Johnson Homomorphisms, the Goldman–Turaev Lie bialgebra and GRT

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Dedicated to the memory of Steven Zucker

October 8, 2019

Geography

algebraic geometry: geometry of curves

arithmetic: motives unramified over $\mathbb{Z}$

$\mathcal{M}_{g,n}$

dynamics: moduli of abelian differentials Kontsevich–Zorich

topology: mapping class groups

Notation

Suppose that $\overline{S}$ is a compact oriented surface of genus g, a set P of n distinct points in $\overline{S} - \partial\overline{S}$ and r boundary circles. Set $S = \overline{S} - (P \cup \partial S)$. We will suppose that S is *hyperbolic*:

$$\chi(S) = -(2g - 2 + r + n) < 0.$$

The mapping class group of S is:

$$\Gamma_S := \pi_0 \operatorname{Diff}^+(\overline{S}, P \cup \partial\overline{S}).$$

Up to an inner automorphism, it depends only on (g, n, r). We'll denote it by $\Gamma_{g,n+\vec{r}}$.

Remark: we can (and will) replace boundary components by non-zero tangent vectors.

Notation: $\mathcal{M}_{g,n+\vec{r}}$ is the moduli space of smooth projective curves C of genus g with:

1. $n + r$ distinct, labelled points $P = \{x_1, \dots, x_n, y_1, \dots, y_r\}$, and
2. r non-zero tangent vectors $V = \{\vec{v}_1, \dots, \vec{v}_r\}$, where $\vec{v}_j \in T_{y_j} C$.

This is a smooth stack over $\mathbb{Z}$. But we'll consider it to be $/\mathbb{C}$ unless otherwise stated.

There is a natural isomorphism

$$\pi_1(\mathcal{M}_{g,n+\vec{r}}, [C; P, V]) \cong \pi_0 \operatorname{Diff}^+(C, P, V).$$

The Torelli Group

The Torelli group consists of those mapping classes that act trivially on $H_1(\overline{S})$:

$$T_{g,n+\vec{r}} = \ker\{\Gamma_{g,n+\vec{r}} \to \mathrm{Sp}(H_1(\overline{S}), \text{int pairing})\}$$

The classifying space BT_g of T_g is the homotopy fiber of the period map period map $\mathcal{M}_g \to \mathcal{A}_g$. It encodes the difference between the geometry of curves and their jacobians.

Problem: Understand Torelli groups.

Johnson homomorphisms

Suppose $(S,\vec{v})$ is a surface of type $(g,n+\vec{1})$. The *Johnson filtration*

$$T_{g,n+\vec{1}} = J^1 T_{g,n+\vec{1}} \supset J^2 T_{g,n+\vec{1}} \supset J^3 T_{g,n+\vec{1}} \supset \cdots$$

is defined by

$$J^k T_{g,n+\vec{1}} := \ker\left\{ T_{g,n+\vec{1}} \to \operatorname{Aut}\left(\pi_1(S,\vec{v})/L^{k+1}\pi_1(S,\vec{v})\right)\right\},$$

where $L^\bullet$ is the lower central series (LCS).

The *higher Johnson homomorphisms* are the injective maps

$$\tau_k : \operatorname{Gr}^k_J T_{g,n+\vec{1}} \hookrightarrow \operatorname{Hom}_{\mathbb{Z}}\left(H_1(\overline{S}), \operatorname{Gr}^{k+1}_L \pi_1(\overline{S},\vec{v})\right).$$

Johnson: when $g \geq 3$, (and $n = 0$)

$$\tau_1 : H_1(T_{g,\vec{1}};\mathbb{Z}) \to J^1/J^2 \cong \Lambda^3 H_1(\overline{S},\mathbb{Z})$$

is surjective with finite kernel of exponent 2.

