Mapping class groups of simply connected algebraic manifolds

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Mapping class group of a manifold

The *mapping class group* Γ_M of a closed orientable manifold M is the group of *isotopy* classes of orientation preserving diffeomorphisms of M:

$$\Gamma_M := \pi_0 \operatorname{Diff}^+ M.$$

The *Torelli group* T_M of M is the subgroup consisting of the mapping classes that act trivially on the homology of M:

$$T_M := \ker\{\Gamma_M \to \operatorname{Aut} H_\bullet(M;\mathbb{Z})\}.$$

Denote the image of $\Gamma_M \to \operatorname{Aut} H_\bullet(M;\mathbb{Z})$ by S_M. The mapping class group Γ_M is an extension

$$1 \to T_M \to \Gamma_M \to S_M \to 1.$$

Examples

If N is a subset of M (e.g., ∂M or a point), one can define

$$\Gamma_{M,N} := \pi_0(\mathrm{Diff}^+(M, N)).$$

- When $M = S^1 \times S^1 \cong \mathbb{R}^2/\mathbb{Z}^2$, the evident homomorphism

  $$\mathrm{SL}_2(\mathbb{Z}) \to \Gamma_{M,0}$$

  is an isomorphism. The Torelli group $T_{M,0}$ is trivial.
- If A is the annulus $S^1 \times [0, 1]$ one has

  $$\Gamma_{A,\partial A} = \{t_A^n : n \in \mathbb{Z}\} \cong \mathbb{Z}.$$

  The generator

  $$t_A : (\theta, t) \mapsto (\theta + 2\pi t, t)$$

  is the called the *Dehn twist* about the curve $S^1 \times \{1/2\}$.

Monodromy homomorphisms

- A locally trivial bundle $X \to T$ with fiber M over a smooth manifold T gives rise to a *monodromy representation*

$$\pi_1(T, t_o) \to \Gamma_M$$

  where we identify the fiber over t_o with M.
- Represent $a \in \pi_1(T, t_o)$ by smooth $\alpha : (S^1, 1) \to (T, t_o)$. Have

$$\begin{array}{ccc} (\alpha^* X, M) & \longrightarrow & (X, M) \\ \downarrow & & \downarrow \\ (S^1, 1) & \longrightarrow & (T, t_o) \end{array}$$

  Lift the vector field $\partial/\partial t$ on $S^1 = [0, 1]/(0 \sim 1)$ to a vector field on $\alpha^* X$. In integrate to get a diffeomorphism of the fiber M over t_o. Its mapping class in Γ_M is well-defined.

The surface case

The case of surfaces is classical. Suppose that M is a compact oriented surface of genus $g \geq 2$.

- Its MCG Γ_M is generated by a finite number of Dehn twists.
- It is finitely presented (algebraic geometry, Thurston, ...).
- Have $S_M = \mathrm{Sp}(H_1(M;\mathbb{Z})) := \mathrm{Aut}(H_1(M;\mathbb{Z}), \langle\ ,\ \rangle)$.
- Its Torelli group T_M is a tough nut to crack:
  - it is a countably generated free group when $g = 2$ (Mess)
  - it is finitely generated when $g \geq 3$ (Johnson)
  - it is conjectured to be finitely presented when $g \gg 3$, but this is not known for any $g \geq 2$.

Uniformization Theorem

- The uniformization theorem says that every oriented surface with negative Euler characteristic has a complete hyperbolic metric.
- Another version says that the universal covering of every Riemann surface with negative Euler characteristic is biholomorphic to the upper half plane

$$\mathfrak{h} = \{z \in \mathbb{C} : \Im(z) > 0\}.$$

- It has the complete hyperbolic metric $(dx^2 + dy^2)/y^2$.
- As $\mathrm{Isom}^+(\mathfrak{h}, \mathrm{hyp}) = \mathrm{Aut}(\mathfrak{h}) = \mathrm{PSL}_2(\mathbb{R})$, this implies that if $g(M) \geq 2$, then

$$\{\text{hyperbolic structures on } M\} = \{\text{complex structures on } M\}.$$

Hyperbolic structures on M—briefly

Fix a hyperbolic structure on M.

