathbb{H} = (\mathbf{R}^1 f_* \mathbb{Z})^{\vee} \text{ and } \mathbb{V} = \Lambda_0^3 \mathbb{H}(-1).$$

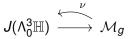
The restriction of the algebraic cycle

$$Z = (\mu_X)_* \mathscr{C} - \iota_*(\mu_X)_* \mathscr{C} \subset \mathcal{J}$$

to the fiber Jac *C* of \mathcal{J} over [C, x] is $C_x - C_x^-$. It gives rise to the admissible normal function



which descends to the Ceresa normal function



It vanishes on the hyperelliptic locus and thus in genus 2.

Applications

Ceresa used it to prove:

Theorem (Ceresa)

If $g \ge 3$, then the Ceresa cycle has infinite order mod algebraic equivalence for general [C] in \mathcal{M}_g .

Nori used Ceresa's result to prove:

Theorem (Nori)

For the general abelian 3-fold A

$$\dim\left(\frac{\textit{algebraic 1-cycles in }A}{\textit{algebraic equivalence}}\right)\otimes\mathbb{Q}=\infty.$$

< □ > < 同 > < Ξ > < Ξ > < Ξ > < Ξ < </p>

The rank of a normal function

Suppose that ν is a normal function section of $J(\mathbb{V}) \to S$. The inclusion $\mathbb{V}_{\mathbb{R}} \hookrightarrow \mathbb{V}_{\mathbb{C}}$ induces a canonical isomorphism

$$\mathbb{V}_{\mathbb{R}}/\mathbb{V}_{\mathbb{Z}} =: J(\mathbb{V}_{\mathbb{R}}) \to J(\mathbb{V}).$$

So ν corresponds to a section $\nu_{\mathbb{R}}$ of $J(\mathbb{V}_{\mathbb{R}})$ over *S*. The bundle $J(\mathbb{V}_{\mathbb{R}})$ is a flat family of compact real tori. So locally $\nu_{\mathbb{R}}$ is a map

 $ilde{
u}_{\mathbb{R}}: U o V_{\mathbb{R}}, \quad ext{where } U \subset S ext{ is contractible.}$

The *rank* of $\nu_{\mathbb{R}}$ at *s* is defined to be the rank of $\nabla \tilde{\nu}_{\mathbb{R}}$ at *s*. Set

$$\mathsf{rk}\,\nu = rac{1}{2}\,\mathsf{max}_{s\in S}\,\mathsf{rk}_{s}\,
u_{\mathbb{R}}.$$

This is an integer. The rank of a torsion section is zero.

Theorem

The rank of the normal function of the Ceresa cycle has maximal rank 3g - 3 for all $g \ge 3$.

The theorem is false in genus 2 as, in that case, the Ceresa normal function is identically zero.

The proof is by induction. The base case is g = 3. I will discuss the proof there and sketch the inductive setup.

(ロ) (同) (三) (三) (三) (○) (○)

I understand that Ziyang Gao has also given a proof using Ax–Schanuel.

Zhang's application

The Gross–Schoen cycle $GS_{C,\xi}$ of a curve *C* and a point $\xi \in Pic^1 C$ is a homologically trivial algebraic 1-cycle in C^3 . Its normal function is an integer multiple (6, I believe) of the Ceresa normal function of (C,ξ) . Below ξ is a (2g - 2)nd root of K_C .

Theorem (S.-W. Zhang)

For each $g \ge 3$, there is a non-empty Zariski open subset U of \mathcal{M}_g/\mathbb{Q} such that:

Northcott property of Bloch–Beilinson height of the Gross–Schoen cycle: for H, D ∈ ℝ₊

 $\#\{[C] \in U(\overline{\mathbb{Q}}) : \deg[C] < D \text{ and } \langle GS_{C,\xi}, GS_{C,\xi} \rangle_{BB} < H\} < \infty;$

▶ for all $[C] \in U(\mathbb{C}) - U(\overline{\mathbb{Q}})$, $GS_{C,\xi}$ has infinite order in $CH^2(C^3)$.

Technical tools

This is a summary of work of Griffiths, Green, Nori, with a few additions. Suppose that $\mathbb{V} \to S$ is a PVHS of weight -1. Let

$$\mathcal{V}=\mathbb{V}\otimes\mathcal{O}_{\mathcal{S}}$$
 and $abla:\mathcal{V} o\mathcal{V}\otimes\Omega^1_{\mathcal{S}}$

be the associated flat bundle and its connection. It satisfies Griffiths transversality

$$abla : F^{p}\mathcal{V} o F^{p}(\mathcal{V}\otimes \Omega^{1}_{\mathcal{S}}) = F^{p-1}\mathcal{V}\otimes \Omega^{1}_{\mathcal{S}}.$$

A basic tool for studying a normal function $\nu : S \to J(\mathbb{V})$ is the complex $\mathcal{V} \otimes \Omega_S^{\bullet}$ and its Hodge graded quotients

$$\mathrm{Gr}_F^p(\mathcal{V}\otimes\Omega^{ullet}_S):\mathbf{0}
ightarrow\mathrm{Gr}_F^p\mathcal{V}
ightarrow\mathrm{Gr}_F^{p-1}\mathcal{V}\otimes\Omega^1_S
ightarrow\mathrm{Gr}_F^{p-2}\mathcal{V}\otimes\Omega^2_S
ightarrow$$

Its differential $\overline{\nabla}$ is \mathcal{O}_{S} -linear. So this is a complex of holomorphic vector bundles.

