Elliptic Motives and Multiple Zeta Values

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Outline

Acknowledgments

Related Work

- Beilinson and Levin (1994): *Elliptic polylogarithms*
- Manin (2005): *Iterated Shimura Integrals*
- Levin and Racinet (2007): *Towards multiple elliptic polylogarithms*
- Calaque, Enriques, Etingof (2007): *Universal elliptic KZB equation*

Collaborators

- Makoto Matsumoto (Hiroshima): Galois theory
- Aaron Pollack (Duke): algebraic de Rham aspects
- Greg Pearlstein (Michigan State U.): Hodge theory
- Tomohide Terasoma (Tokyo): Hodge theory

Outline

Unipotent completion

For F a field of characteristic zero and Γ a finitely generated discrete group, have group algebra $F\Gamma$, its augmentation $\epsilon : F\Gamma \to F$ and its augmentation ideal $J = \ker\{\epsilon : F\Gamma \to F\}$. The J-adic completion of $F\Gamma$ is

$$F\Gamma^{\wedge} : \varprojlim_n F\Gamma/J^n.$$

This is a complete Hopf algebra.

Unipotent Completion

The set of F-points of the unipotent completion of Γ F is the prounipotent group

$$\Gamma^{\mathrm{un}}(F) = \{\text{group-like elets of } F\Gamma^{\wedge}\} = \{x \in 1 + J^{\wedge} : \Delta x = x \otimes x\}.$$

Example: completion of a free group

If Γ is the free group $\Gamma = \langle u, v \rangle$ and $ad - bc \neq 0$, then

$$\theta : F\Gamma^\wedge \xrightarrow{\simeq} F\langle\langle X, Y \rangle\rangle$$

is a (complete) Hopf algebra isomorphism when

$$\theta(u) = \exp U \text{ and } \theta(v) = \exp V$$

where $U, V \in \mathbb{L}(X, Y)^\wedge$ and

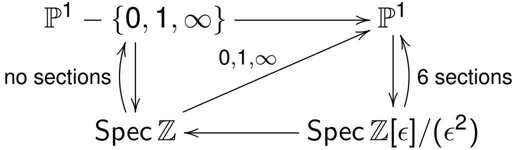
$$U \equiv aX + bY \text{ and } V \equiv cX + dY \mod J^2.$$

Theta induces an isomorphism $\Gamma^{\mathrm{un}}(F) \xrightarrow{\simeq} \exp \mathbb{L}(X, Y)^\wedge$

Outline

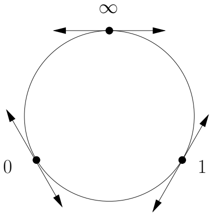
Integral base points

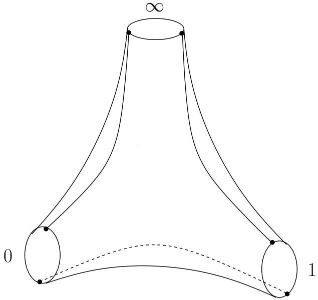
$\mathbb{P}^1 - \{0, 1, \infty\}$ has no points over $\mathbb{Z}$, and $\mathbb{P}^1$ has only 6 everywhere non-zero tangent vectors over $\mathbb{Z}$:



These tangent vectors are $\partial/\partial z \in T_0\mathbb{P}^1$ and its translates under $\operatorname{Aut}(\mathbb{P}^1, \{0, 1, \infty\})$.

Real blow-up





Drinfeld Associator

The Drinfeld associator $\Phi(Y, Z) \in \mathbb{C}\langle\langle Y, Z\rangle\rangle$ is the regularized value of the parallel transport, along the unit interval, of the *KZ*-connection

$$\nabla f = df - f\omega$$

where

$$\omega = \frac{dz}{z} Y + \frac{dz}{z-1} Z \in H^0\big(\Omega^1_{\mathbb{P}^1}(\log\{0, 1, \infty\}\big) \otimes \mathbb{L}(Y, Z).$$