- Each simple closed curve on M that does not bound a disk is homotopic to a simple closed geodesic.
- Fix a "pants decomposition" of M. We can assume that each curve in the decomposition is a geodesic.
- For a fixed pair of pants P, the function

  $$\{\text{hyperbolic structures on } P \text{ with geodesic boundary}\} \to \mathbb{R}^3_+$$

  that takes the hyperbolic structure to lengths of the 3 boundary components is a bijection.
- Hyperbolic pants with geodesic boundary can be glued to get a hyperbolic surface provided the lengths of the corresponding boundary components match.

Teichmüller space $\mathscr{T}_g$

- A *marked Riemann surface* is a homotopy class of diffeomorphisms $f : M \to X$ of M with a compact Riemann surface, or equivalently, a hyperbolic surface.
- The set of marked Riemann surfaces of genus g is a manifold $\mathscr{T}_g$ that is diffeomorphic to $\mathbb{R}^{6g-6}$.
- To see why, decompose M into "pairs of pants".
- Since $\chi(P) = -1$ and $\chi(\partial P) = 0$, we have

$$\#P = -\chi(M) = 2g - 2.$$

- Since each pair of pants has 3 boundary circles and since each circle bounds 2 pants of pants, the number of curves in the "pants decomposition is $3/2 \times \#P = 3g - 3$.
- The map $\mathscr{T}_g \to \mathbb{R}_+^{3g-3} \times \widetilde{S^1}^{3g-3} \to \mathbb{R}^{6g-6}$ that takes the metric to the length and "twist angles" is a bijection.

Moduli of compact Riemann surfaces

- The mapping class group Γ_M acts on Teichmüller space $\mathscr{T}_g$:

$$\Gamma_M \quad \circlearrowright M \xrightarrow{f} X \qquad [\phi] : [f] \mapsto [f \circ \phi^{-1}].$$

- This action is properly discontinuous and virtually free.
- The moduli space of compact Riemann surfaces is the orbifold quotient:

$$\mathcal{M}_g = \Gamma_M \backslash \mathscr{T}_g.$$

- It is the orbifold classifying space $B\Gamma_g$ of Γ_M. That is, the topology of $\mathcal{M}_g$ is determined by Γ_M.
- One manifestation of this is the isomorphism

$$H^\bullet(\Gamma_M; \mathbb{Q}) \cong H^\bullet(\mathcal{M}_g; \mathbb{Q}).$$

- Much geometry of algebraic curves is encoded in the cohomology and structure of Γ_M.

Higher dimensions

To what extent does this hold in higher dimensions?

- Suppose that $\mathscr{M}_M$ is a moduli space that parameterizes a natural family of complex algebraic structures on M and that there is a universal family $\mathscr{X} \to \mathscr{M}_M$.
- Suppose X is a complex algebraic manifold and that $\phi : M \to X$ is an orientation preserving diffeomorphism.
- Denote the point in $\mathscr{M}_M$ that corresponds to X by $[X]$. We have a monodromy representation

$$\pi_1(\mathscr{M}_M, [X]) \to \Gamma_X \xrightarrow{\simeq} \Gamma_M$$

- Is it close to being an isomorphism? (Image of finite index? Finite kernel?)
- Is $\mathscr{M}_M$ an Eilenberg–MacLane space $K(\Gamma_M, 1) = B\Gamma_M$?

The case of hypersurfaces

Projective space:

$$\mathbb{P}^{n+1} = (\mathbb{C}^{n+2} - \{0\})/\mathbb{C}^\times.$$

Coordinates $\mathbf{x} = (x_0, \ldots, x_{n+1}) \in \mathbb{C}^{n+2}$ and $[\mathbf{x}] \in \mathbb{P}^{n+1}$.

