Hodge Correlators, the Goldman–Turaev Lie Bialgebra and Johnson Homomorphisms

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Les Diablerets
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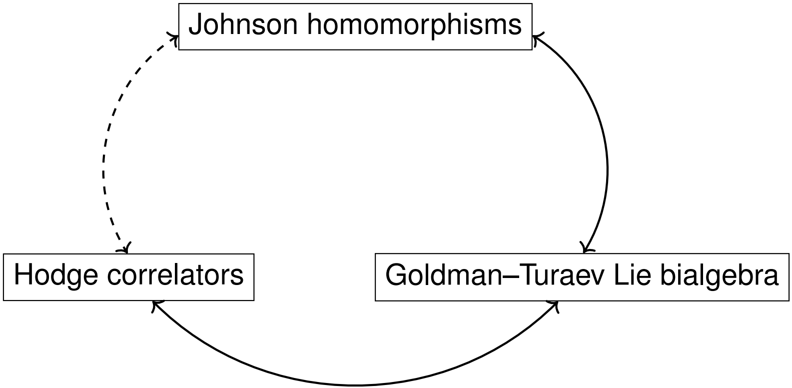

Johnson homomorphisms
Hodge correlators
Goldman–Turaev Lie bialgebra

Overview

- A central problem is to determine the image of (geometric) Johnson homomorphisms.
- More generally, and perhaps more naturally, we want to bound the image *arithmetic* Johnson homomorphisms.
- Over $\mathbb{R}$, this is what is generated by the image of the geometric Johnson homomorphism and also the image of the Lie algebra of the real “Mumford–Tate group”.
- Goncharov’s Hodge correlators provide a method of computing the image of the real MT Lie algebra.
- The Goldman–Turaev Lie bialgebra plays a central (if somewhat hidden) role in both stories.
- If you do not understand any of this, don’t worry — all will be explained!

Outline

I: The Goldman–Turaev Lie Bialgebra

II: Johnson Homomorphisms

III: Goncharov's Hodge Correlators

Initial setting

- For a topological space X, define $\lambda(X) = [S^1, X]$.
- When X is path connected (as it will be from now on)

$$\lambda(X) = \text{ conjugacy classes in } \pi_1(X, x).$$

- For a commutative ring $\mathbb{k}$ (for us $\mathbb{Z}$ or a field of char 0) set

$$\mathbb{k}\lambda(X) = \text{free } \mathbb{k}\text{-module generated by } \lambda(X).$$

- There is an inclusion $\mathbb{k} \to \mathbb{k}\lambda(X)$ that takes 1 to the boundary of a disk and a projection $\mathbb{k}\lambda(X) \to \mathbb{k}$ that takes each loop to 1. This gives a natural decomposition

$$\mathbb{k}\lambda(X) = \mathbb{k} \oplus I_{\mathbb{k}}\lambda(X)$$

- The *cyclic quotient* of an associative $\mathbb{k}$-algebra A is

$$\mathscr{C}(A) = A/\operatorname{span}\{uv - vu : u, v \in A\}.$$

- For example the cyclic quotient of the free associative algebra $\mathbb{k}\langle x : x \in \mathscr{X}\rangle$ is spanned by the "cyclic words" in the elements x of the alphabet $\mathscr{X}$:

$$x_1 x_2 \ldots x_m \sim x_2 \ldots x_m x_1 .$$

- We have $\mathbb{k}\lambda(X) = \mathscr{C}(\mathbb{k}\pi_1(X, x))$.

The Goldman–Turaev Lie bialgebra

The *Goldman bracket* is a map

$$\{\ ,\ \} : \Bbbk\lambda(X) \otimes \Bbbk\lambda(X) \to \Bbbk\lambda(X)$$

that makes $\Bbbk\lambda(X)$ into a Lie algebra. The *Turaev cobracket* is a map

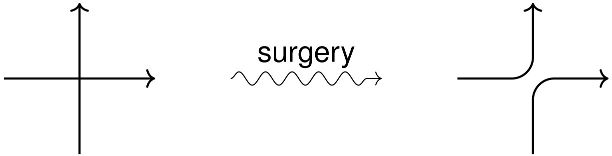
$$\delta_\xi : \Bbbk\lambda(X) \to \Bbbk\lambda(X) \otimes \Bbbk\lambda(X)$$

that depends on a framing ξ (a nowhere vanishing vector field) on X. Together they form a *Lie bialgebra*:

$$\delta_\xi\{u, v\} = u \cdot \delta_\xi(v) - v \cdot \delta_\xi(u)$$

where $w \cdot (x \otimes y) = \{w, x\} \otimes y + x \otimes \{w, y\}$.

The bracket and cobracket are defined using elementary surgery: Each element of $\lambda(X)$ can be represented by an immersed circle with simple normal crossings. (So no triple points, etc). One can perform surgery at a double point:



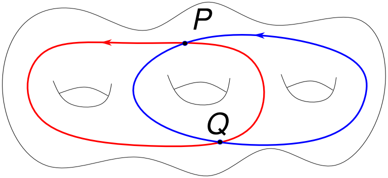
Goldman bracket

To define the Goldman bracket of $\alpha, \beta \in \lambda(X)$, represent them by oriented, transversally intersecting, immersed circles. Their Goldman bracket is

$$\{\alpha, \beta\} = \sum_P \epsilon_P \, \alpha \#_P \beta$$

where P ranges over the points where α intersects β, $\epsilon_P = \pm 1$ is the local intersection number at P and $\alpha \#_P \beta$ is the loop obtained by simple surgery at P.

