Hecke actions on loops and periods of iterated Shimura integrals

Richard Hain

Duke University

IMAG
Université de Montpellier
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Outline

Iterated line integrals

These are functions on the path space PM of a smooth manifold M. Suppose that $\omega_1, \ldots, \omega_r$ are smooth $\mathbb{k}$-valued 1-forms on M. The function

$$\int \omega_1 \ldots \omega_r : PM \to \mathbb{k}$$

takes the value

$$\int_\alpha \omega_1 \ldots \omega_r := \int_{0 \le t_1 \le \cdots \le t_r \le 1} f_1(t_1) \ldots f_r(t_r) dt_1 \ldots dt_r$$

on the piecewise smooth path $\alpha : [0, 1] \to M$, where

$$\gamma^* \omega_j = f_j(t) dt.$$

An iterated line integral is a linear combination of such functions.

An iterated (line) integral is *closed* if its value on each path depends only on its homotopy class relative to its endpoints. Closed iterated integrals on M induce functions

$$\pi_1(M, x) \to \mathbb{k}.$$

Iterated integrals of holomorphic 1-forms on a complex curve are closed.

Example: Take $M = \mathbb{P}^1 - \{0, 1, \infty\}$, $\omega_0 = dz/z$ and $\omega_1 = dz/(1-z)$. Then

$$\mathrm{Li}_k(z) = \int_0^z \omega_1 \overbrace{\omega_0 \ldots \omega_0}^{k-1}$$

is the kth polylogarithm, a multivalued function. On the principal branch $\mathrm{Li}_k(1) = \zeta(k)$.

Iterated Shimura integrals (Manin, 2005)

Suppose that $f_1, \dots, f_m$ are modular forms of $\Gamma \leq \mathrm{SL}_2(\mathbb{Z})$. Set

$$\omega_j = f_j(\tau)\tau^{k_j - 1} d\tau \text{ where } 0 < k_j < (\text{weight of } f_j).$$

An *iterated Shimura integral* is a linear combination of iterated iterated integrals of the form

$$\int \omega_1 \omega_2 \dots \omega_m.$$

They are closed and define holomorphic functions

$$\tau \mapsto \int_{\tau_0}^{\tau} \omega_1 \omega_2 \dots \omega_m$$

on the upper half plane $\mathfrak{h}$, and thus functions

$$\Gamma \to \mathbb{C}, \quad \gamma \mapsto \int_{\tau_0}^{\gamma\tau_0} \omega_1 \omega_2 \dots \omega_m.$$

Multiple modular values (Brown, 2014)

A *multiple modular value* is the regularized value of

$$\int_0^{i\infty} \omega_1 \dots \omega_m$$
$$= i^{k_1+\cdots+k_m} \int_{0\leq y_1\leq\cdots\leq y_m} f_1(iy_1)y_1^{k_1-1} \dots f_m(iy_m)y_m^{k_m-1} dy_1 \dots dy_m.$$

These include periods of cusp forms ($m = 1$) and all multiple zeta values:

Take $\Gamma = \Gamma(2)$ and f_j to be Eisenstein series of weight 2. Then $\Gamma\backslash\mathfrak{h} = \mathbb{P}^1 - \{0, 1, \infty\}$ and (for example)

$$\zeta(a, b) := \sum_{0<m<n} \frac{1}{m^a n^b} = \int_0^1 \omega_1 \overbrace{\omega_0 \dots \omega_0}^{a-1} \omega_1 \overbrace{\omega_0 \dots \omega_0}^{b-1}.$$

Twice iterated integrals of Eisenstein series

- Define (for $0 < a < m$, $0 < b < n$)

$$\Lambda(\mathbb{G}_m, \mathbb{G}_n; a, b) = \int_0^{i\infty} \mathbb{G}_m(\tau)\tau^{a-1} d\tau \; \mathbb{G}_n(\tau)\tau^{b-1} d\tau$$

  where

$$\mathbb{G}_{2k}(\tau) = -\frac{B_{2k}}{4k} + \sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^n.$$

- Brown (2014) showed that certain linear combinations of twice iterated integrals of Eisenstein series are non-critical periods of cusp forms. For example:

$$600\Lambda(\mathbb{G}_4, \mathbb{G}_{10}; 2, 5) + 480\Lambda(\mathbb{G}_4, \mathbb{G}_{10}; 3, 4) = \frac{1}{\pi} \int_0^{i\infty} \Delta(\tau)\tau^{12} d\tau.$$

  where Δ is the Ramanujan τ-function. Other $\mathbb{Q}$-linear combinations are multiple zeta values.