The Green–Griffiths infinitesimal invariant

- Locally a normal function *ν* : S → J(V) can be lifted to a holomorphic section *ṽ* of *V*. It is well defined up to a section of *F*⁰*V*.
- The Griffiths infinitesimal invariant δ(ν) of ν is the image of ∇ν̃ in

$$H^0(\mathcal{S}, \mathcal{H}^1(\mathcal{F}^0(\mathcal{V}\otimes\Omega^{\bullet}_{\mathcal{S}}))).$$

(日) (日) (日) (日) (日) (日) (日)

Green's variant — the Green–Griffiths invariant — is its image δ(ν) in H⁰(S, H¹(Gr⁰_F(V ⊗ Ω[•]_S))).

A canonical cocycle representative of $\overline{\delta}(\nu)$

The (1,0) component of the derivative ∇ν_R of a *real* lift of ν is an element ∇'ν_R of

$$H^0(S, \operatorname{Gr}_F^{-1} \mathcal{V} \otimes \Omega^1_S)$$

that provides a *canonical* 1-cocycle that represents $\overline{\delta}(\nu)$.

For each $s \in S$, it can be regarded as a \mathbb{C} linear map

$$abla'
u_{\mathbb{R}}: T_s S o \operatorname{Gr}_F^{-1} V_s$$

< □ > < 同 > < Ξ > < Ξ > < Ξ > < Ξ < </p>

This has rank equal to the rank of ν at *s*.

Genus 3: introduction

Every non-hyperelliptic curve C of genus 3 is a plane quartic via its canonical embedding

 $C \to \mathbb{P}(H^0(\Omega^1_C))^{\vee}.$

Collino and Pirola showed that away from the hyperelliptic locus

 $\mathcal{H}^1(\mathrm{Gr}^0_F(\mathcal{V}\otimes\Omega^{\bullet}))$

is a vector bundle with fiber $S^4 H^0(\Omega^1_C) \otimes \det H^0(\Omega^1_C)^{\vee}$ over [C].

Theorem (Collino-Pirola, 1995)

If C is a non-hyperelliptic curve of genus 3, the Green–Griffiths invariant $\overline{\nabla}\nu$ at C is a defining equation of the canonical image of C.

Linear algebra

Suppose that *C* is a genus 3 curve. Set

$$A = F^0 H_1(C)$$
 and $B = \operatorname{Gr}_F^{-1} H_1(C) = H^0(\Omega_C^1)$.

The intersection pairing induces an isomorphism $A \cong B^{\vee}$. If *C* is non-hyperelliptic, there are natural isomorphisms

$$T^{\vee}_{[C]}\mathcal{M}_3\cong H^0(\Omega^{\otimes 2}_C)\cong S^2H^0(\Omega^2_C)\cong T^{\vee}_{[\operatorname{Jac} C]}\mathcal{A}_3\cong S^2B.$$

• The fiber over [C] of the complex $Gr_F^0(\mathcal{V} \otimes \Omega^{\bullet}_{\mathcal{M}_3})$ is

$$\frac{A\otimes \Lambda^2 B}{B} \to \frac{\Lambda^2 A\otimes B}{A}\otimes S^2 B \to \Lambda^3 A\otimes \Lambda^2 S^2 B.$$

(日) (日) (日) (日) (日) (日) (日)

It is a complex of GL(B) modules.

The differential T̄ is induced by B → A ⊗ S²B adjoint to B^{⊗2} → S²B. A little representation theory shows that the group of 1-cocycles is

 $S^2S^2B \otimes \det A = S^4B \otimes \det A + S^2A \otimes \det B.$

and the group of 1-coboundaries is $S^2A \otimes \det B$.

- This gives the Collino–Pirola computation $H^1(\operatorname{Gr}_F^0(V \otimes \Omega^{\bullet})) = S^4 B \otimes \det A.$
- The computation implies that, away from the hyperelliptic locus, ∇[']ν_ℝ is a symmetric bilinear form

$$S^2A \otimes S^2A \to \det A.$$

(日) (日) (日) (日) (日) (日) (日)

Its rank is the rank of ν at C.

• The part coming from $S^4B \otimes \det A$ is

 $S^2A \otimes S^2A \rightarrow \mathbb{C}, \quad u \otimes v \mapsto f(uv), \quad u, v \in S^2A$

where $f \in S^4B$ is a quartic defining equation of *C*. This and its rank are easily computed from *f*.

Proposition

If C is the Klein quartic, then the coboundary component of $\nabla' \nu_{\mathbb{R}}$ vanishes and the other part has rank 6.

Corollary

The genus 3 Ceresa normal function has maximal rank on a dense open subset of \mathcal{M}_3 .

(ロ) (同) (三) (三) (三) (○) (○)

Conjecture

If C is not hyperelliptic, then the component of $\nabla' \nu_{\mathbb{R}}$ in $S^2 A \otimes \det B$ vanishes.

- ► I believe I have a proof. It uses recent work of Cléry, Faber and van der Geer. If this component were not zero, it would have to be a Teichmüllar modular form of type (0, 2, -1), of which there are none.
- The component of ∇'ν_R with values in S⁴B ⊗ det A is a non-zero multiple of the Teichmüller modular form χ_{4,0,−1} that plays a significant role in their paper. This should also yield a new proof of the Collino–Pirola theorem.
- If the conjecture is true, one can explicitly compute the rank of ν at all non-hyperelliptic curves.

References

- F. Cléry, C. Faber, G. van der Geer: Concomitants of ternary quartics and vector-valued Siegel and Teichmüller modular forms of genus three. Selecta Math. 26 (2020).
- R. Hain: The Rank of the normal function of the Ceresa cycle, notes (February, 2024) and manuscript (in preparation, 2024).
- S.-W. Zhang: A Northcott property for Gross–Schoen cycles and Ceresa cycles, manuscript, April, 2024.

(ロ) (同) (三) (三) (三) (○) (○)