An example

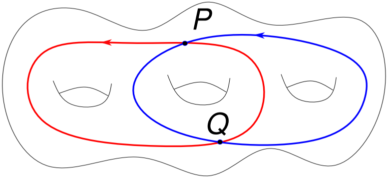


$\epsilon_P = 1$

$\epsilon_Q = -1$

$$\{\alpha, \beta\} = \epsilon_P \, \alpha \#_P \beta + \epsilon_Q \, \alpha \#_Q \beta$$

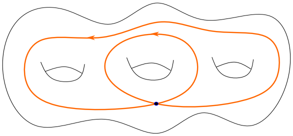
An example



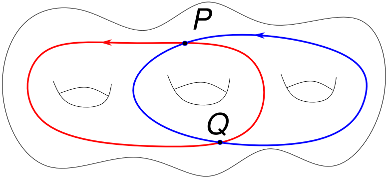
$\epsilon_P = 1$

$\epsilon_Q = -1$

$\alpha \#_P \beta$



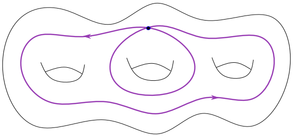
An example



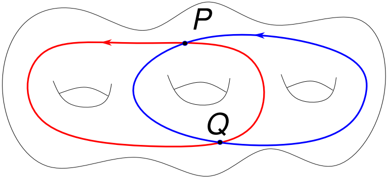
$\epsilon_P = 1$

$\epsilon_Q = -1$

$\alpha \#_Q \beta$



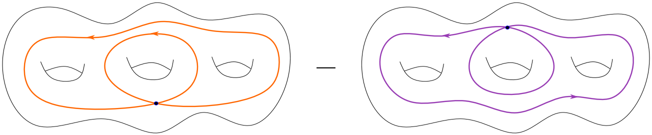
An example



$\epsilon_P = 1$

$\epsilon_Q = -1$

$$\{\alpha, \beta\} = \epsilon_P\, \alpha \#_P \beta + \epsilon_Q\, \alpha \#_Q \beta = \alpha \#_P \beta - \alpha \#_Q \beta$$

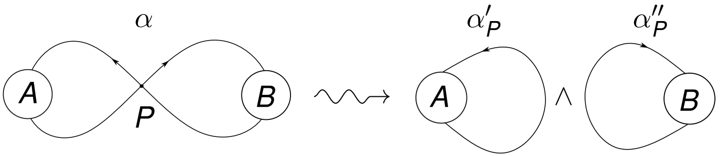


The Turaev cobracket

For convenience, we denote the element $v \otimes w - w \otimes v$ of $V^{\otimes 2}$ by $v \wedge w$. Suppose that α is an immersed circle with simple normal crossings. The first step in defining the cobracket is to define for each double point P of α

$$\delta_P(\alpha) = \alpha'_P \wedge \alpha''_P$$

where



To define $\delta_\xi(\alpha)$ represent α by an immersed loop with simple normal crossings **and trivial winding number** with respect to the framing:

$$\mathrm{rot}_\xi\, \alpha = 0.$$

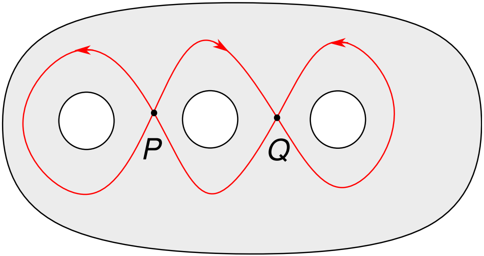
(Add some “backflips” as necessary.) The cobracket is defined by

$$\delta_\xi(\alpha) = \sum_{\text{double points } P} \epsilon_P\, \delta_P(\alpha)$$

where $\epsilon_P = \pm 1$ is the local intersection number of the initial arcs of α'_P and α''_P (in that order).

Sample cobracket

To compute the cobracket of

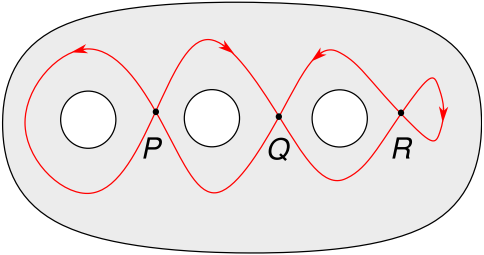


$\xi = \partial/\partial x$

$\text{rot}_\xi\, \alpha = 1$

Sample cobracket

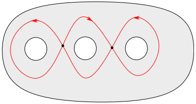
represent it by



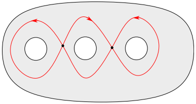
$\xi = \partial/\partial x$

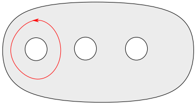
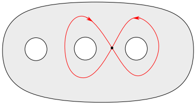
$\text{rot}_\xi\, \alpha = 0$

Sample cobracket

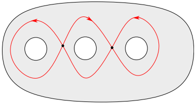
to see that δ_ξ takes  to

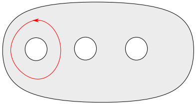
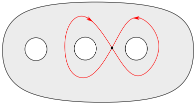
Sample cobracket

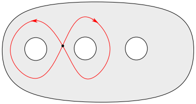
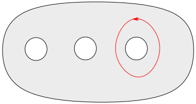
to see that δ_ξ takes  to

$-$  $\wedge$ 

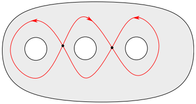
Sample cobracket

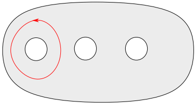
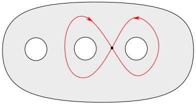
to see that δ_ξ takes  to

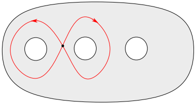
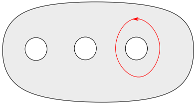
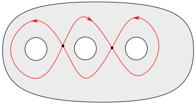
$-$  $\wedge$ 

$+$  $\wedge$ 

Sample cobracket

to see that δ_ξ takes  to

$-$  $\wedge$ 

$+$  $\wedge$  $-$  $\wedge 1$

- The Goldman–Turaev Lie bialgebra is *involutive*. That is

$$\Bbbk\lambda(X) \xrightarrow{\delta_\xi} \Bbbk\lambda(X) \otimes \Bbbk\lambda(X) \xrightarrow{\{\ ,\ \}} \Bbbk\lambda(X)$$

  is zero.
- The cobracket δ_ξ induces a map

$$\overline{\delta} : \Bbbk\lambda(X)/\Bbbk \to (\Bbbk\lambda(X)/\Bbbk)^{\otimes 2}$$

  It does not depend on the framing ξ. This is called the *reduced cobracket*.