Questions

1. Where do iterated Shimura integrals arise? What is the significance of multiple modular values?
2. Can Brown's computations of periods of twice iterated integrals of Eisenstein series be proved using a Hecke action on (say) iterated Shimura integrals?
3. What is the relationship of MMVs for $\Gamma(N)$, $N > 1$, to Goncharov's work on "higher cyclotomy"? ($N = 1$: Brown + Hain–Matsumoto.)

I'll address the first — the other two are works in progress.

Outline

Executive summary

Relative unipotent completion replaces a discrete group, such as $\mathrm{SL}_2(\mathbb{Z})$, by an affine $\mathbb{Q}$ group (i.e., a pro-algebraic group) $\mathcal{G}$ and a Zariski dense homomorphism $\tilde{\rho} : \Gamma \to \mathcal{G}(\mathbb{Q})$. It is an extension of a (possibly pro-) reductive group, to be specified in advance, by a prounipotent group.

The point is that relative completion replaces the discrete group Γ (not motivic) by a vector space — the ring of functions $\mathcal{O}(\mathcal{G})$ on $\mathcal{G}$. With the right choices, this Hopf algebra is "motivic" in the sense that it has a natural mixed Hodge structure (MHS) and, after tensoring with $\mathbb{Q}_\ell$, a Galois action.

The connection to iterated Shimura integrals is that $\mathcal{O}(\mathcal{G})$ is a Hopf algebra of closed iterated integrals which contains all iterated Shimura integrals.

Brief definition

- Γ a discrete group, R a (pro)reductive $\mathbb{Q}$-group, $\rho : \Gamma \to R(\mathbb{Q})$ a Zariski representation
- Extensions of affine groups (over $\mathbb{Q}$) of the form

$$1 \to U \to G \to R \to 1$$

where U is (pro)unipotent plus a homomorphism $\tilde{\rho} : \Gamma \to G(\mathbb{Q})$ that lift ρ

$$\begin{array}{ccccccccc} & & & & \Gamma & & & & \\ & & & & \downarrow{\scriptstyle \rho} & \searrow^{\tilde{\rho}} & & & \\ 1 & \longrightarrow & U & \longrightarrow & G & \longrightarrow & R & \longrightarrow & 1 \end{array}$$

form a category. The relative completion of Γ (with respect to ρ) is the initial object of this category: $\Gamma \to \mathcal{G}(\mathbb{Q})$ where

$$1 \to \mathcal{U} \to \mathcal{G} \to R \to 1.$$

Setup for relative completion of $\mathrm{SL}_2(\mathbb{Z})$

- Denote the modular curve $\mathrm{SL}_2(\mathbb{Z})\backslash\mathfrak{h}$ by Y. It will be regarded as a stack. (That is, we work $\mathrm{SL}_2(\mathbb{Z})$ equivariantly on $\mathfrak{h}$.)
- The choice of a base point $\tau_0 \in \mathfrak{h}$ determines an isomorphism
$$\mathrm{SL}_2(\mathbb{Z}) \to \pi_1(Y, \tau_0).$$
The element γ maps to the loop that corresponds to the unique homotopy class c_γ of paths from τ_0 to $\gamma\tau_0$ in $\mathfrak{h}$.
- The most natural choice of a base point is the tangent vector $\partial/\partial q$ at the cusp. (That is, $i[y, \infty)$, $y \gg 0$.)
- There are two natural choices for the relative completion of $\mathrm{SL}_2(\mathbb{Z})$ — the "small" and the "large".