- A non-zero polynomial $f(\mathbf{x}) \in \mathrm{Sym}^d \, \mathbb{C}^{n+2}$ defines a hypersurface

  $$X_f := \{[\mathbf{x}] \in \mathbb{P}^{n+1} : f(\mathbf{x}) = 0\}.$$

  of degree d in $\mathbb{P}^{n+1}$.
- It is smooth when $f(\mathbf{x})$ has nowhere vanishing discriminant:

  $$f(\mathbf{x}) = 0 \text{ and } \nabla f(\mathbf{x}) = 0 \text{ implies } \mathbf{x} = 0.$$

Moduli of hypersurfaces

- Let $\mathscr{U}_{n,d}$ be the space of homogeneous polynomials of degree d in $n+2$ variables with non-vanishing discriminant.
- The group $\mathrm{GL}_{n+2}(\mathbb{C})$ acts on it. The (stack) quotient is the moduli space $\mathscr{H}_{n,d}$ of hypersurfaces in $\mathbb{P}^{n+1}$ of degree d.
- The map $\mathscr{U}_{n,d} \to \mathscr{H}_{n,d}$ is a principal $\mathrm{GL}_{n+2}(\mathbb{C})$ bundle, so we have a central extension

$$0 \to \mathbb{Z} \to \pi_1(\mathscr{U}_{n,d}, f) \to \pi_1(\mathscr{H}_{n,d}, [X_f]) \to 1$$

where X_f denotes the hypersurface in $\mathbb{P}^{n+1}$ defined by the homogeneous polynomial f.

Lefschetz hyperplane theorem

Theorem (Lefschetz, special case)

If $n \geq 2$ and X is a smooth hypersurface in $\mathbb{P}^{n+1}$, then

1. *X is simply connected,*
2. *the restriction map*

$$H^j(\mathbb{P}^{n+1}; \mathbb{Q}) \to H^j(X; \mathbb{Q})$$

   is an isomorphism when $j \neq n$,

3. *in degree n we have an exact sequence*

$$0 \to H^n(\mathbb{P}^{n+1}; \mathbb{Q}) \to H^n(X; \mathbb{Q}) \to H^n_o(X; \mathbb{Q}) \to 0$$

The cokernel is the *primitive cohomology* of X. It has a non-degenerate $(-1)^n$ symmetric bilinear form $\langle \ , \ \rangle$.

- The monodromy homomorphisms are:

$$\pi_1(\mathscr{U}_{n,d}, f) \to \pi_1(\mathscr{H}_{n,d}, [X_f]) \to \Gamma_M \to \mathrm{Aut}(H^n_o(X_f;\mathbb{Z}); \langle\ ,\ \rangle).$$

- Beauville (1986) computed the images of $\pi_1(\mathscr{M}, [X])$ and Γ_M. Both have finite index in $\mathrm{Aut}(H^n_o(X;\mathbb{Z}); \langle\ ,\ \rangle)$.
- When $n = 3$, we have

$$\dim H^n_o(X;\mathbb{Q}) = \frac{(d-1)^5+1}{d} - 1$$

which is positive for all $d \geq 3$. The only interesting part of the monodromy representation is

$$\pi_1(\mathscr{U}_{3,d}, f) \to \mathrm{Sp}(H^n_o(X_f);\mathbb{Z}).$$

- Since Beauville computed S_M, the problem is to understand or compute the Torelli group T_M.
- We will skip $\dim_{\mathbb{C}} X = 2$ as 4-manifold topology is harder and more subtle. But there are recent results in dimension 4 by Konno–Lin and Konno–Mallick–Taniguchi.
- The first steps have been taken by Kreck and Su in complex dimension 3.

The result of Kreck and Su

Every complex projective manifold is a compact Kähler manifold. A simplified version of their main result is:

Theorem (Kreck–Su, 2022)

If M is a simply connected compact Kähler 3-fold with $b_2 = 1$, then there is a homomorphism

$$\delta_M : T_M \to H^3(M; \mathbb{Q})$$

whose image is a lattice of full rank and whose kernel is finite.

They give a complete computation of T_M for simply connected Kähler 3-folds.