The Kawazumi–Kuno action and Turaev coaction

- Let $\vec{v}$ be a tangential base point — equivalently, a base point in the boundary of X.
- Kawazumi and Kuno extended the constructions of Goldman and Turaev to define an action

  $$\kappa_{\vec{v}} : \mathbb{k}\lambda(X) \to \operatorname{Der} \mathbb{k}\pi_1(X, \vec{v}).$$

  Turaev defined a coaction

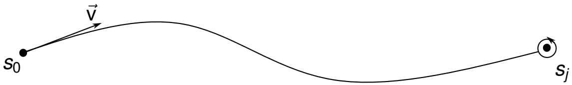
  $$\mathbb{k}\pi_1(X; \vec{v}) \to \mathbb{k}\lambda(X) \otimes \mathbb{k}\pi_1(X; \vec{v}).$$

Special derivations

A derivation D of $\mathbb{k}\pi_1(X, \vec{v})$ is *special* if there are $\mu_1, \ldots, \mu_n \in \mathbb{k}\pi_1(X, \vec{v})$ (resp., its completion) such that $D(\gamma_0) = 0$ and

$$D(\gamma_j) = [\gamma_j, \mu_j] := \gamma_j \mu_j - \mu_j \gamma_j \text{ when } j > 0.$$

Here γ_j is any path of the form



Loops act as special derivations, so

$$\kappa_{\vec{v}} : \mathbb{k}\lambda(X) \to \mathrm{SDer}\, \mathbb{k}\pi_1(X, \vec{v}).$$

Completions

- From now on, $\mathbb{k}$ is a field of characteristic zero.
- Denote the augmentation idea of $\mathbb{k}\pi_1(X, \vec{v})$ by I.
- The I-adic completion of $\mathbb{k}\pi_1(X, \vec{v})$ is

$$\mathbb{k}\pi_1(X, \vec{v})^\wedge := \varprojlim_m \mathbb{k}\pi_1(X, \vec{v})/I^m.$$

- Give $\mathbb{k}\lambda(X)$ the quotient topology via $\mathbb{k}\pi_1(X, \vec{v}) \to \mathbb{k}\lambda(X)$. Its I-adic completion is

$$\mathbb{k}\lambda(X)^\wedge = \mathscr{C}(\mathbb{k}\pi_1(X, \vec{v})^\wedge).$$

The completed GT Lie bialgebra

- Kawazumi and Kuno showed that the Goldman bracket and Turaev cobracket are continuous in the *I*-adic topology and thus induce maps

$$\{\ ,\ \} : \mathbb{k}\lambda(X)^{\wedge} \otimes \mathbb{k}\lambda(X)^{\wedge} \to \mathbb{k}\lambda(X)^{\wedge}$$

  and

$$\delta_{\xi} : \mathbb{k}\lambda(X)^{\wedge} \to \mathbb{k}\lambda(X)^{\wedge} \widehat{\otimes}\, \mathbb{k}\lambda(X)^{\wedge}$$

  This is the *completed GT Lie bialgebra*.
- They also showed that their action is continuous, so that there is a continuous Lie algebra homomorphism

$$\kappa_{\vec{\mathsf{v}}} : \mathbb{k}\lambda(X)^{\wedge} \to \mathrm{SDer}\, \mathbb{k}\pi_1(X, \vec{\mathsf{v}})^{\wedge}$$

Mixed Hodge structures for non-specialists

- Suppose that $\mathbb{k}$ is a subfield of $\mathbb{R}$, such as $\mathbb{Q}$. A $\mathbb{k}$-MHS A is a finite dimensional $\mathbb{k}$ vector space with additional structure. Part of that is an increasing *weight filtration*

$$0 = W_M A \subseteq \cdots \subseteq W_r A \subseteq W_{r+1} A \subseteq \cdots \subseteq W_N A = A.$$

  This is often topologically defined, as it is in our setting.
- The category of $\mathbb{k}$-MHS is *tannakian*, which means that it is equivalent to the category of finite dimensional representations of an affine (aka, proalgebraic) group

$$G_{\mathbb{k}} = \pi_1(\mathrm{MHS}_{\mathbb{k}})$$

  defined over $\mathbb{k}$. So every $\mathbb{k}$-MHS A has an action by $G_{\mathbb{k}}$ and all morphisms of MHS are $G_{\mathbb{k}}$-equivariant.

This group has a family of cocharacters $\chi : \mathbb{G}_m \to G$. If we fix one, then every $\Bbbk$-MHS decomposes

$$A = \bigoplus_{m \in \mathbb{Z}} A_m$$

where $t \in \mathbb{G}_m(\Bbbk) = \Bbbk^\times$ acts on A_m by t^m. We have

$$W_r A = \bigoplus_{m \leq r} A_m.$$

This implies the exactness of the “weight graded” functor:

$$A \to \mathrm{Gr}_r^W A := W_r A / W_{r-1}.$$

This is a fundamental and very useful fact.

Updated setup

- Suppose that $X = \overline{X} - S$ where $\overline{X}$ is a compact oriented surface, $S = \{s_0, \ldots, s_n\}$ with $n \geq 0$ and $\vec{v} \in T_{s_0}\overline{X}$, $\vec{v} \neq 0$.
- We have the exact sequence

$$0 \to H_2(\overline{X}) \to H_0(S) \to H_1(X) \to H_1(\overline{X}) \to 0.$$

- Image $\mathbf{e}_j$ of s_j in $H_1(X)$ is a small positive loop about s_j.
- We have the relation $\mathbf{e}_0 + \cdots + \mathbf{e}_n = 0$.
- Set $E_0 = \mathrm{span}\{\mathbf{e}_1, \ldots, \mathbf{e}_n\} \subseteq H_1(X)$.
- Have the "weight" filtration

$$0 = W_{-3}H_1(X) \subseteq W_{-2}H_1(X) \subseteq W_{-1}H_1(X) = H_1(X)$$

where $W_{-2}H_1(X) = E_0$.