The "small" completion of $\mathrm{SL}_2(\mathbb{Z})$

- Here $R = \mathrm{SL}_{2/\mathbb{Q}}$ and ρ is the inclusion. It is an extension

$$1 \to \mathcal{U} \to \mathcal{G} \to \mathrm{SL}_2 \to 1.$$

Its coordinate ring $\mathcal{O}(\mathcal{G})$ consists of all closed iterated integrals of elements of

$$\left\{ \begin{array}{l} \text{“smooth modular forms” of level 1 on} \\ \mathfrak{h} \text{ with a “log singularity” at the cusp} \end{array} \right\}$$

It contains all iterated Shimura integrals of level 1. The homomorphism

$$\tilde{\rho} : \mathrm{SL}_2(\mathbb{Z}) \to \mathcal{G}(\mathbb{C})$$

takes $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ to

$$\left\{ \begin{array}{l} \text{the maximal ideal of such iterated integrals} \\ \text{that vanish on the path } c_\gamma \text{ from } \tau_0 \text{ to } \gamma\tau_0 \end{array} \right\}$$

The “large” completion of $\mathrm{SL}_2(\mathbb{Z})$

- Every profinite group can be regarded as a pro-reductive group in natural way.
- To get the *large completion*, take $R = \mathrm{SL}_{2/\mathbb{Q}} \times \mathrm{SL}_2(\widehat{\mathbb{Z}})$ and ρ to be the diagonal inclusion.
- It is an extension

$$1 \to \widehat{\mathcal{U}} \to \widehat{\mathcal{G}} \to \mathrm{SL}_{2/\mathbb{Q}} \times \mathrm{SL}_2(\widehat{\mathbb{Z}}) \to 1.$$

- Its coordinate ring contains iterated Shimura integrals of all levels as well as all continuous functions $\mathrm{SL}_2(\widehat{\mathbb{Z}}) \to \mathbb{k}$.
- $\mathcal{G}$ is a quotient of $\widehat{\mathcal{G}}$.

Motivic structures

- For each choice of $\tau_0 \in \mathfrak{h}$, there are natural MHSs on $\mathcal{O}(\mathcal{G})$ and $\mathcal{O}(\widehat{\mathcal{G}})$. We will take $\vec{v} := \partial/\partial q$, the most natural choice.
- After tensoring with $\mathbb{Q}_\ell$, there is a natural $G_{\mathbb{Q}}$ action. So $\mathcal{O}(\widehat{\mathcal{G}})$ looks like a motive as it has compatible Hodge and étale realizations.
- There is also a natural $\mathbb{Q}$ DR structure, so we have periods.
- The periods $\mathcal{O}(\widehat{\mathcal{G}})$ contain Brown's multiple modular values (MMVs), which are iterated Shimura integrals evaluated on the imaginary axis. This is c_γ for

$$\gamma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

- The graded quotients of the weight filtration of both are sums of "motives" of the form

$$\mathrm{Sym}^{r_1} V_{f_1} \otimes \cdots \otimes \mathrm{Sym}^{r_m} V_{f_m} \otimes S^n H(d) \qquad (*)$$

  where V_f denotes the motive (Hodge structure, Galois representation) associated to a Hecke eigen form f and $H = H^1(E_{\tau_0})$. (Note that $H_{\partial/\partial q} = \mathbb{Z}(0) \oplus \mathbb{Z}(1)$.)
- Thus (after rearranging and taking SL_2 invariants) one generates lots of extensions of $\mathbb{Q}(0)$ by the "motives"

$$\mathrm{Sym}^{r_1} V_{f_1} \otimes \cdots \otimes \mathrm{Sym}^{r_m} V_{f_m}(d).$$

- Do these conform to Beilinson's conjectures? Wrinkle: Brown [1,§17] observed that they cannot quite conform.
- Does the Hecke algebra act on $\mathbb{Z}[\mathrm{SL}_2(\mathbb{Z})]$, $\mathcal{O}(\widehat{\mathcal{G}})$ or its periods? If so, can one explain Brown's computation of periods of twice iterated integrals of Eisenstein?

Outline

Étale correspondences

Call a correspondence

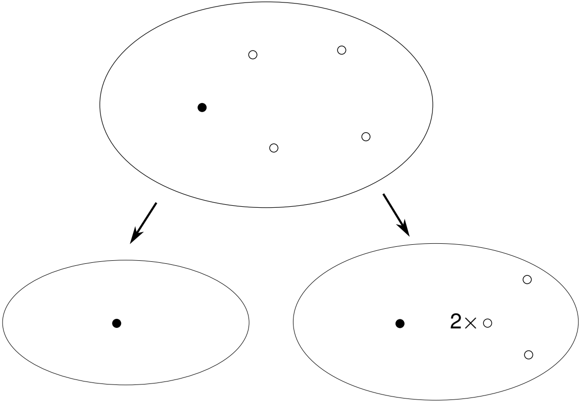
$$F = \left[\begin{array}{ccc} & U & \\ & {}^{g}\swarrow \quad \searrow^{h} & \\ X & & Y \end{array} \right]$$

étale if both g and h are étale. It acts on (say) homology by the formula

$$F_* = h_* \circ g^*.$$

Do étale correspondences act on $\mathbb{Z}\pi_1(X, x)$?

