Relative higher Albanese manifolds over modular curves

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- Albanese varieties were defined by André Blanchard in 1956.[1] They were named in honour of Giacomo Albanese (1890–1947).
- Zucker and I defined “unipotent higher Albanese varieties” about 40 years ago(!)
- Their *p*-adic version plays a central role in Minhyong Kim’s non-abelian Chabauty method for bounding rational and/or integral points on a variety.
- The unipotent Albanese varieties of X are trivial when $H^1(X) = 0$. Can you generalize Chabauty–Kim in this case?
- I’ll introduce a generalization of unipotent Albanese varieties for complex algebraic varieties. Is there a parallel *p*-adic version that is useful in Chabauty–Kim?

[1] Sur les variétés analytiques complexes, Ann. Sci. École Norm. Sup. (3) 73 (1956), 157–202.

Albanese varieties

The Albanese variety of a smooth projective variety X is defined by

$$\operatorname{Alb} X := H^0(X, \Omega^1_X)^\vee / H_1(X; \mathbb{Z}).$$

Rewrite this:

$$H_1(X; \mathbb{C}) = H^{1,0}(X)^\vee \oplus H^{0,1}(X)^\vee = H_1(X)^{-1,0} \oplus H_1(X)^{0,-1}.$$

So

$$H^0(X, \Omega^1_X)^\vee = H_1(X; \mathbb{C}) / F^0 H_1(X).$$

and

$$\operatorname{Alb} X = H_1(X; \mathbb{Z}) \backslash H_1(X; \mathbb{C}) / F^0 H_1(X). \tag{$*$}$$

The definition $(*)$ applies to all smooth (even normal) varieties, not just projective varieties.

The Albanese mapping

If X is smooth and $x_0 \in X$, then have the Albanese mapping $(X, x_0) \to (\operatorname{Alb} X, 0)$, which is defined by

$$\alpha^0 : x \mapsto \int_{x_0}^{x} \quad \in H_1(X;\mathbb{Z})\backslash H^0(\Omega_X^1)^\vee.$$

- If X is a smooth projective curve, $\operatorname{Alb} X = \operatorname{Jac} X := \operatorname{Pic}^0 X$ and α^0 is the Abel–Jacobi mapping.
- Also have an Albanese mapping when X is not projective. For example
$$\operatorname{Alb}(\mathbb{P}^1 - \{0, 1, \infty\}) = \mathbb{C}^\times \times \mathbb{C}^\times.$$
The Albanese mapping is $x \mapsto (x, 1 - x)$.
- Albanese mappings are universal for maps $(X, x_0) \to (A, 0)$ of X to an algebraic torus (e.g., semi-abelian varieties).

Unipotent completion

The higher Albanese manifolds of a variety (X,x) are constructed from the unipotent fundamental group of (X,x) — the unipotent completion of $\pi_1(X,x)$.

The **unipotent completion** Γ^{un} of a finitely generated group Γ is a pro-unipotent group defined over $\mathbb{Q}$. The group algebra $\mathbb{Q}\Gamma$ has a natural topology defined by the powers of its augmentation ideal

$$I := \ker\{\epsilon : \mathbb{Q}\Gamma \to \mathbb{Q}\}, \quad \epsilon(\gamma) = 1,\ \gamma \in \Gamma.$$

The ring of functions on Γ^{un} is the Hopf algebra of continuous continuous $\mathbb{Q}\Gamma \to \mathbb{Q}$

$$\mathcal{O}(\Gamma^{\mathrm{un}}) = \mathrm{Hom}^{\mathrm{cts}}(\mathbb{Q}\Gamma, \mathbb{Q}) := \varinjlim \mathrm{Hom}(\mathbb{Q}\Gamma/I^m, \mathbb{Q}).$$

The canonical map $\Gamma \to \Gamma^{\mathrm{un}}(\mathbb{Q})$ takes $\gamma \in \Gamma$ to the maximal ideal of continuous functions on $\mathbb{Q}\Gamma$ that vanish at γ.

Dual to $\mathcal{O}(\Gamma^{\mathrm{un}})$ is the completed group algebra

$$\mathbb{Q}\Gamma^{\wedge} := \varprojlim \mathbb{Q}\Gamma/I^m.$$

It is a complete Hopf algebra with coproduct induced by $\Delta(\gamma) = \gamma \otimes \gamma$ for all $\gamma \in \Gamma$.

The group of $\mathbb{k}$ rational points ($\mathrm{char}\, \mathbb{k} = 0$) of Γ^{un} is

$$\Gamma^{\mathrm{un}}(\mathbb{k}) = \{\text{group-like elements of } \mathbb{k}\Gamma^{\wedge}\} := \{u : \Delta u = u \otimes u\}.$$

Its Lie algebra $\mathfrak{g}$ is the set of primitive elements:

$$\mathfrak{g} = \{u \in I^{\wedge} : \Delta u = 1 \otimes u + u \otimes 1\}.$$

It is pro-nilpotent. The exponential mapping

$$\exp : \mathfrak{g} \otimes \mathbb{k} \to \Gamma^{\mathrm{un}}(\mathbb{k})$$

is an isomorphism of pro-varieties for all $\mathbb{k}/\mathbb{Q}$.