Tannakian categories

Briefly: a (neutral) tannakian category $\mathscr{T}$ is an abelian category that has the formal properties of the category of representations of a group (discrete, algebraic, Lie, ...) in finite dimensional vector spaces over a field F of char 0. Essential features: have

- tensor products and duals,
- trivial object 1_F; natural isomorphism $V \otimes 1_F \cong V$
- dual objects $V^\vee = \mathrm{Hom}(V, 1_F)$
- a *fiber functor* — a faithful tensor functor $\omega : \mathscr{T} \to \mathrm{Vec}_F$

Tannaka duality: if $\mathscr{T}$ is a neutral F-linear tannakian category,

1. $\pi_1(\mathscr{T}, \omega) := \mathrm{Aut}^\otimes \omega$ is an affine F-group
2. if V is in $\mathscr{T}$, then $\omega(V)$ is naturally a $\pi_1(\mathscr{T}, \omega)$-module
3. $\mathscr{T}$ is equivalent to the category of $\pi_1(\mathscr{T}, \omega)$-modules.

Relative unipotent completion

Input:

1. a discrete group Γ
2. a field F of characteristic 0
3. a reductive group R/F
4. a homomorphism $\rho : \Gamma \to R(F)$, Zariski dense

Let $\mathscr{R}(\Gamma, \rho)$ be the category of Γ-modules V (finite dim/F) that admit a filtration

$$0 = V_0 \subset V_1 \subset \ldots V_n = V$$

by Γ-submodules such that each V_j/V_{j-1} is an R-module and Γ acts on it via $\Gamma \to R(F) \to \mathrm{Aut}(V_j/V_{j-1})$. This is tannakian.

The *completion of* Γ *relative to* ρ is $\mathcal{G}^{\mathrm{rel}} := \pi_1(\mathscr{R}(\Gamma, \rho), \omega)$.

Properties of Relative Completion

1. Reduces to unipotent completion when R is trivial.
2. It is an extension $1 \to \mathcal{U}^{\mathrm{rel}} \to \mathcal{G}^{\mathrm{rel}} \to R \to 1$, where $\mathcal{U}^{\mathrm{rel}}$ is prounipotent.
3. This extension splits (Levi), so $\mathcal{G}^{\mathrm{rel}} \cong R \ltimes \mathcal{U}^{\mathrm{rel}}$.
4. $H^1(\mathfrak{u}^{\mathrm{rel}}) \cong \bigoplus_{\alpha \in \check{R}} H^1(\Gamma, V_\alpha) \otimes V_\alpha^\vee$

Theorem

1. *The relative completion $\mathcal{G}_C^{\mathrm{rel}}$ of $\pi_1(\mathcal{M}_{g,n+\vec{r}}, [C])$ has a natural $\mathbb{Q}$-MHS. That is, its coordinate ring $\mathcal{O}(\mathcal{G}_C^{\mathrm{rel}})$ and Lie algebra $\mathfrak{g}_C$ are Hopf algebra and Lie algebra in the category (ind or pro) MHS.*
2. *If C is defined over a number field K, then $\mathrm{Gal}(\overline{\mathbb{Q}}/K)$ acts on $\mathcal{O}(\mathcal{G}_C^{\mathrm{rel}}) \otimes \mathbb{Q}_\ell$ and $\mathfrak{g}_C \otimes \mathbb{Q}_\ell$.*

Relative completions of mapping class groups

Take S a surface of type $(g, n + \vec{r})$. Take $F = \mathbb{Q}$ and ρ to be the natural action $\Gamma_S \to \mathrm{Sp}(H_1(\overline{S}))$. Denote the relative completion of Γ_S with respect to ρ by $\mathcal{G}_S^{\mathrm{rel}}$ (or $\mathcal{G}_{g,n+\vec{r}}$ and its prounipotent radical by $\mathcal{U}_S^{\mathrm{rel}} = \mathcal{U}_{g,n+\vec{r}}$. Denote their Lie algebras by $\mathfrak{g}_S^{\mathrm{rel}}$ and $\mathfrak{u}_S^{\mathrm{rel}}$.