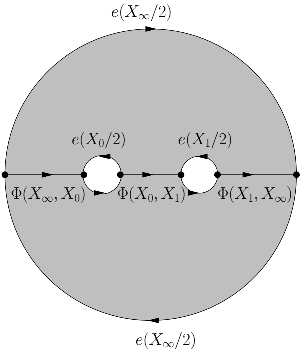
Its coefficients are multi-zeta numbers:

$$\begin{aligned}\Phi(Y, Z) = 1 + \zeta(2)[Y, Z] - \zeta(3)[Y, [Y, Z]] + \zeta(1, 2)[[Y, Z], Z] \\ - \zeta(4)[Y, [Y, [Y, Z]]] - \zeta(1, 3)[Y, [[Y, Z], Z]] \\ + \zeta(1, 1, 2)[[[Y, Z], Z], Z] + \frac{1}{2}\zeta(2)^2[Y, Z]^2 + \cdots\end{aligned}$$

Fundamental groupoid

The Drinfeld associator defines a functor from the fundamental groupoid of $\mathbb{P}^1 - \{0, 1, \infty\}$ to the group-like elements of

$$\mathbb{C}\langle\langle X_0, X_1\rangle\rangle \cong \mathbb{C}\langle\langle X_0, X_1, X_\infty\rangle\rangle/(X_0 + X_1 + X_\infty).$$



Outline

Punctured elliptic curves

Want to study the motivic structure on the unipotent completion of a punctured elliptic curve.

- Suppose $E = (E, 0)$ is an elliptic curve over $\mathbb{C}$.
- Set $E' := E - \{0\}$
- For $x \in E'$, have $\pi_1(E', x)$, a free group of rank 2.

Likewise, for $\vec{v} \in T_{\mathrm{id}} E - \{0\}$, we have $\pi_1(E', \vec{v})$

The fundamental torsor

We would like to generalize the genus 0 story to genus 1. But in genus 1, there are many elliptic curves. So we consider all at once:

$$\mathcal{E} \to \mathcal{M}_{1,1}$$

is the universal punctured elliptic curve. Over $\mathcal{E}'$, the universal punctured elliptic curve, we have the torsor

$$\boldsymbol{\mathcal{P}} \to \mathcal{E}'$$

whose fiber over $[E, x]$ is the unipotent completion of $\pi_1(E', x)$.

The universal elliptic curve

To describe $\mathcal{P}$, we need an explicit description of $\mathcal{E}$. It is the orbifold quotient

$$\mathcal{E} = \big(\mathrm{SL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2\big)\backslash\big(\mathbb{C} \times \mathfrak{h}\big)$$

where

$$(m, n) : (\xi, \tau) \mapsto (\xi + m\tau + n, \tau)$$

and

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : (\xi, \tau) \mapsto \big(\xi/(c\tau + d), (a\tau + b)/(c\tau + d)\big).$$

Outline

The Jacobi form $F(\xi, \eta, \tau)$

A certain Jacobi modular form F is fundamental in writing down the connection on $\mathcal{P}$. Geometrically, F is a section of a line bundle over the total space of

$$\overline{\mathcal{E}} \times_{\overline{\mathcal{M}}_{1,1}} \overline{\mathcal{E}} \to \overline{\mathcal{M}}_{1,1}.$$

whose divisor is

$$\iota^* \Delta - Z_1 - Z_2$$

where $\iota^* \Delta$ is the graph of the elliptic involution $\iota : x \mapsto -x$ and Z_j is the locus $z_j = 0$, where (z_1, z_2) are the fiber coordinates.

Formula for F

F is a meromorphic function on $\mathbb{C} \times \mathbb{C} \times \mathfrak{h}$, first defined by Kronecker (1881) and rediscovered by Zagier (1991):

$$F(\xi, \eta, \tau) = \frac{\theta'(0;\tau)\theta(\xi+\eta;\tau)}{\theta(\xi,\tau)\theta(\eta,\tau)}$$
$$= \frac{1}{\xi} + \frac{1}{\eta} - 2\sum_{r,s=0}^{\infty} (2\pi i)^{1+\max\{r,s\}} \left(\frac{\partial}{\partial\tau}\right)^{\min\{r,s\}} G_{|r-s|+1}(\tau)\, \frac{\xi^r}{r!}\frac{\eta^s}{s!},$$

where $G_{\mathrm{odd}} = 0$ and

$$G_{2m}(\tau) = -\frac{B_{2m}}{4m} + \sum_{n=1}^{\infty} \sigma_{2m-1}(n) q^n$$

with $q = \exp(2\pi i\tau)$. The function F is modular in τ and is elliptic in ξ and η.