Hodge theory

- Now suppose that $\overline{X}$ is a compact Riemann surface.
- There is a canonical MHS on $H_1(X)$ with the weight filtration above:

$$\mathrm{Gr}^W_{-1} H_1(X) = H_1(\overline{X}), \quad \mathrm{Gr}^W_{-2} H_1(X) = E_0.$$

- There is a canonical pro-mixed Hodge structure (MHS) on $\mathbb{Q}\pi_1(X, \vec{v})^\wedge$. It induces a canonical pro-MHS on $\mathbb{Q}\lambda(X)^\wedge$.
- The MHS on $\mathbb{Q}\lambda(X)^\wedge$ does not depend on $\vec{v}$, only on X.
- Set $V = \mathrm{Gr}^W_\bullet H_1(X) = H \oplus E_0$, where $H = H_1(\overline{X})$.
- There are canonical isomorphisms

$$\mathrm{Gr}^W_\bullet \mathbb{Q}\pi_1(X, \vec{v})^\wedge \cong T(V)^\wedge \text{ and } \mathrm{Gr}^W_\bullet \mathbb{Q}\lambda(X)^\wedge \cong \mathscr{C}(T(V))^\wedge.$$

Theorem (H: G&T 2020, JEMS 2021)

After tensoring $\mathbb{Q}\lambda(X)^\wedge$ with $\mathbb{Q}(-1)$, the completed Goldman bracket

$$\{\ ,\ \} : \Bbbk\lambda(X)^\wedge \otimes \Bbbk\lambda(X)^\wedge \to \Bbbk\lambda(X)^\wedge,$$

the completed Turaev cobracket (when ξ is meromorphic on $\overline{X}$ and nowhere vanishing holomorphic on X)

$$\delta_\xi : \Bbbk\lambda(X)^\wedge \to \Bbbk\lambda(X)^\wedge \widehat{\otimes} \Bbbk\lambda(X)^\wedge,$$

and the Kawazumi–Kuno action

$$\kappa_{\vec{v}} : \Bbbk\lambda(X)^\wedge \to \mathrm{SDer}\, \Bbbk\pi_1(X, \vec{v})^\wedge$$

are all morphisms of pro-MHS.

The mixed Hodge structure $\mathbb{Q}(-1)$ is the one dimensional Hodge structure of weight $+2$. Tensoring with it shifts the weight filtration by 2.

- Fix a symplectic basis $\mathbf{p}_1, \ldots, \mathbf{p}_g, \mathbf{q}_1, \ldots, \mathbf{q}_g$ of $H_1(\overline{X})$.
- Then

$$T(V) = \mathbb{Q}\langle \mathbf{p}_1, \ldots, \mathbf{p}_g, \mathbf{q}_1, \ldots, \mathbf{q}_g, \mathbf{e}_1, \ldots, \mathbf{e}_n \rangle,$$

  where each $\mathbf{p}_j, \mathbf{q}_j$ has weight -1 and each $\mathbf{e}_k$ weight -2.
- Define $\mathbf{e}_0 \in T(V)$ so that

$$\mathbf{e}_0 + \mathbf{e}_1 + \cdots + \mathbf{e}_n + \sum_{j=1}^{g} [\mathbf{p}_j, \mathbf{q}_j] = 0.$$

- A derivation D of $T(V)$ is *special* if $D(\mathbf{e}_0) = 0$ and there are $\mathbf{u}_k \in T(V)$ such that $D(\mathbf{e}_k) = [\mathbf{e}_k, \mathbf{u}_k]$ when $k \neq 0$.

Splitting the weight filtration

Hodge theory gives natural isomorphisms (so compatible with the Goldman bracket and $\kappa_{\vec{v}}$)

$$\mathbb{Q}\pi_1(X,\vec{v})^\wedge \cong \prod_{m\le 0} \mathrm{Gr}^W_m \mathbb{Q}\pi_1(X,\vec{v})^\wedge \cong T(V)^\wedge$$

$$\mathbb{Q}\lambda(X)^\wedge \cong \prod_{m\le 0} \mathrm{Gr}^W_m \mathbb{Q}\lambda(X)^\wedge \cong \mathscr{C}(T(V)^\wedge)$$

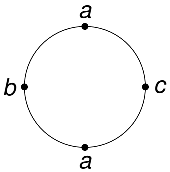
$$\mathrm{SDer}\,\mathbb{Q}\pi_1(X,\vec{v})^\wedge \cong \mathrm{SDer}\,\mathrm{Gr}^W_\bullet \mathbb{Q}\pi_1(X,\vec{v})^\wedge \cong \mathrm{SDer}\,T(V)^\wedge.$$

So we need only find formulas for $\{\ ,\ \}$ and $\kappa_{\vec{v}}$ on the associated weight gradeds.

Formula for the KK-action

When $A = \mathbb{k}\langle a_1, \dots, a_m \rangle$, have operators $\frac{\partial}{\partial a_j} : \mathscr{C}(A) \to A$ of degree -1. For example:

$$\frac{\partial}{\partial a} :$$



$$\mapsto \quad bac + cab$$

For $F \in \mathscr{C}(T(V))$, $\Phi_0(F) \in \mathrm{SDer}\, T(V)$ is defined by

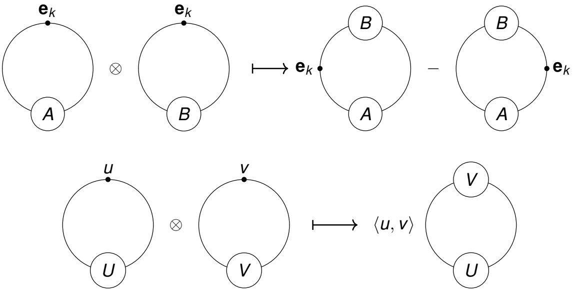
$$\Phi_0(F) : \begin{cases} \mathbf{p}_j \mapsto -\partial F / \partial \mathbf{q}_j, \\ \mathbf{q}_j \mapsto \quad \partial F / \partial \mathbf{p}_j, \\ \mathbf{e}_k \mapsto [\mathbf{e}_k, \partial F / \partial \mathbf{e}_k] \quad k \neq 0. \end{cases}$$

Its kernel is spanned by $\mathbf{e}_k^m$ where $j \neq 0$ and $m \geq 0$.