1. When S has genus 0, $\mathcal{G}_S = \mathcal{U}_S^{\mathrm{rel}}$ is just unipotent completion. (Well understood.) When $g \geq 3$, there is a non-trivial extension
$$0 \to \mathbb{G}_a \to T_S^{\mathrm{un}} \to \mathcal{U}_S^{\mathrm{rel}} \to 1.$$
2. $\mathfrak{u}_S^{\mathrm{rel}}$ is finitely generated except in genus 1. (Generating set known for all $g \geq 0$)
3. (Finite) presentations of the $\mathfrak{u}_S$ are known for all $g \neq 2$. A partial presentation is known in genus 2 — Watanabe using Petersen.

The geometric Johnson homomorphisms

Suppose that $(S, \vec{v})$ is a surface of type $(g, n + \vec{1})$. Set

$$\mathfrak{p}(S, \vec{v}) = \text{ Lie algebra of } \pi_1^{\mathrm{un}}(S, \vec{v}).$$

This is an object of $\mathscr{R}(\Gamma_S, \rho)$. So $\mathcal{G}_{S,\vec{v}}$ acts on it. The *geometric Johnson homomorphism* is the induced Lie algebra homomorphism

$$\phi_S : \mathfrak{g}_{S,\vec{v}} \to \operatorname{Der} \mathfrak{p}(S, \vec{v}).$$

Problem: Is ϕ_S injective? What is its image ?

Hodge theory

The category MHS of (graded polarizable) $\mathbb{Q}$-mixed Hodge structures is tannakian. It is the category of representations of an affine $\mathbb{Q}$-group $\pi_1(\mathrm{MHS})$. This is an extension

$$1 \to \mathcal{U}^{\mathrm{MHS}} \to \pi_1(\mathrm{MHS}) \to \pi_1(\mathrm{MHS}^{\mathrm{ss}}) \to 1$$

where $\mathrm{MHS}^{\mathrm{ss}}$ is the full sub-category of semi-simple MHS. The group $\pi_1(\mathrm{MHS}^{\mathrm{ss}})$ is reductive and $\mathcal{U}^{\mathrm{MHS}}$ is prounipotent.

There is a canonical *central* cocharacter $\chi : \mathbb{G}_m \to \pi_1(\mathrm{MHS}^{\mathrm{ss}})$ that corresponds to the functor $\mathrm{MHS}^{\mathrm{ss}}$ to graded vector spaces. By Levi's Theorem, this lifts to a cocharacter

$$\tilde{\chi} : \mathbb{G}_m \to \pi_1(\mathrm{MHS}).$$

It is unique up to conjugation by an element of $\mathcal{U}^{\mathrm{MHS}}(\mathbb{Q})$.

Each MHS V has a canonical *weight filtration* by sub-MHS:

$$0 = W_a V \subseteq W_{a+1} V \subseteq \cdots \subseteq W_{b-1} V \subseteq W_b V = V$$

When $V_{\mathbb{Q}}$ is finite dimensional, each choice of the lift $\tilde{\chi}$ determines a splitting

$$V_{\mathbb{Q}} \cong \operatorname{Gr}^W_\bullet V_{\mathbb{Q}} := \bigoplus_{n\in\mathbb{Z}} \operatorname{Gr}^W_n V$$

of the weight filtration that is preserved by morphisms of MHS and is compatible with tensor products and duals.

For a pro-MHS V, the choice of $\tilde{\chi}$ determines an isomorphism

$$V_{\mathbb{Q}} \cong \widehat{\operatorname{Gr}}^W_\bullet V_{\mathbb{Q}} := \prod_{n\in\mathbb{Z}} \operatorname{Gr}^W_n V.$$

These splittings are natural but not canonical!

Sample presentations

Suppose that $(S,\vec{v})$ is of type $(g,\vec{1})$. Set $H = H_1(\overline{S})$. There are natural isomorphisms

$$\mathfrak{p}(S,\vec{v}) \cong \widehat{\mathrm{Gr}}^W_\bullet\, \mathfrak{p}(S,\vec{v}) \cong \mathbb{L}(H)^\wedge$$

and, when $g \geq 4$,

$$\mathfrak{u}_{S,\vec{v}} \cong \widehat{\mathrm{Gr}}^W_\bullet\, \mathfrak{u}_{S,\vec{v}} \cong \mathbb{L}(\Lambda^3 H)^\wedge/(\text{quad relns})$$

The relations are almost determined by

$$\mathbb{L}_2(\Lambda^3 H) = \Lambda^2\Lambda^3 H/(\mathit{relns}) \cong V_{[2,2]} + V_{[1,1]}.$$

Both graded Lie algebras and the map $\mathrm{Gr}^W_\bullet\, \mathfrak{u}_{S,\vec{v}} \to \mathrm{Der}\,\mathbb{L}(H)$ can be described in terms of planar trivalent graphs in the standard way.