The Levin-Racinet connection

Denote the group-like elements of $\mathbb{C}\langle\langle T, A\rangle\rangle$ by $\mathcal{P}$ and its Lie algebra $\mathbb{L}(T, A)^\wedge$ by $\mathfrak{p}$. Then the 1-form

$$\omega = 2\pi i A \frac{\partial}{\partial T}\, d\tau + \psi + \nu \in H^0(\Omega^1_{\mathbb{C}\times\mathfrak{h}}) \hat{\otimes} \operatorname{Der} \mathfrak{p}$$

defines a connection on $\mathcal{P} \times \mathbb{C} \times \mathfrak{h} \to \mathfrak{h}$ by $\nabla f = df + \omega f$, where

$$\psi = 4\pi i \sum_{m \geq 1} \left[\frac{G_{2m+2}(\tau)}{(2m)!} d\tau \sum_{\substack{j+k=2m+1 \\ j,k>0}} (-1)^j [\operatorname{ad}_T^j(A), \operatorname{ad}_T^k(A)] \frac{\partial}{\partial A} \right]$$

and

$$\nu = \operatorname{ad}_T F(\xi, \operatorname{ad}_T/2\pi i, \tau)(A)\, d\xi + \left(\frac{1}{\operatorname{ad}_T} + \operatorname{ad}_T \frac{\partial F}{\partial T}(\xi, \operatorname{ad}_T/2\pi i, \tau) \right)(A)\, d\tau.$$

Description of $\mathcal{P}^{\mathrm{DR}}$

The action of $\mathrm{SL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$ on $\mathbb{C} \times \mathfrak{h}$ can be lifted to this bundle. The quotient is, by definition, $\mathcal{P}^{\mathrm{DR}}$. For $\gamma \in \mathrm{SL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$ define

$$\gamma(u, \xi, \tau) = \big(\widetilde{M}_\gamma(\xi, \tau)u, g(\xi, \tau)\big)$$

where $\widetilde{M}_\gamma(\xi, \tau) = e^{-m \operatorname{ad}_T}$ when $\gamma = (m, n) \in \mathbb{Z}^2$ and, when $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$,

$$\widetilde{M}_\gamma(\xi, \tau) = M_\gamma(\tau) \circ \exp\big(c\xi \operatorname{ad}_T /(c\tau + d)\big)$$

where

$$\mathcal{M}_\gamma(\tau) : \begin{cases} A & \mapsto (c\tau + d)^{-1} A + cT \\ T & \mapsto (c\tau + d) T \end{cases}$$

Flatness and descent

Theorem (Levin-Racinet)

1. *The connection is invariant: for all* $\gamma \in \mathrm{SL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$

$$\gamma^*\omega = \mathrm{Ad}(\gamma)\omega - d\widetilde{M}_\gamma \widetilde{M}_\gamma^{-1}$$

2. *The connection is flat:*

$$d\omega + \frac{1}{2}[\omega, \omega] = 0.$$

So the connection descends to a flat meromorphic connection on the principal $\mathcal{P}^{\mathrm{DR}}$ bundle $(\mathrm{SL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2)\backslash(\mathcal{P} \times \mathbb{C} \times \mathfrak{h})$ over $\mathcal{E}$.

On each 2-pointed elliptic curve $(E_\tau, 0, x)$, the connection restricts to the flat connection

$$\nabla = d + \operatorname{ad}_T F(\xi, \operatorname{ad}_T / 2\pi i, \tau)(A)\, d\xi.$$

Parallel transport induces a homomorphism

$$\pi_1(E'_\tau, x) \to \mathcal{P}$$

that is an isomorphism as

$$\mathbf{a} \mapsto 2\pi i A \text{ and } \mathbf{b} \mapsto 2\pi i \tau A - T \bmod [\mathcal{P}, \mathcal{P}].$$

Theorem (Rigidity)

The flat bundles $\mathcal{P}^{\mathrm{DR}}$ and $\mathcal{P}$ are isomorphic.

Outline

Introduction

Our goals are to:

1. find suitable integrally defined base points that are everywhere non-zero;
2. for these, compute the monodromy isomorphism
$$\pi_1(E_o', x_o)^{\mathrm{un}} \to \mathcal{P};$$
3. "compute" the corresponding monodromy representation
$$\pi_1(\mathcal{M}_{1,\vec{1}}, o) \to \operatorname{Aut} \mathcal{P}.$$

To do this, we will localize the LR connection about E_o, which will turn out to be the first order Tate curve.