Formula for the bracket

For $F, G \in \mathscr{C}(T(V))$

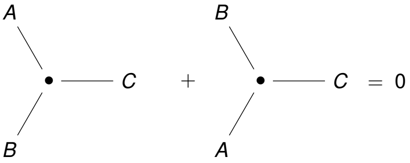
$$\{F, G\}_0 = \left| \sum_{k \neq 0} \left[\frac{\partial F}{\partial \mathbf{e}_k}, \frac{\partial G}{\partial \mathbf{e}_k} \right] \mathbf{e}_k + \sum_{j=1}^{g} \left(\frac{\partial F}{\partial \mathbf{p}_j} \frac{\partial G}{\partial \mathbf{q}_j} - \frac{\partial G}{\partial \mathbf{p}_j} \frac{\partial F}{\partial \mathbf{q}_j} \right) \right|$$



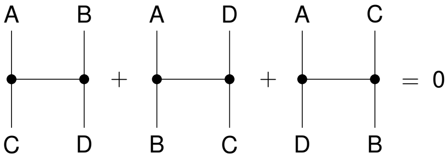
Here $k \neq 0$, $u, v \in H$ and $A, B, U, V \in T(V)$.

The Lie algebra $\mathscr{C}(\mathbb{L}(V))$

The Lie algebra $\mathscr{C}(\mathbb{L}(V))$ is defined to be the Lie algebra of V-decorated trivalent planar graphs modulo the AS-relation



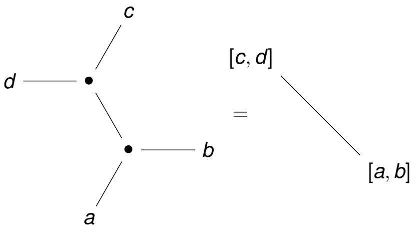
and the IHX-relation



The homomorphism $\mathscr{C}(\mathbb{L}(V)) \to \mathscr{C}(T(V))$

Expanding V-labelled planar trivalent trees defines an injective Lie algebra homomorphism

$$\mathscr{C}(\mathbb{L}(V)) \to \mathscr{C}(T(V))$$



$$\mapsto \quad \left|(ab - ba)(cd - dc)\right|$$

The PBW Theorem gives a coalgebra isomorphism:

$$T(V) = U\mathbb{L}(V) = \bigoplus_{m \geq 0} \operatorname{Sym}^m \mathbb{L}(V).$$

"Cutting" an edge of a decorated tree defines a well-defined map $\mathscr{C}(\mathbb{L}(V)) \to |\operatorname{Sym}^2 \mathbb{L}(V)|$. It has an obvious inverse, so we have a Lie algebra isomorphism

$$\mathscr{C}(\mathbb{L}(V)) \cong |\operatorname{Sym}^2 \mathbb{L}(V)|.$$

The restriction of $\mathscr{C}(T(V)) \to \operatorname{SDer} \mathbb{L}(V)$ to $\mathscr{C}(\mathbb{L}(V))$ is surjective and has kernel

$$\operatorname{span}\{\mathbf{e}_1^2, \ldots, \mathbf{e}_n^2\}.$$

Lifting $\mathscr{C}(\mathbb{L}(V))$ to $\mathbb{Q}\lambda(X)^\wedge$

- The set of primitive elements of $\mathbb{Q}\pi_1(X,\vec{v})^\wedge$ is the Lie algebra $\mathfrak{p}(X,\vec{v})$ of the unipotent (aka, Malcev) completion of $\pi_1(X,\vec{v})$. Its associated weight graded is canonically isomorphic to $\mathbb{L}(V)$.
- The (completed) enveloping algebra of $\mathfrak{p}(X,\vec{v})$ is $\mathbb{Q}\pi_1(X,\vec{v})^\wedge$.
- PBW gives an isomorphism (even in pro-$\mathsf{MHS}_\mathbb{Q}$)

$$\mathbb{Q}\pi_1(X,\vec{v})^\wedge \cong \prod_{m\geq 0} \operatorname{Sym}^m \mathfrak{p}(X,\vec{v}).$$

- The image of $\operatorname{Sym}^2 \mathfrak{p}(X,\vec{v})$ in $\mathbb{Q}\lambda(X)^\wedge$ is a sub-MHS. Its associated weight graded is $|\operatorname{Sym}^2 \mathbb{L}(V)|$.

- We have central extensions

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{span}\{\mathbf{e}_j^2 : j \neq 0\} & \longrightarrow & \mathscr{C}(\mathbb{L}(V)) & \to & \mathrm{SDer}\,\mathbb{L}(V) & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \mathrm{span}\{\mathbf{e}_j^m : j \neq 0, m \geq 0\} & \to & \mathscr{C}(T(V)) & \to & \mathrm{SDer}\,T(V) & \to & 0 \end{array}$$

- We conclude that there is a central extension

$$0 \to \mathrm{span}\{(\log \sigma_j)^2 : j \neq 0\} \to |\,\mathrm{Sym}^2\,\mathfrak{p}(X, \vec{\mathrm{v}})| \to \mathrm{SDer}\,\mathfrak{p}(X, \vec{\mathrm{v}}) \to 0$$

of pro-MHS, where σ_j is a small loop about s_j. It spans a copy of $\mathbb{Q}(1)$.

Outline

Mapping class groups

- Denote the mapping class group of $(\overline{X}; S, \vec{v})$ by $\Gamma_{X,\vec{v}}$. It is a mapping class group of type $(g, n + \vec{1})$.
- Assume that X is hyperbolic: $2g - 2 + n + 1 > 0$.
- Its Torelli subgroup $T_{X,\vec{v}}$ is the kernel of the homomorphism $\Gamma_{X,\vec{v}} \to \mathrm{Sp}(H_{\Bbbk})$, where $H = H_1(\overline{X}; \Bbbk)$.
- We have the extension

$$1 \to T_{X,\vec{v}} \to \Gamma_{X,\vec{v}} \to \mathrm{Sp}(H_{\mathbb{Z}}) \to 1.$$

and the natural representation $\Gamma_{X,\vec{v}} \to \mathrm{Aut}\, \pi_1(X, \vec{v})$.

Relative completion of mapping class groups

The relative completion of $\Gamma_{X,\vec{v}}$ consists of an affine (aka proalgebraic) group $\mathcal{G}_{X,\vec{v}}$ defined over $\mathbb{Q}$ and a homomorphism

$$\rho : \Gamma_{X,\vec{v}} \to \mathcal{G}_{X,\vec{v}}(\mathbb{Q}).$$

This group is an extension

$$1 \to \mathcal{U}_{X,\vec{v}} \to \mathcal{G}_{X,\vec{v}} \to \mathrm{Sp}(H_{\mathbb{Q}}) \to 1$$

where $\mathcal{U}_{X,\vec{v}}$ is prounipotent. The composite

$$\Gamma_{X,\vec{v}} \to \mathcal{G}_{X,\vec{v}}(\mathbb{Q}) \to \mathrm{Sp}(H_{\mathbb{Q}})$$

is the canonical homomorphism. Such extensions form a category. The relative completion is the initial object of this category.