For each choice of a complex structure on $(S, \vec{v})$, the geometric Johnson homomorphism

$$\phi_S : \mathfrak{g}_{S,\vec{v}} \to \operatorname{Der} \mathfrak{p}(S, \vec{v})$$

is a morphism of MHS. After choosing a lift $\tilde{\chi}$ of χ, we can replace it by its weight graded:

$$\begin{array}{ccc} \mathfrak{g}_{S,\vec{v}} & \xrightarrow{\phi_S} & \operatorname{Der} \mathfrak{p}(S, \vec{v}) \\ \downarrow \cong & & \downarrow \cong \\ \widehat{\operatorname{Gr}}^W_\bullet \mathfrak{g}_{S,\vec{v}} & \xrightarrow{\operatorname{Gr}^W_\bullet \phi_S} & \operatorname{Der} \widehat{\operatorname{Gr}}^W_\bullet \mathfrak{p}(S, \vec{v}) \end{array}$$

When $n = 0$, this recovers the Johnson homomorphism:

$$\begin{array}{ccc} (\operatorname{Gr}^n_J T_S) \otimes \mathbb{Q} & \xrightarrow{\tau_n} & (\operatorname{Gr}^n_J T_S) \otimes \mathbb{Q} \\ \downarrow \cong & & \downarrow \cong \\ \operatorname{Gr}^W_{-n} \overline{\mathfrak{g}}_{S,\vec{v}} & \xrightarrow{\operatorname{Gr}^W_{-n} \phi_S} & \operatorname{Hom}(H_1(S), \operatorname{Gr}^{n+1}_L \pi_1(S, \vec{v})) \otimes \mathbb{Q} \end{array}$$

where $\overline{\mathfrak{g}}_{S,\vec{v}}$ is the image of ϕ_S.

Arithmetic Johnson homomorphisms

Suppose that $(S, \vec{v})$ is a surface of type $(g, n + \vec{1})$. As above, $\overline{\mathfrak{g}}_{g,n+\vec{1}} \subset \operatorname{Der} \mathfrak{p}(S, \vec{v})$ is the image of the geometric Johnson homomorphism.

Let $\mathfrak{mhs}$ be the Lie algebra of $\pi_1(\mathrm{MHS})$. Fix a complex structure on $(S, \vec{v})$. This acts on $\mathfrak{p}(S, \vec{v})$ and $\mathfrak{g}_{S,\vec{v}}$. Since the geometric Johnson homomorphism is a morphism of MHS, it extends to a homomorphism

$$\mathfrak{mhs} \ltimes \mathfrak{g}_{S,\vec{v}} \to \operatorname{Der} \mathfrak{p}(S, \vec{v}).$$

Denote its image by

$$\widehat{\mathfrak{g}}_{S,\vec{v}} \subset \operatorname{Der} \mathfrak{p}(S, \vec{v}).$$

This inclusion is the *arithmetic Johnson homomorphism*.

The category MTM of mixed Tate motives unramified over $\mathbb{Z}$ is tannakian. Its fundamental group is an extension

$$1 \to \mathcal{K} \to \pi_1(\mathrm{MTM}) \to \mathbb{G}_m \to 1$$

where $\mathcal{K}$ is the prounipotent group with Lie algebra

$$\mathfrak{k} = \mathbb{L}(\sigma_3, \sigma_5, \sigma_7, \sigma_9, \dots)^\wedge.$$

Let $U = \mathbb{P}^1 - \{0, 1, \infty\}$ and $\vec{v} = \partial/\partial z \in T_1\mathbb{P}^1$.