Integral base points

The natural coordinate on $\overline{\mathcal{M}}_{1,1}$ about the cusp is $q := \exp(2\pi i\tau)$. The fiber of $\overline{\mathcal{E}} \to \mathcal{M}_{1,1}$ over the cusp $q = 0$ is the nodal cubic. This can be identified with $\mathbb{P}^1$ with $0 \sim \infty$. There is a unique parameter w on the nodal cubic that takes the value 1 at the identity.

- There are no integral points $\operatorname{Spec}\mathbb{Z} \to \mathcal{M}_{1,\vec{1}}$.
- The only everywhere non-zero integral tangent vectors are $\operatorname{Spec}\mathbb{Z}[\epsilon]/(\epsilon^2) \to \mathcal{M}'_{1,\vec{1}}$ are $\pm\frac{\partial}{\partial q} \pm \frac{\partial}{\partial w}$.

Denote the fiber of $\mathcal{E}$ over $\partial/\partial q$ by $E_{\partial/\partial q}$. It is the first order Tate curve. As a base point we choose $\vec{v} = \partial/\partial q + \partial/\partial w$.

Tangential base points

Since all of our base points will be tangential, we will restrict the torsor $\mathcal{P}$ to the punctured relative tangent bundle of the identity section:

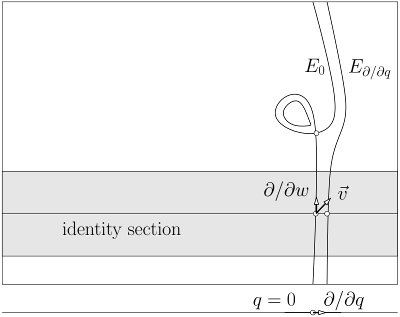
$$\mathcal{M}_{1,\vec{1}} \hookrightarrow \mathcal{E}',$$

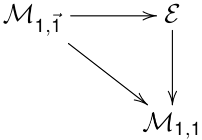
This is the $\mathbb{C}^*$-bundle associated to the dual of the Hodge bundle over $\mathcal{M}_{1,1}$:

$$\mathcal{M}_{1,\vec{1}} = \{(E,\vec{v})\}/\text{isomorphism} = \mathrm{SL}_2(\mathbb{Z})\backslash(\mathbb{C}\times\mathfrak{h})$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : (\xi,\tau) \mapsto \big(\xi/(c\tau+d),(a\tau+b)/(c\tau+d)\big)$$

Topologists’ picture





Restriction of LR-connection to $E_{\partial/\partial q}$

This is $\nabla = d + \omega_0$, where

$$\omega_0 = N_q \frac{dq}{q} + [T,A]\frac{dw}{w-1} + \left(\frac{\mathrm{ad}_T}{e^{\mathrm{ad}_T}-1}\right)(A)\frac{dw}{w}.$$

where

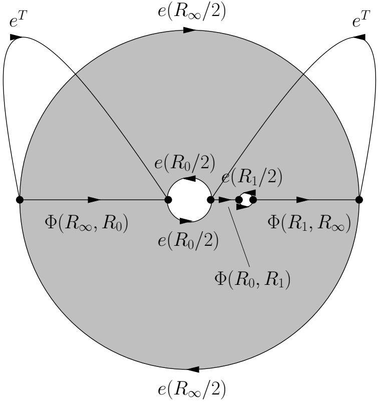
$$N_q = A\frac{\partial}{\partial T} + \sum_{m=1}^{\infty} \frac{B_{2m}}{2m(2m-2)!}\left(\mathrm{ad}_T^{2m-1}(A) - \sum_{\substack{j+k=2m-1\\ j>k>0}} (-1)^j [\mathrm{ad}_T^j(A), \mathrm{ad}_T^k(A)]\frac{\partial}{\partial A}\right)$$

Set

$$R_0 = \left(\frac{\mathrm{ad}_T}{e^{\mathrm{ad}_T}-1}\right)(A), \quad R_1 = [T,A], \quad R_\infty = \left(\frac{\mathrm{ad}_T}{e^{-\mathrm{ad}_T}-1}\right)(A)$$

Note that R_0 is the generating function for Bernoulli numbers.

Fundamental group of $E'_{\partial/\partial q}$



Identify outer and inner circles to obtain $E_{\partial/\partial q}$. The diagram gives a well defined homomorphism $\pi_1(E'_{\partial/\partial q}, \vec{v}) \to \mathcal{P}$ because of the cylinder relation:

$$e^T\, e(\lambda R_0)\, e^{-T}\, e(\lambda R_\infty) = 1$$

which holds for all $\lambda \in \mathbb{C}$.