The Johnson homomorphism

- Since unipotent completion is functorial, the action of $\Gamma_{X,\vec{v}}$ on $\pi_1(X,\vec{v})$ induces a homomorphism

$$\Gamma_{X,\vec{v}} \to \operatorname{Aut}\mathfrak{p}(X,\vec{v})$$

- The universal mapping property of relative completion implies that it induces a homomorphism $\mathcal{G}_{X,\vec{v}} \to \operatorname{Aut}\mathfrak{p}(X,\vec{v})$ such that the diagram

$$\begin{array}{ccccc}
T_{X,\vec{v}} & \hookrightarrow & \Gamma_{X,\vec{v}} & \hookrightarrow & \operatorname{Aut}\pi_1(X,\vec{v}) \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{U}_{X,\vec{v}}(\mathbb{Q}) & \hookrightarrow & \mathcal{G}_{X,\vec{v}}(\mathbb{Q}) & \longrightarrow & \operatorname{Aut}\mathfrak{p}(X,\vec{v})
\end{array}$$

commutes.

- The homomorphism $\mathcal{G}_{X,\vec{v}} \to \operatorname{Aut} \mathfrak{p}(X, \vec{v})$ induces a Lie algebra homomorphism

$$\mathfrak{g}_{X,\vec{v}} \to \operatorname{SDer} \mathfrak{p}(X, \vec{v}) \qquad (*)$$

- This is (for me) the (*geometric*) Johnson homomorphism.
- For each complex structure on $(\overline{X}, S, \vec{v})$, there is a canonical MHS on $\mathfrak{g}_{X,\vec{v}}$ and $(*)$ is a morphism of MHS.
- So $(*)$ determines (and is determined by) the homomorphism of associated weight graded Lie algebras

$$\operatorname{Gr}^W_\bullet \mathfrak{g}_{X,\vec{v}} \to \operatorname{SDer} \operatorname{Gr}^W_\bullet \mathfrak{p}(X, \vec{v}) \cong \operatorname{SDer} \mathbb{L}(V).$$

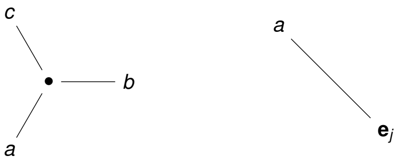
Known results

- For all $g \geq 0$, $\mathfrak{g}_{X,\vec{v}}$ has weights ≤ 0 and $\mathrm{Gr}_0^W \mathfrak{g}_{X,\vec{v}} = \mathfrak{sp}(H)$.
- The homomorphism $T_{X,\vec{v}} \to \mathcal{U}_{X,\vec{v}}(\mathbb{Q})$ induces $T_{X,\vec{v}}^{\mathrm{un}} \to \mathcal{U}_{X,\vec{v}}$.
- This induces a homomorphism $\mathfrak{t}_{X,\vec{v}} \to \mathfrak{u}_{X,\vec{v}}$.
- It is surjective when $g \geq 2$. When $g \geq 3$ it has a 1-dimensional kernel (in weight -2).
- When $g \geq 3$, $\mathfrak{u}_{X,\vec{v}}$ (and therefore the image of the Johnson homomorphism) is generated by

$$\mathrm{Gr}_{-1}^W \mathfrak{u}_{X,\vec{v}} \cong H_1(T_{X,\vec{v}}) \cong \Lambda^3 H \oplus H^{\oplus n}$$

Generators of the Johnson image

Denote the images of the geometric Johnson homomorphism by $\overline{\mathfrak{g}}_{X,\vec{v}}$. When $g \geq 3$, $\overline{\mathfrak{g}}_{X,\vec{v}}$ is generated in weight -1 by



Here $a, b, c \in H$ and $j \geq 1$. The first kind arise from bounding pair maps; the second kind from "point pushing".

Injectivity and surjectivity

- It is not know whether $\mathfrak{g}_{X,\vec{v}} \to \mathrm{SDer}\,\mathfrak{p}(X,\vec{v})$ is injective when $g > 0$.
- When $n = 0$ and $g \gg 3$, results of Morita–Sakasai–Suzuki imply that it is injective in weights ≥ -6 .
- Kupers–Randal-Williams show that, *stably*, the kernel is central, thus contained in the $\mathrm{Sp}(H)$ trivial part of $\mathrm{Gr}^W_\bullet\,\mathfrak{u}_{X,\vec{v}}$.
- It is known (Morita, Enomoto–Satoh, Conant) that the image of

$$\mathrm{Gr}^W_\bullet\,\mathfrak{u}_{X,\vec{v}} \to \mathrm{SDer}\,\mathrm{Gr}^W_\bullet\,\mathfrak{p}(X,\vec{v}) \cong \mathrm{SDer}\,\mathbb{L}(V)$$

is “small”. (ie, have a “large Johnson cokernel”)

The arithmetic Johnson homomorphism

- There is also a homomorphism (for $\Bbbk = \mathbb{Q}, \mathbb{R}$).

$$\mathfrak{mhs}_{\Bbbk} \to \operatorname{Der} \mathfrak{p}(X, \vec{v})$$

  where $\mathfrak{mhs}_{\Bbbk}$ is the Lie algebra of $G_{\Bbbk} = \pi_1(\mathrm{MHS}_{\Bbbk})$.
- Since $\mathfrak{mhs}_{\Bbbk}$ acts on $\mathfrak{g}_{X,\vec{v}}$, we have

$$\mathfrak{mhs}_{\Bbbk} \ltimes \mathfrak{g}_{X,\vec{v}}$$

- Since $\mathfrak{mhs}_{\Bbbk}$ acts on $\mathfrak{p}(X, \vec{v})$, the Johnson homomorphism extends to

$$\mathfrak{mhs}_{\Bbbk} \ltimes \mathfrak{g}_{X,\vec{v}} \to \operatorname{Der} \mathfrak{p}(X, \vec{v})$$