An example to illustrate the problem:



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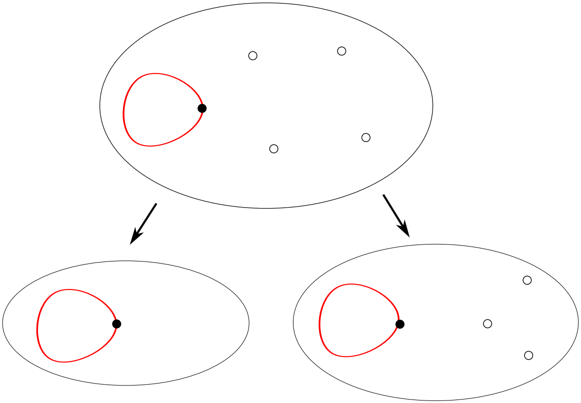

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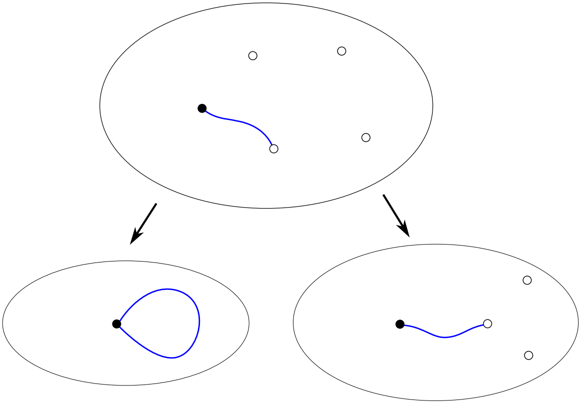

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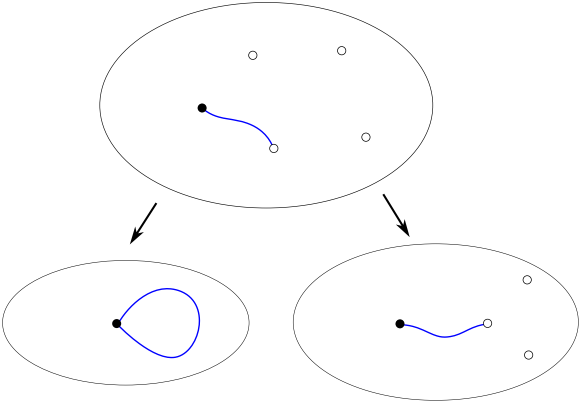


This problem can be avoided by working with conjugacy classes — equivalently, with unbased loops.

For a topological space define

$$\lambda(X) = \{\text{free homotopy classes of maps } S^1 \to X\}.$$

For a group Γ define

$$\boldsymbol{\lambda}(\Gamma) = \{\text{conjugacy classes in } \Gamma\}.$$

If X is connected, then $\lambda(X) = \boldsymbol{\lambda}(\pi_1(X, x))$.

Denote the free $\Bbbk$ modules they generate by

$$\Bbbk\lambda(X) \text{ and } \Bbbk\boldsymbol{\lambda}(\Gamma).$$

These are clearly covariant under maps $Y \to X$ and group homomorphisms $\Gamma' \to \Gamma$.