Monodromy computation

Define $\widetilde{\mathrm{SL}}_2(\mathbb{Z})$ to be the inverse image of $\mathrm{SL}_2(\mathbb{Z})$ in the universal covering group of $\mathrm{SL}_2(\mathbb{R})$. It is an extension

$$0 \to \mathbb{Z} \to \widetilde{\mathrm{SL}}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}) \to 1.$$

and has presentation $\langle S, U : S^2 = U^3 \rangle$. It is isomorphic to B_3.

$\pi_1(\mathcal{M}_{1,\vec{1}}, \vec{v})$ is isomorphic to $\widetilde{\mathrm{SL}}_2(\mathbb{Z})$. It is generated by:

1. the Dehn twist about $q = 0$, which acts as $\exp(2\pi i N_q)$;
2. any lift σ of $\tau \mapsto -1/\tau$ to $\widetilde{\mathrm{SL}}_2(\mathbb{Z})$, which acts via a formal series of iterated integrals of Eisenstein series, and acts via a representation of one of Manin's non-abelian modular symbols.

Remarks

1. The condition that $\exp(N_q) : \mathcal{P} \to \mathcal{P}$ preserve the image of $\pi_1(E'_{\partial/\partial q}, \vec{v})$ and fix the image of $\pi_1(\mathbb{P}^1 - \{0, 1, \infty\}, \partial/\partial w)$ appears to be strong. For example, it implies that $N_q(\Phi(R_0, R_1))\Phi(R_1, R_0)$ must commute with $\exp(2\pi i R_0)$. Pollack and I are currently investigating whether this implies, for example, the double shuffle relations.
2. That the monodromy $\Theta(\sigma) : \mathcal{P} \to \mathcal{P}$ preserves the image of $\pi_1(E'_{\partial/\partial q}, \vec{v})$ appears to be deeper. This may impose relations on MZN, but more likely it will give a computation of Manin's non-abelian modular symbols of Eisenstein series in terms of MZN. If all periods of mixed Tate motives over $\mathbb{Z}$ are MZNs, then (I believe) the coefficients of the Manin symbol will be MZNs.

Outline

Informal description

A *mixed elliptic motive over* $\mathbb{Z}$ should be a "motivic local system" $\mathbb{V}$ of $\mathbb{Q}$-vector spaces over $\mathcal{M}_{1,1/\mathbb{Z}}$ with a weight filtration $W_\bullet$ that satisfies:

1. each weight graded quotient of $\mathbb{V}$ is a sum of the simple local systems $S^n\mathbb{H}(m)$, where $\mathbb{H} = R^1\pi_*\mathbb{Q}(0)$ and $\pi : \mathcal{E} \to \mathcal{M}_{1,1}$ is the universal elliptic curve;
2. the fiber $V_{\partial/\partial q}$ of $\mathbb{V}$ over $\partial/\partial q$ is in MTM($\mathbb{Z}$).

A basic example of an Ind-object of MEM($\mathbb{Z}$) should be the local system consisting of the coordinate rings of the unipotent completions of the $\pi_1(E', \vec{v})$

Mixed elliptic motives

Conjecture

There is a tannakian category $\mathsf{MEM}(\mathbb{Z})$ of mixed elliptic motives over $\operatorname{Spec}\mathbb{Z}$ that contains $\mathsf{MTM}(\mathbb{Z})$ as a full subcategory and satisfies:

1. There is a fiber functor $\mathsf{MEM}(\mathbb{Z}) \to \mathsf{MTM}(\mathbb{Z})$ whose restriction to $\mathsf{MTM}(\mathbb{Z})$ is the identity.
2. There are realization functors, *Betti, Hodge, ℓ-adic, $\mathbb{Q}$-de Rham,...* to $\mathbb{Q}$-local systems, VMHS, lisse ℓ-adic sheaves, $\mathbb{Q}$-connections, ...over $\mathcal{M}_{1,1}$ that commute with the fiber functor to $\mathsf{MTM}(\mathbb{Z})$.
3. All weight graded quotients are direct sums of Tate twists of symmetric powers of $\mathbb{H}$.