Theorem

1. *(Deligne–Goncharov)* $\mathfrak{p}(U, \vec{v})$ *is a pro-object of* MTM
2. *(Brown) the action* $\mathfrak{k} \to \mathrm{Der}\,\mathfrak{p}(U, \vec{v})$ *is faithful*

The action of factors $\mathfrak{mhs} \twoheadrightarrow \mathfrak{k} \hookrightarrow \mathrm{Der}\,\mathfrak{p}(U)$

Theorem

If S is a hyperbolic surface of type $(g, n + \vec{1})$, then

1. $\widehat{\mathfrak{g}}_S$ *does not depend on the complex structure on S*
2. *there is a natural isomorphism* $\widehat{\mathfrak{g}}_S/\overline{\mathfrak{g}}_S \cong \mathfrak{k}$

The second part follows the proof of Oda's Conjecture by Takao after previous work by various combinations of Ihara, Matsumoto and Nakamura. It implies that $\widehat{\mathfrak{g}}_S/\overline{\mathfrak{g}}_S$ does not depend on g or n. The $(0, 2 + \vec{1})$ case follows from Brown's Theorem.

The Goldman–Turaev Lie bialgebra

This part is mainly based on work Alekseev, Kawazumi, Kuno and Naef (AKKN) and the subset KK.

Suppose that S is a hyperbolic surface. Set

$$\lambda(S) = \{\text{closed geodesics in } S\} = \text{ conj classes in } \pi_1(S,*).$$

For $R = \mathbb{Z}, \mathbb{Q}, \ldots$, set

$$R\lambda(S) = \text{ free } R\text{-module generated by } \lambda(S) = |R\pi_1(S,*)|$$

where $|A|$ denotes the *cyclic quotient*

$$|A| := A/\{uv - vu : u, v \in A\}$$

of the associative algebra A. The completion of $R\lambda(S)$ is

$$R\lambda(S)^\wedge := |R\pi_1(S,*)^\wedge| = \varprojlim_n |R\pi_1(S,*)/I^n|.$$

- $\mathbb{Z}\lambda(S)$ has a Lie bracket, the *Goldman bracket*

$$\{\ ,\ \} : \mathbb{Z}\lambda(S) \otimes \mathbb{Z}\lambda(S) \to \mathbb{Z}\lambda(S)$$

- and a cobracket: the *Turaev cobracket* [framing ξ]

$$\delta_\xi : \mathbb{Z}\lambda(S) \to \mathbb{Z}\lambda(S) \otimes \mathbb{Z}\lambda(S).$$

- compatibility:

$$\delta_\xi\{u, v\} = u \cdot \delta_\xi(v) = v \cdot \delta_\xi(u)$$

  where the dot denotes the natural adjoint action of $\mathbb{Z}\lambda(S)$ on $\mathbb{Z}\lambda(S) \otimes \mathbb{Z}\lambda(S)$.

Goldman bracket

The bracket of two conjugacy classes *a* and *b* is

$$\{a, b\} = \sum_{p} \epsilon_p(\alpha, \beta)\,[\alpha \#_p \beta]$$

where

1. α and β are transversely intersecting immersed circles representing *a* and *b*,
2. *p* is a point of intersection of α and β,
3. $\epsilon_p(\alpha, \beta) \in \{\pm 1\}$ is the local intersection number of α and β at *p*,
4. $[\alpha \#_p \beta]$ is the free homotopy class of the loop $\alpha \#_p \beta$ obtained by joining α and β at *p* by a simple surgery.

Turaev cobracket

The cobracket of $a \in \lambda(X)$ is

$$\delta_\xi(a) = \sum_p \epsilon_p (a'_p \otimes a''_p - a''_p \otimes a'_p),$$

where

1. α is an immersed circle representing a with transverse self intersections and trivial winding number with respect to ξ,
2. p is a double point of α,
3. α'_p and α''_p are the two loops obtained by doing simple surgery on α at p,
4. ϵ_p is the intersection number of the initial arcs of α'_p and α''_p at p,
5. a'_p is the homotopy class of α'_p and a''_p is the class of α''_p.