Pullback

When $\pi : Y \to X$ is étale, there is a pullback map

$$\pi^* : \mathbb{Z}\lambda(X) \to \mathbb{Z}\lambda(Y).$$

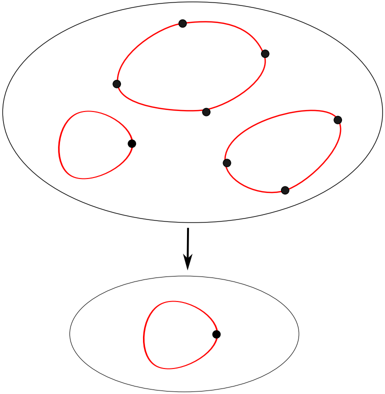
To compute its value on $\alpha : S^1 \to X$ observe that the pullback covering $\alpha^* Y \to S^1$

$$\begin{array}{ccc} \alpha^* Y & \longrightarrow & Y \\ \downarrow & & \downarrow \pi \\ S^1 & \xrightarrow{\alpha} & X \end{array}$$

is a disjoint union of circles $\tilde{\alpha}_j : S^1 \to Y$. Define

$$\pi^* \alpha = \sum_j \tilde{\alpha}_j \in \lambda(Y).$$

An example:



Observe that $\deg \pi = 8$ and that $\pi_* \pi^* \alpha = \alpha + \alpha^3 + \alpha^4 \neq 8\alpha$.

Étale correspondences act on $\mathbb{Z}\lambda(X)$

Proposition

An étale, the correspondence

$$F = [X \xleftarrow{g} U \xrightarrow{h} Y]$$

induces a homomorphism $F_* : \mathbb{Z}\lambda(X) \to \mathbb{Z}\lambda(Y)$*. Namely, the composite*

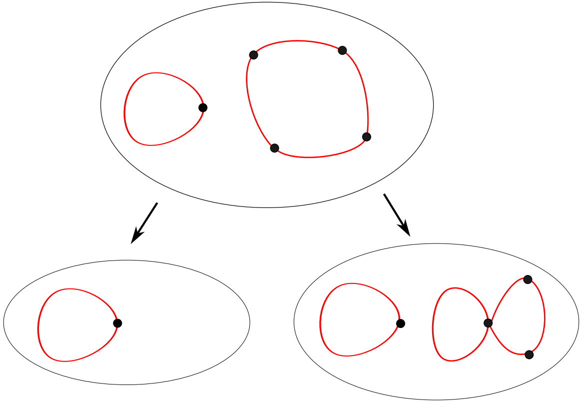
$$\mathbb{Z}\lambda(X) \xrightarrow{g^*} \mathbb{Z}\lambda(U) \xrightarrow{h_*} \mathbb{Z}\lambda(Y).$$

If F *and* G *are composable étale correspondences, then* $G \circ F$ *is étale and* $(G \circ F)_* = G_* \circ F_*$.

An example:

$$\alpha \overset{g^*}{\longmapsto} g^*\alpha \overset{h_*}{\longmapsto} h_* \circ g^*\alpha$$

$$\alpha \overset{F_*}{\dashrightarrow} h_* \circ g^*\alpha$$



Hecke operators

Denote the moduli space of degree $N \geq 1$ isogenies $E' \to E$ of elliptic curves by Cov_N. The Hecke operator T_N is the étale correspondence

$$T_N := \begin{bmatrix} & \mathrm{Cov}_N & \\ \swarrow & & \searrow \\ Y & & Y \end{bmatrix}, \quad \begin{bmatrix} & [E' \to E] & \\ \swarrow & & \searrow \\ [E] & & [E'] \end{bmatrix}$$

When $N = p$, a prime, $\mathrm{Cov}_N = Y_0(p)$.

Proposition

The Hecke operators T_N, $N \in \mathbb{N}$, act on $\mathbb{Z}\boldsymbol{\lambda}(\mathrm{SL}_2(\mathbb{Z}))$. The operators T_N and T_M commute when M and N are relatively prime.

For each prime p define

$$\mathbf{e}_p : \mathbb{Z}\boldsymbol{\lambda}(\mathrm{SL}_2(\mathbb{Z})) \to \mathbb{Z}\boldsymbol{\lambda}(\mathrm{SL}_2(\mathbb{Z}))$$

by $\mathbf{e}_p = \pi_*\pi^* - \mathrm{id}$ where $\pi : Y_0(p) \to Y$. It is a non-commutative generalization of p.