- This is the *arithmetic* Johnson homomorphism

Arithmetic versus geometric Johnson image

- Denote the image of the arithmetic Johnson homomorphism by $\widehat{\mathfrak{g}}_{X,\vec{v}}$.
- Denote their pronilpotent radicals of $\overline{\mathfrak{g}}_{X,\vec{v}}$ and $\widehat{\mathfrak{g}}_{X,\vec{v}}$ by $\overline{\mathfrak{u}}_{X,\vec{v}}$ and $\widehat{\mathfrak{u}}_{X,\vec{v}}$, respectively.
- The proof of Oda's Conjecture by Takao (+ Ihara, Matsumoto, Nakamura, ...), Hodge theory and Brown's fundamental theorem (on mixed Tate motives) give:

Theorem

The Lie algebras $\overline{\mathfrak{g}}_{X,\vec{v}}$ and $\widehat{\mathfrak{g}}_{X,\vec{v}}$ have natural MHS and the inclusion is a morphism. For $\mathbb{k} = \mathbb{Q}, \mathbb{R}$, and all $g, n \geq 0$ there is a SES

$$0 \to \mathrm{Gr}^W_\bullet \overline{\mathfrak{u}}_{X,\vec{v}} \to \mathrm{Gr}^W_\bullet \widehat{\mathfrak{u}}_{X,\vec{v}} \to \mathbb{L}(\sigma_3, \sigma_5, \sigma_7, \dots) \to 0$$

Constraining the Johnson image

It may be more natural to find constraints on the arithmetic Johnson image $\widehat{\mathfrak{g}}_{X,\vec{v}}$ — equivalently, on $\mathrm{Gr}^W_\bullet \widehat{\mathfrak{u}}_{X,\vec{v}}$. Consider the diagram

$$\begin{array}{ccccccccc}
 & & & & & & \widehat{\mathfrak{u}}_{X,\vec{v}} & & \\
 & & & & & \swarrow & \downarrow & & \\
0 & \longrightarrow & \mathrm{span}\{\mathbf{e}_j^2 : j \neq 0\} & \longrightarrow & |\,\mathrm{Sym}^2\,\mathfrak{p}(X,\vec{v})| & \longrightarrow & \mathrm{SDer}\,\mathfrak{p}(X,\vec{v}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{span}\{1, \mathbf{e}_j^m : j \neq 0\} & \longrightarrow & \mathbb{Q}\lambda(X)^\wedge & \longrightarrow & \mathrm{SDer}\,\mathbb{Q}\pi_1(X,\vec{v})^\wedge & \longrightarrow & 0
\end{array}$$

with exact rows. The generalized Picard–Lefschetz formula of Kawazumi–Kuno gives a lift on $\mathfrak{g}_{X,\vec{v}}$. It extends to $\widehat{\mathfrak{u}}_{X,\vec{v}}$ as the kernel has weight -2 and the new generators σ_{2m+1} all have weights ≤ -6

Constraints, ctd

Kawazumi and Kuno observed that the Turaev cobracket constrains the geometric Johnson image. It also constrains the *arithmetic* Johnson image. Suppose that ξ is an *algebraic* framing. Since the kernel of

$$|\operatorname{Sym}^2 \mathfrak{p}(X,\vec{v})|(-1) \to \operatorname{SDer} \mathfrak{p}(X,\vec{v})$$

has weight -2, the cobracket induces

$$D_\xi : W_{-3} \operatorname{SDer} \mathbb{Q}\pi_1(X,\vec{v})^\wedge \to \big(\operatorname{SDer} \mathbb{Q}\pi_1(X,\vec{v})^\wedge\big)^{\otimes 2}$$

Theorem (special case)

If $g \geq 3$ and $m \geq 3$, then $W_{-m}[\widehat{\mathfrak{u}}_{X,\vec{v}}, \widehat{\mathfrak{u}}_{X,\vec{v}}] \subseteq \ker D_\xi$.

Morita's proposal

Here we suppose that X is a surface of type $(g, \vec{1})$ with $g \geq 3$. Morita has defined derivations $\mu_{2m+1} \in \mathrm{Gr}^W_{-4m-2} \operatorname{SDer} \mathfrak{p}(X, \vec{\mathrm{v}})$ that he conjectures equal (up to a non-zero multiple) the images of the σ_{2n+1} mod the geometric derivations $\bar{\mathfrak{g}}_{X,\vec{\mathrm{v}}}$.

Proposition (Morita)

For each $m \geq 0$ there is a unique copy of $\operatorname{Sym}^{2m+1} H$ in $\mathrm{Gr}^W_{-2m-1} \operatorname{SDer} \mathfrak{p}(X, \vec{\mathrm{v}})$. When $m > 0$, is not in the Johnson image.

There is a unique copy of the trivial representation in $\Lambda^2 \operatorname{Sym}^{2m+1} H$. The derivation μ_{2m+1} is the image of a generator of this trivial representation under the bracket map:

$$\mathbb{k}\,\mu_{2m+1} = [\Lambda^2 \operatorname{Sym}^{2m+1} H]^{\mathrm{Sp}(H)} \longrightarrow \mathrm{Gr}^W_{-4m-2} \operatorname{SDer} \mathbb{L}(H).$$

How might one approach proving this?

Outline

I: The Goldman–Turaev Lie Bialgebra

II: Johnson Homomorphisms

III: Goncharov's Hodge Correlators

- Oda's conjecture (a theorem) is true for both $\mathbb{k} = \mathbb{Q}$ and $\mathbb{k} = \mathbb{R}$.
- This implies that one can define $\widehat{\mathfrak{g}}_{X,\vec{v}}$ using either.
- The group $G_{\mathbb{k}} := \pi_1(\mathrm{MHS}_{\mathbb{k}})$ can be computed when $\mathbb{k} = \mathbb{R}$ but appears intractable when $\mathbb{k} = \mathbb{Q}$.
- Goncharov's Hodge correlators give a method for computing $\mathfrak{mhs}_{\mathbb{R}} \to \mathrm{SDer}\,\mathfrak{p}(X, \vec{v})$.
- I'll give a very brief introduction to Hodge correlators.