Kawazumi and Kuno showed that the Goldman bracket and Turaev cobracket are continuous in the I-adic topology and thus induce

$$\{\ ,\ \} : \mathbb{Q}\lambda(S)^\wedge \otimes \mathbb{Q}\lambda(S)^\wedge \to \mathbb{Q}\lambda(S)^\wedge$$

and

$$\delta_\xi : \mathbb{Q}\lambda(S)^\wedge \to \mathbb{Q}\lambda(S)^\wedge \mathbin{\widehat{\otimes}} \mathbb{Q}\lambda(S)^\wedge$$

So $\mathbb{Q}\lambda(S)^\wedge$ is a *completed Lie bialgebra*.

Theorem

If X is a smooth algebraic curve over $\mathbb{C}$, then

1. $\mathbb{Q}\lambda(X)^\wedge$ *has a natural (pro) MHS (with negative weights),*
2. *the (completed) Goldman bracket is a morphism of MHS of type* $(1,1)$*,*
3. *if ξ is a "quasi-algebraic framing" of X, then the completed cobracket δ_ξ is a morphism of MHS of type* $(1,1)$.

Except in genus 1, every framed surface (S,ξ) admits a complex structure where ξ is quasi-algebraic. (Kawazumi + H, perhaps well-known?)

Corollary

For every framed surface (S, ξ) that admits a complex structure with ξ quasi-algebraic[a], the completed Goldman–Turaev Lie bialgebra is isomorphic to its associated weight graded Lie bialgebra:

$$\mathbb{Q}\lambda(S)^{\wedge} \cong \prod_{m \leq 0} \mathrm{Gr}^W_m \mathbb{Q}\lambda(S).$$

[a] So everything, except in genus 1.

The bracket and cobracket both increase weights (which are negative) by 2.

Kawazumi–Kuno action

Kawazumi and Kuno generalize the Goldman bracket to define an "action"

$$\kappa_{\vec{\mathrm{v}}} : \mathbb{Z}\lambda(S) \to \operatorname{Der} \mathbb{Z}\pi_1(S, \vec{\mathrm{v}}).$$

It induces a map on completions. The completion is a morphism of MHS.

Theorem (Combination of results of KK and H)

For all surfaces of type $(g, n + \vec{1})$*, there is a Lie algebra homomorphism* $\widehat{\mathfrak{g}}_S \to \mathbb{Q}\lambda(S)^\wedge$ *such that*

$$\begin{array}{ccc} \widehat{\mathfrak{g}}_S & \xrightarrow{\text{Johnson}} & \operatorname{Der} \mathfrak{p}(S, \vec{\mathrm{v}}) \\ \downarrow & & \downarrow \\ \mathbb{Q}\lambda(S)^\wedge & \xrightarrow{\kappa_{\vec{\mathrm{v}}}} & \operatorname{Der} \mathbb{Q}\pi_1(S, \vec{\mathrm{v}})^\wedge \end{array}$$

commutes. All are morphisms of MHS when $(S, \vec{\mathrm{v}})$ *is algebraic.*

The kernel of the cobracket

Results of Kawazumi and Kuno imply that the image of $\mathfrak{g}_S$ under δ_ξ is H. More generally, we have:

Theorem (KK + AKKN + H)

The image of $\widehat{\mathfrak{g}}_S$ under

$$\widehat{\mathfrak{g}}_S \longrightarrow \mathbb{Q}\lambda(S)^\wedge \xrightarrow{\delta_\xi} \mathbb{Q}\lambda(S)^\wedge \,\widehat{\otimes}\, \mathbb{Q}\lambda(S)^\wedge$$

is isomorphic to

$$H_1(S) \oplus H_1(\mathfrak{k}) \cong H \oplus \prod_{n\geq 1} \mathbb{Q}(2n+1).$$

Conjecture: $\ker \delta_\xi \subseteq \widehat{\mathfrak{g}}$.