Theorem

The actions of the Hecke operators T_{p^n} on $\mathbb{Z}\boldsymbol{\lambda}(\mathrm{SL}_2(\mathbb{Z}))$ satisfy

$$T_{p^n} \circ T_p = T_{p^{n+1}} + T_{p^{n-1}} \circ \mathbf{e}_p. \qquad (\dagger)$$

Note that T_p does *not* commute with $\mathbf{e}_p$. Since

$$T_p^2 = T_{p^2} + \mathbf{e}_p$$

we have

$$[T_p, T_{p^2}] = -[T_p, \mathbf{e}_p] \neq 0.$$

Generalized Hecke algebra

Each $\mathbf{e}_p$ satisfies a polynomial relation. Let $m_p(x)$ be the monic generator of the ideal

$$\{h(x) \in \mathbb{Q}[x] : h(\mathbf{e}_p) = 0\} \subset \mathbb{Q}[x].$$

Then

$$m_p(x) = \begin{cases} x(x+1)(x-2) & p = 2 \\ x(x^2-1)(x-p) & p \text{ odd.} \end{cases}$$

Define a non commutative Hecke algebra $\widehat{\mathbb{T}}$ to be the restricted tensor product of the non-commutative algebras

$$\widehat{\mathbb{T}}_p := \mathbb{Z}\langle T_p, \mathbf{e}_p \rangle / (m_p(\mathbf{e}_p)).$$

For $m > 1$, define $T_{p^m} \in \widehat{\mathbb{T}}_p$ using (†). Then $\widehat{\mathbb{T}}$ acts on $\mathbb{Z}[\mathrm{SL}_2(\mathbb{Z})]$.

Dual version

Set $\mathscr{C}\ell(\widehat{\mathcal{G}}) = \mathcal{O}(\widehat{\mathcal{G}})^{\mathrm{conj}} = \{$class functions $\widehat{\mathcal{G}} \to \mathbb{C}\}$.

Length 0: generated by $\mathtt{tr}$ and characters of $\mathrm{SL}_2(\mathbb{Z}/N)$.

Length 1: Suppose f is a modular form of weight $2n$, level 1. Denote the corresponding $S^{2n-2}H$ valued form by ω_f. Since

$$\mathcal{O}(\mathrm{SL}_2) = \bigoplus_{m\geq 0} (\mathrm{End}\, S^m H)^\vee \subset \mathcal{O}(\widehat{\mathcal{G}}), \qquad \text{(Peter–Weyl)}$$

$\mathcal{O}(\mathrm{SL}_2)$ contains countable copies of $S^{2n-2}H$ for each $n \geq 0$. Fix an SL_2 invariant function $\varphi : S^{2n-2}H \to \mathcal{O}(\mathrm{SL}_2)$. Set

$$\omega_f(\varphi) = \varphi \circ \omega_f, \quad \text{a 1-form with values in } \mathcal{O}(\mathrm{SL}_2).$$

Then

$$F_{f,\varphi} : \alpha \mapsto \left\langle \int_\alpha \omega_f(\varphi), \alpha \right\rangle \quad \text{is in } \mathscr{C}\ell(\widehat{\mathcal{G}}).$$

Proposition

The ring $\mathscr{C}\ell(\widehat{\mathcal{G}})$ of class functions on $\widehat{\mathcal{G}}$ carries a natural mixed Hodge structure as well as a natural $G_{\mathbb{Q}}$ action after tensoring with $\mathbb{Q}_\ell$. Neither depends on the choice of the base point.

The weight graded quotients of $\mathscr{C}\ell(\widehat{\mathcal{G}}) \otimes \mathbb{R}$ are sums of "motives" of the form

$$\mathrm{Sym}^{r_1} V_{f_1} \otimes \cdots \otimes \mathrm{Sym}^{r_m} V_{f_m}(d)$$

where $f_1, \ldots, f_m$ are modular forms of arbitrary weight and level.

Theorem

Each Hecke correspondence T_N induces a (dual) Hecke operator

$$\check{T}_N : \mathscr{C}\ell(\widehat{\mathcal{G}}) \to \mathscr{C}\ell(\widehat{\mathcal{G}})$$

which is a morphism of mixed Hodge structures and, after tensoring with $\mathbb{Q}_\ell$, is Galois equivariant. This action is dual to the action on $\mathbb{Z}\boldsymbol{\lambda}(\mathrm{SL}_2(\mathbb{Z}))$ in the sense that

$$\langle \check{T}_N F, \alpha \rangle = \langle F, T_N(\alpha) \rangle.$$

In addition, the Adams operations

$$\psi^m : \mathscr{C}\ell(\widehat{\mathcal{G}}) \to \mathscr{C}\ell(\widehat{\mathcal{G}})$$

defined by

$$\langle \psi^m F, \alpha \rangle := \langle F, \alpha^m \rangle$$

are morphisms of MHS and commute with the Galois action.