Real MHS

- Deligne showed that $\pi_1(\mathrm{MHS}_\mathbb{R})$ is an extension

$$0 \to \mathcal{N} \to \pi_1(\mathrm{MHS}_\mathbb{R}) \to R_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m \to 1$$

  where
  - $R_{\mathbb{C}/\mathbb{R}}$ is Weil restriction:

$$R_{\mathbb{C}/\mathbb{R}}(\mathbb{R}) = \mathbb{C}^\times \text{ and } R_{\mathbb{C}/\mathbb{R}}(\mathbb{C}) = \mathbb{C}^\times \times \mathbb{C}^\times,$$

  - $\mathfrak{n} \otimes \mathbb{C} \cong \mathbb{L}(\mathbf{z}^{p,q} : p, q < 0)^\wedge$ and $w \in \mathbb{C}^\times = R_{\mathbb{C}/\mathbb{R}}(\mathbb{R})$ acts on it via

$$w : \mathbf{z}^{p,q} \mapsto w^p \overline{w}^q \mathbf{z}^{p,q}.$$

- It was later observed (by Goncharov) that there is a canonical (or at least very natural) choice of the $\mathbf{z}^{p,q}$ that also satisfy $\overline{\mathbf{z}^{p,q}} = -\mathbf{z}^{q,p}$. ($\mathbf{z}^{p,q} + \mathbf{z}^{q,p}$ is purely imaginary.)
- The image of $\mathbf{z}^{-2n-1,-2n-1}$ under $\mathfrak{n} \to \mathrm{SDer}\,\mathfrak{p}(X, \vec{\mathrm{v}})$ is congruent to σ_{2n+1} mod geometric derivations $\overline{\mathfrak{g}}_{X,\vec{\mathrm{v}}}$.

- For every (framed) real MHS V, there is a unique, purely imaginary derivation

$$D_V \in i\, W_{-2}\, \mathrm{End}\, \mathrm{Gr}^W_\bullet\, V_\mathbb{R}$$

with $D_V^{p,q} = 0$ unless both $p, q < 0$ such that V is isomorphic to the MHS

$$\exp D_V : \mathrm{Gr}^W_\bullet\, V_\mathbb{R} \to \mathrm{Gr}^W_\bullet\, V_\mathbb{C}.$$

Here the isomorphism of V with this MHS is required to be the identity on $\mathrm{Gr}^W_\bullet$.
- The action $\mathfrak{n} \to \mathrm{End}\, V_\mathbb{C}$ takes $\mathbf{z}^{p,q}$ to $D_V^{p,q}$.
- This is "easy" to prove, once one knows the statement.

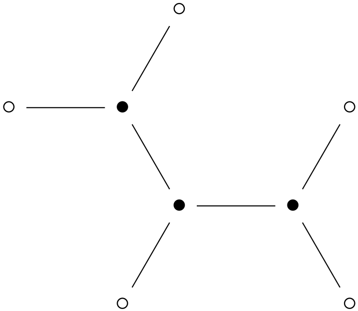
Hodge correlators

When $V = \mathfrak{p}(X, \vec{v})$, the derivation $D = D_{\mathfrak{p}(X.\vec{v})}$ and its (p, q) components can be computed using Hodge correlators. The setup:

- $\mathscr{C}(T(H_1(X)))^\vee$ consists of all cyclic words in elements of $\Omega^1(\overline{X}) \oplus \overline{\Omega^1(X)}$, the complex harmonic 1-forms.
- Observe: the external vertices of a trivalent, planar tree T are cyclically ordered.
- Begin by decorating the leaves of planar trivalent graphs with elements of $\mathscr{C}(T(H_1(X)))^\vee$.

Decorated planar trivalent trees

A trivalent planar tree T with m internal vertices can be labelled by a cyclic word $\mathbf{w}$ of length $\ell(\mathbf{w}) = m + 2$. Example, $m = 3$:



$$\mathbf{w} = \left|\omega_1\omega_2\omega_3\omega_4\omega_5\right|$$

Decorated planar trivalent trees

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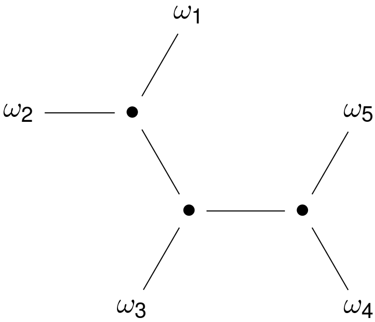

$$\mathbf{w} = |\omega_1\omega_2\omega_3\omega_4\omega_5|$$

Sketch of construction

- Suppose that **w** is a cyclic word in $\mathscr{C}(T(H_1(X)))^\vee$ of length $\ell(\mathbf{w}) = m + 2$.
- For each **w**-decorated (planar trivalent) graph T, one constructs a $2m$-current $\Omega_T(\mathbf{w})$ on X^m from the 1-forms that occur in **w** and derivatives of a Green's operator associated to each interior edge of T.
- The *correlator* associated to **w** is defined to be

$$\mathrm{Cor}(\mathbf{w}) := \sum_{T \vdash \mathbf{w}} \int_{X^m} \Omega_T(\mathbf{w}) \in \mathbb{C}$$

  where the sum ranges over all trivalent planar trees T decorated by **w**.

- The Hodge correlator $\mathrm{Cor}_{X,\vec{v}}$ of $(X, \vec{v})$ is:

$$\sum_{\mathbf{w}} \mathrm{Cor}(\mathbf{w}) \in \mathrm{Hom}_{\mathbb{C}}(\mathscr{C}(T(H_1(X)))^{\vee}, \mathbb{C}) \cong \mathscr{C}(T(H_1(X))).$$

- $\mathrm{Cor}_{X,\vec{v}}$ determines a derivation $D \in \mathrm{SDer}\, T(H_1(X))$, which, one can show, lies in $\mathrm{SDer}\, \mathbb{L}(H_1(X))$, is pure imaginary and has no components of type (p, q) where p or q is ≥ 0.
- This is the derivation $D_{\mathfrak{p}(X,\vec{v})}$ that determines the real MHS on $\mathfrak{p}(X, \vec{v})$. (I have not yet checked this.)
- Hope is that one can show that $D^{-2m-1,-2m-1}$ is a multiple of Morita's class μ_{2m+1} mod $\bar{\mathfrak{g}}_{\bullet}$ for one curve $(X, \vec{v})$. (That will suffice.)

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