Special case: genus 0

Theorem (AKKN)

When $(S,\xi,\vec{v}) = \big(\mathbb{P}^1 - \{0,1,\infty\}, \partial/\partial z, (\partial/\partial z)_1\big)$*, there is a canonical Lie algebra homomorphism*

$$\mathrm{SDer}\,\mathfrak{p}(\mathbb{P}^1 - \{0,1,\infty\}) \hookrightarrow \mathbb{Q}\lambda(\mathbb{P}^1 - \{0,1,\infty\})^\wedge$$

whose composition with

$$\kappa_{\vec{v}} : \mathbb{Q}\lambda(\mathbb{P}^1 - \{0,1,\infty\})^\wedge \to \mathrm{Der}\,\mathfrak{p}(\mathbb{P}^1 - \{0,1,\infty\})$$

is the inclusion, and

$$\ker\{\mathfrak{grt} \to H_1(\mathfrak{k})\} \subseteq \ker\{\mathfrak{grt} \to H_1(\mathfrak{k})\} = \ker\delta_\xi \cap \mathrm{SDer}\,\mathfrak{p}(\mathbb{P}^1 - \{0,1,\infty\})$$

This is conjectured to be an equality.

Pants decompositions

Graphic: an indexed pair of pants.

- A pants decomposition of a hyperbolic surface S of type (g, n) gives a maximally degenerate stable curve S_0 of type (g, n) by collapsing each circle to a point.
- There is a universal family of smoothings of S_0 indexed by Δ^N, where $N = \dim \mathcal{M}_{g,n} = 3g - 3 + n$. Very roughly speaking, the parameters in Δ^N are $q_j = \ell_j e^{i\theta_j}$ where ℓ_j is the length of the jth circle and θ_j is the glueing angle (relative to the marking).

Indexed pants decompositions

Definition

An *indexed pants decomposition*[a] is a pants decomposition where all of the angles are 0 or π. They correspond to the first order smoothings of S_0 given by the tangent vectors

$$\vec{w} = \sum_{j=1}^{N} \pm\partial/\partial q_j \in T_{S_0}\overline{\mathcal{M}}_{g,n}.$$

[a] Called a *pillow decomposition* by Nakamura–Schneps.

You can think of the circles in the pants decompositions as having length 0 and each pair of pants a a “bordification” of the hyperbolic surface $\mathbb{P}^1 - \{0, 1, \infty\}$.

Ihara Curves

An *Ihara curve* $C_{\vec{w}}$ is the first order smoothing of a nodal curve that corresponds to an indexed pants decomposition. (This can be made precise using the generalization of Tate's elliptic curve over $\mathbb{Z}[[q]]$ to higher genus curves by Ihara and Nakamura.)

Theorem

If $(X,\vec{v},\dots)$ *is a decorated Ihara curve, then* $\overline{\mathfrak{g}}_X$, $\mathfrak{p}(X,\vec{v})$, $\mathbb{Q}\lambda(X)^\wedge$, $\operatorname{Der}\mathfrak{p}(X,\vec{v})$, *... are all pro-objects of* MTM.

This means that the action of $\pi_1(\text{MHS})$ on each factors through the quotient map $\pi_1(\text{MHS}) \to \pi_1(\text{MTM})$.

A category

Objects: The moduli points $\vec{w} \in T\overline{\mathcal{M}}_{g,n+\vec{r}}$ corresponding to decorated Ihara curves $C_{\vec{w}}$.

Morphisms: For $\vec{w}, \vec{w}' \in T\overline{\mathcal{M}}_{g,n}$ corresponding to Ihara curves, let

$$\mathrm{Hom}(\vec{w}, \vec{w}') = \mathcal{G}^{\mathrm{rel}}_{\mathcal{M}_{g,n}}(\vec{w}, \vec{w}')$$

the relative completion of the torsor of paths in $\mathcal{M}_{g,n}$ from $\vec{w}$ to $\vec{w}'$.

There are other morphisms that correspond, for example, to "the morphism" $\mathcal{M}_{g',n'+\vec{1}} \times \mathcal{M}_{g'',n''+\vec{1}} \to \mathcal{M}_{g'+g'',n'+n''}$.