Sullivan's general results

Theorem (Sullivan, 1977)

If M is a simply connected closed manifold of (real) dimension ≥ 5, there is an affine algebraic group $\mathcal{G}_M$, defined over $\mathbb{Q}$, that is a central extension

$$1 \to \mathcal{D}_M \to \mathcal{G}_M \to \mathcal{G}_M^h \to 1$$

and a homomorphism $\Gamma_M \to \mathcal{G}_M(\mathbb{Q})$ with arithmetic image and finite kernel. Here $\mathcal{D}_M$ (the "Pontryagin distortion group") is a quotient of

$$\bigoplus_{4k \leq \dim_{\mathbb{R}} M} H^{4k-1}(M;\mathbb{Q})$$

and $\mathcal{G}_M^h$ is the group of homotopy self equivalences of Sullivan's algebraic model of the rational homotopy type of M.

The group $\mathcal{G}_M^h$ is an extension

$$1 \to \mathcal{U}_M^h \to \mathcal{G}_M^h \to \mathcal{S}_M \to 1$$

where U_M^h is unipotent and $\mathcal{S}_M$ is a subgroup of $\mathrm{Aut}\, H^\bullet(M;\mathbb{Q})$. The group S_M is an arithmetic subgroup of $\mathcal{S}_M$. The group $\mathcal{S}_M$ is typically not reductive, such as when $M = U(9)$.

Theorem (H, 2023)

Suppose that $\mathbb{k}$ is a subfield of $\mathbb{R}$. If M is a compact Kähler manifold with Kähler class $\omega \in H^2(M;\mathbb{k})$, then the automorphism group of its cohomology ring that fixes ω is a reductive $\mathbb{k}$ group.

Corollary

If M is a complex projective manifold of complex dimension ≥ 3, then S_M is a lattice in a reductive $\mathbb{Q}$ group.

Johnson homomorphisms of Algebraic 3-folds

If M is simply connected, $\pi_3(M) \otimes \mathbb{Q}$ is an extension.

$$0 \to \operatorname{Sym}^2 H_2(M;\mathbb{Q})/\operatorname{im}\Delta \to \pi_3(M,x_o) \otimes \mathbb{Q} \to H_3(M;\mathbb{Q}) \to 0,$$

where $\Delta : H_4(M;\mathbb{Q}) \to S^2 H_2(M;\mathbb{Q})$ is the dual of the cup product. The action of T_M on this gives rise to Johnson homomorphism.

Theorem (H, 2023) (updated)

If M is a simply connected compact Kähler 3-fold, there is a surjective S_M invariant homomorphism

$$\tau_M : H_1(T_M;\mathbb{Q}) \to \operatorname{Hom}(H_3(M;\mathbb{Q}), \operatorname{Sym}^2 H_2(M;\mathbb{Q})/\operatorname{im}\Delta)$$

whose kernel contains the distortion group $\mathcal{D}_M = H^3(M;\mathbb{Q})$.

Theorem (H, 2023)

If X is a hypersurface of degree d in $\mathbb{P}^4$, then the image of $\pi_1(\mathscr{H}_{3,d}, [X]) \to \mathcal{G}_X(\mathbb{Q})$ does not intersect the distortion subgroup $\mathcal{D}_X$ of $\mathcal{G}_X$. In particular, the image of $\pi_1(\mathscr{H}_{3,d}, [X])$ in Γ_X has infinite index.

Theorem (Carlson–Toledo, 1999)

Suppose that X is a smooth hypersurface of degree d in $\mathbb{P}^{n+1}$. If $d \geq 3$ and $n > 1$, the kernel of the representation $\pi_1(\mathscr{H}_{n,d}, [X]) \to \operatorname{Aut} H^n(X; \mathbb{Q})$ surjects onto a lattice in a non compact, almost simple $\mathbb{R}$-group of rank ≥ 2. In particular, it contains a non-abelian free subgroup.

In short, when $d \geq 3$, the homomorphism

$$\pi_1(\mathscr{H}_{3,d}, [X]) \to \Gamma_X$$

has a large kernel and "cokernel".