Outline

Fundamental group of MEM($\mathbb{Z}$)

This will be a proalgebraic $\mathbb{Q}$-group that is a split extension

$$1 \to \mathcal{G} \to \pi_1(\text{MEM}) \to \pi_1(\text{MTM}) \to 1.$$

where $\mathcal{G}$ (the "geometric fundamental group") is an extension

$$1 \to \mathcal{U} \to \mathcal{G} \to \mathrm{SL}_2 \to 1$$

with $\mathcal{U}$ prounipotent. There will be Zariski dense repns:

$$\mathrm{SL}_2(\mathbb{Z}) \to \mathcal{G}(\mathbb{Q}) \text{ and } \pi_1(\mathcal{M}_{1,1/\mathbb{Q}}) \to \pi_1(\text{MEM})(\mathbb{Q}_\ell).$$

Thus $\mathcal{G}$ will be a quotient of the relative unipotent completion of $\mathrm{SL}_2(\mathbb{Z})$ and $\pi_1(\text{MEM})$ will be a quotient of the ℓ-adic relative unipotent completion of $\pi_1(\mathcal{M}_{1,1/\mathbb{Q}})$. The coordinate ring $\mathcal{O}(\mathcal{G})$ will be an Ind object of MTM($\mathbb{Z}$).

Outline

The Lie algebra of $\mathcal{U}$

A computation (with Matsumoto) of the relative unipotent completion of $\pi_1(\mathcal{M}_{1,\vec{1}/\mathbb{Q}}, \partial/\partial q)$ implies that the Lie algebra $\mathfrak{u}$ of $\mathcal{U}$ (if it exists) has presentation of the form

$$\mathrm{Gr}^W_\bullet \mathfrak{u} = \mathbb{L}\big((\oplus_{m\geq 1} S^{2m-2}H(2m-1)\big)/(\rho_{f,n} : n \geq 1)$$

where f ranges over the cusp forms of $\mathrm{SL}_2(\mathbb{Z})$. Let e_{2m} be a highest weight vector of $S^{2m-2}H(2m-1)$ with respect to the torus that is diagonal in the basis A, T of H. It has motivic weight $-2m$.

For $m \geq 1$, set

$$\epsilon_{2m} := \mathrm{ad}_T^{2m-1}(A) - \sum_{\substack{j+k=2m-1\\ j>k>0}} (-1)^j [\mathrm{ad}_T^j(A), \mathrm{ad}_T^k(A)] \frac{\partial}{\partial A}.$$

The homomorphism $\mathfrak{u} \to \mathrm{Der}\,\mathfrak{p}$ will take e_{2m} to ϵ_{2m}. Relations satisfied by the e_{2m} will be satisfied by the ϵ_{2m}.

Modular symbols

The *modular symbol* associated to a cusp form f of $\mathrm{SL}_2(\mathbb{Z})$ of weight $2m$ is the homogeneous polynomial

$$r_f(x, y) := \int_0^{i\infty} f(\tau)(x - \tau y)^{m-2} d\tau.$$

of degree $2m - 2$. The even bidegree part is

$$r_f^+(x, y) = \big(r_f(x, y) + r_f(x, -y)\big)/2.$$

This can be generalized to Eisenstein series.

Theorem (Pollack)

If m_j, n_j are positive integers satisfying $2m_j + 2n_j = 2k - 2$, then

$$\sum_j a_j \left[\epsilon_{2m_j+2}, \epsilon_{2n_j+2}\right] = 0 \text{ in } \operatorname{Der}\mathfrak{p}$$

if and only if there is a a modular form f of weight $2k+2$ such that

$$r_f^+(x,y) = \sum_j a_j\left(x^{2m_j}y^{2n_j} - x^{2n_j}y^{2m_j}\right).$$

1. $c(x^{2n} - y^{2n})$ is the period polynomial of G_{2n+2}. This gives the relation $[\epsilon_2, \epsilon_{2n}] = 0$ for all $n \geq 1$.
2. From the Ramanujan τ function: $[\epsilon_{10}, \epsilon_4] - 3[\epsilon_8, \epsilon_6] = 0$

Remark

Compare with results of:

- Schneps (2005): relates modular symbols of cusp forms to congruences between certain integral elements in $\operatorname{Der}\operatorname{Gr}^W_\bullet \pi_1(\mathbb{P}^1 - \{0,1,\infty\})^{\mathrm{un}}$.
- Gangl-Kaneko-Zagier (2006): Relations between double zeta values and period polynomials of cusp forms.

All three results should be manifestations of one result.