Functors: to (say) Lie (bi)algebras:

$$\vec{w} \mapsto \operatorname{Der}\mathfrak{p}(C_{\vec{w}}, \vec{v}), \quad \vec{w} \mapsto \mathbb{Q}\lambda(C_{\vec{w}})^{\wedge}, \quad \vec{w} \mapsto \overline{\mathfrak{g}}_{C_{\vec{w}}}, \quad \vec{w} \mapsto \widehat{\mathfrak{g}}_{C_{\vec{w}}}.$$

Natural transformations

There are many:

$$\mathfrak{g}_{C_{\vec{w}}} \to \mathbb{Q}\lambda(C_{\vec{w}}) \to \operatorname{OutDer}\mathfrak{p}(C_{\vec{w}}).$$

Automorphisms of *all* of these natural transformations:

$\pi_1(\mathrm{MTM})$. Drinfeld's GRT is the automorphism group if we restrict to genus 0. Let

$$\mathrm{GRT}^{\mathrm{new}} = \text{ Automorphism group all genera}$$

Is $\mathrm{GRT} = \mathrm{GRT}^{\mathrm{new}}$?

Is $\mathrm{GRT}^{\mathrm{new}}$ a *proper* subgroup of GRT? Elements of $\mathrm{GRT}^{\mathrm{new}}$ satisfy at least one essentially new condition:

When S is an Ihara curve, it acts on $\overline{\mathfrak{g}}_S$ and the inclusion $\overline{\mathfrak{g}} \hookrightarrow \mathbb{Q}\lambda(S)^\wedge$ is $\mathrm{GRT}^{\mathrm{new}}$-equivariant.

I do not see why GRT should stabilize the image of $\overline{\mathfrak{g}}_S$ unless one can see that $\mathfrak{k} = \mathfrak{grt}^1$.

Opinion: GRT is very much a genus 0 gadget. I find it hard to imaging how it can see higher genus phenomena like the Johnson homomorphism. Perhaps it does, but I do not see how.

Parting comments

Today I have focused on bounding $\widehat{\mathfrak{g}}_S$ and the image of the Johnson homomorphism by the kernel of the cobracket. Another (and complementary) approach is to try to prove that the cohomology $H^\bullet(\mathfrak{u}_{g,n})$ is stably pure — that is, for each fixed n, if $g \gg 0$, then $H^m(\mathfrak{u}_{g,n}) = \mathrm{Gr}^W_m H^m(\mathfrak{u}_{g,n})$. One can then compute the stable values of $\mathrm{Gr}^W_\bullet \mathfrak{u}_{g,n}$ in the representation ring of Sp.[1] This approach might be used to disprove the injectivity of $\mathfrak{g}_{g,\vec{1}} \to \mathrm{Der}\,\mathfrak{p}(S, \vec{\mathrm{v}})$, if that indeed is the case. So far, there is very little evidence ($0 \le m \le 7$), most of it due to Morita, Sakasai and Suzuki.

This is explained in my Johnson homomorphism survey.

[1] Garoufalidis–Getzler (with help from Petersen + Randal-Williams) have computed the pure part stably, which is a quadratic algebra.

Footnotes

Interlocutors

- Francis Brown (ongoing, e.g, modular inverter)
- Alekseev, Kawazumi, Kuno, and Naef: lots of help with understanding their work

Survey of Johnson homomorphisms: [arXiv:1909.03914]. It describes the cohomological version of computing the stable value of $\mathrm{Gr}^W_{-m}\,\mathfrak{u}_{g,*}$ in the representation ring of $\mathrm{Sp}_g(H)$, where $* \in \{0, 1, \vec{1}\}$. This will give an upper bound on the stable value of $\mathrm{Gr}^W_\bullet\,\overline{\mathfrak{g}}_{g,*}$ and is conjecturally equal to it. This will require proving that $H^\bullet(\mathfrak{u}_g)$ is *stably pure*. Equivalently,

$$H^1(\mathfrak{u}_g)^{\otimes m} \to H^m(\mathfrak{u}_g)$$

is surjective when $g \gg 0$.

References

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