A simple example

The Adams operator ψ^m acts on the periods of $\mathscr{C}\ell(\widehat{\mathcal{G}})$ by

$$\psi^m : \langle F, \alpha\rangle \mapsto \langle F, \alpha^m\rangle.$$

If f is a modular form of weight $2n$ and level 1 and if $\alpha \in \mathrm{SL}_2(\mathbb{Z})$ acts on $\mathbb{P}^1(\mathbb{F}_p)$ with one orbit, then

$$T_p\langle F_{f,\varphi}, \alpha\rangle := \langle F_{f,\varphi}, T_p(\alpha)\rangle = \frac{\psi^{p+1}}{p^{n-1}(p+1)}\langle F_{T_p(f),\varphi}, \alpha\rangle.$$

So if $f = \sum a_n q^n$ is a normalized Hecke eigenform, then $\langle F_{f,\varphi}, \alpha\rangle$ will be an "eigen-period" of T_p with "eigenvalue"

$$\frac{a_p}{p^{n-1}(p+1)}\,\psi^{p+1}.$$

Two questions

1. Does $\mathscr{C}\ell(\widehat{\mathcal{G}})$ generate MMM? This is closely related to Brown's question. As mentioned before, this generation statement is slightly inconsistent with Beilinson's conjecture.

2. Have we thrown out the baby with the bathwater when we replaced $\mathcal{O}(\widehat{\mathcal{G}})$ by $\mathscr{C}\ell(\widehat{\mathcal{G}})$? At the other extreme, do they generate the same tannakian subcategory of (say) MHS, in which case there their rings of periods are the same?

Mumford–Tate groups

The category $\mathsf{MHS}_{\mathbb{Q}}$ of $\mathbb{Q}$-Mixed Hodge structures is a $\mathbb{Q}$-linear tannakian category. It is therefore equivalent to the category of representations of an affine $\mathbb{Q}$-group $\pi_1(\mathsf{MHS})$. The Mumford–Tate group of a $\mathbb{Q}$-MHS V is the image of the homomorphism

$$\pi_1(\mathsf{MHS}) \to \mathsf{Aut}\, V_{\mathbb{Q}}.$$

It is an affine algebraic group. Denote it by MT_V.

Since $\mathscr{C}\ell(\widehat{\mathcal{G}}) \subset \mathcal{O}(\widehat{\mathcal{G}})$, the homomorphism

$$\mathrm{MT}_{\mathcal{O}(\widehat{\mathcal{G}})} \to \mathrm{MT}_{\mathscr{C}\ell(\widehat{\mathcal{G}})}$$

is surjective. The question is whether this homomorphism is also injective.

The unipotent case

I am inclined to think that it is based on the unipotent case.

Theorem

If X is a smooth affine curve and $\vec{v}$ a non-zero tangent vector at a cusp, then

$$W_{-3}\mathrm{MT}_{\mathcal{O}(\pi_1^{\mathrm{un}}(X,\vec{v}))} \to W_{-3}\mathrm{MT}_{\mathscr{C}\ell(\pi_1^{\mathrm{un}}(X,\vec{v}))}$$

is an isomorphism.

The proof uses the unipotent completions of the Goldman bracket

$$\{\ ,\ \} : \mathbb{Z}\lambda(X) \otimes \mathbb{Z}\lambda(X) \to \mathbb{Z}\lambda(X),$$

which makes $\mathbb{Z}\lambda(X)$ into a Lie algebra, and the *Kawazumi–Kuno action*

$$\kappa : \mathbb{Z}(\lambda(X)) \to \mathrm{Der}\,\mathbb{Z}\pi_1(X, \vec{v}).$$

References

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[2] R. Hain: *The Hodge–de Rham theory of modular groups*. Recent advances in Hodge theory, 422–514, London Math. Soc. Lecture Note Ser., 427, 2016. [arXiv:1403.6443]

[3] R. Hain: *Hecke Actions on loops and periods of iterated Shimura integrals*, [arXiv:2303.00143]