When X is a connected manifold, Chen's de Rham theorem says that

$$\mathcal{O}(\pi_1^{\mathrm{un}}(X,x)) = \{\text{closed iterated integrals of 1-forms on } X\}.$$

When X is a smooth variety, $\mathcal{O}(\pi_1^{\mathrm{un}}(X,x))$ has a natural mixed Hodge structure (MHS) for which the product and coproduct are morphisms. This implies that the Lie algebra $\mathfrak{g}(X,x)$ of $\pi_1^{\mathrm{un}}(X,x)$ has a MHS. Its Hodge filtration satisfies

$$\mathfrak{g} \supseteq \cdots \supseteq F^{-2}\mathfrak{g} \supseteq F^{-1}\mathfrak{g} \supseteq F^{0}\mathfrak{g} \supseteq F^{1}\mathfrak{g} = 0.$$

Set $G = \pi_1^{\mathrm{un}}(X,x)$. Denote its subgroup with Lie algebra $F^p\mathfrak{g}$ by F^pG

Set $G_m = G/L^{m+1}G$, where $L^k G$ is the kth term of the LCS of G. This is a unipotent group. Set

$$\Gamma_m = \mathrm{im}\{\pi_1(X, x) \to G_m(\mathbb{Q})\}.$$

The mth higher Albanese of (X, x) is

$$\mathrm{Alb}^m(X, x) = \Gamma_m \backslash G_m(\mathbb{C})/F^0 G(\mathbb{C})$$

This is a complex manifold (generally *not* algebraic). When $m = 1$ it is the classical Albanese manifold. One can use iterated integrals of length $\leq m$ to define higher Albanese mappings

$$\alpha_x^m : (X, x) \to \mathrm{Alb}^m(X, x).$$

These are holomorphic.

A p-adic version of this construction (especially with $m = 2$) plays a central role in Kim's non-abelian Chabauty method.

Relatively unipotent version

For some non-simply connected varieties of interest (e.g., the moduli stack $\mathcal{M}_g$ of smooth projective curves of genus $g \geq 2$), π_1^{un} is trivial.

For this reason, it is useful to introduce a relative version of unipotent completion.

Input:

- a discrete group Γ
- a reductive group S defined over $\mathbb{Q}$
- a Zariski dense homomorphism $\Gamma \to S(\mathbb{Q})$

Examples

group	S	ρ	comment
Γ	trivial	trivial	unipotent completion
$\mathrm{SL}_2(\mathbb{Z})$	$\mathrm{SL}_2/\mathbb{Q}$	inclusion	
congruence subgroup	$\mathrm{SL}_2/\mathbb{Q}$	inclusion	
$\pi_1(\mathcal{M}_g, [C])$	$\mathrm{Sp}(H_1(C))/\mathbb{Q}$	monodromy	

The definition

The **completion of Γ relative to** $\rho : \Gamma \to S(\mathbb{Q})$ consists of

- an affine (ie, proalgebraic) group $\mathcal{G}$ defined over $\mathbb{Q}$ which is an extension
$$1 \to \mathcal{U} \to \mathcal{G} \to S \to 1$$
where $\mathcal{U}$ is prounipotent, and
- a homomorphism $\hat{\rho} : \Gamma \to \mathcal{G}(\mathbb{Q})$ such that ρ is the composition

$$\Gamma \xrightarrow{\hat{\rho}} \mathcal{G}(\mathbb{Q}) \longrightarrow S(\mathbb{Q})$$

UMP: if G is an affine $\mathbb{Q}$ group that is an extension of S by a (pro) unipotent group and if $\phi : \Gamma \to G(\mathbb{Q})$ is a homomorphism whose composition with $G(\mathbb{Q}) \to S(\mathbb{Q})$ is ρ, then there is a *unique* homomorphism $\theta : \mathcal{G} \to G$ of $\mathbb{Q}$-groups that commutes with their projections to S such that ϕ is the composite

$$\Gamma \xrightarrow{\hat{\rho}} \mathcal{G}(\mathbb{Q}) \xrightarrow{\theta} G(\mathbb{Q}).$$

Some basic results

The Lie algebra $\mathfrak{g}$ of $\mathcal{G}$ is an extension

$$0 \to \mathfrak{u} \to \mathfrak{g} \to \mathfrak{s} \to 0$$

where $\mathfrak{u}$ is pronilpotent. The (co)homology $H^\bullet(\mathfrak{u})$ is an S-module.

- There is an S-module isomorphism

$$H_1(\mathfrak{u}) = \prod_\alpha H^1(\Gamma, V_\alpha)^\vee \otimes V_\alpha$$

  where V_α ranges over representatives of the isomorphism classes of irreducible S-modules.
- If X is a smooth variety and $\rho : \pi_1(X, x) \to S(\mathbb{Q}) \subset \mathrm{Aut}\, V_x$ is the monodromy representation of a variation of Hodge structure of geometric origin (or a PVHS), then $\mathcal{O}(\mathcal{G}_x)$ is a Hopf algebra in the category of MHS and the Lie algebra $\mathfrak{g} = \mathfrak{g}_x$ of $\mathcal{G}_x$ has a natural (pro) MHS.
- One can also take relative completions $\mathcal{G}_{x,y}$ of path torsors $\pi_1(M; x, y)$. These are torsors under $\mathcal{G}_x$ (and $\mathcal{G}_y$). Their coordinate rings have a natural (ind) MHS.

Example: congruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$

Suppose that Γ is a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$. Denote the cusps of the modular curve $Y_\Gamma := \Gamma\backslash\mathfrak{h}$ by C. Denote the defining representation of SL_2 by H.

- The irreducible representations of SL_2 are the symmetric powers $S^m H$. So
$$H_1(\mathfrak{u}) = \prod_{m\geq 0} H^1(\Gamma, S^m H)^\vee \otimes S^m H.$$
- By Eichler–Shimura and Manin–Drinfeld (when $m > 0$)
$$\begin{aligned} H^1(\Gamma, S^m H) &= H^1_{\mathrm{cusp}}(\Gamma, S^m H) \oplus H_0(C) \\ &= \bigoplus_f V_f \oplus \bigoplus_C \mathbb{Q}(-m-1). \end{aligned}$$
  where f ranges over equivalence classes of normalized Hecke eigen cusp forms.
- The Lie algebra $\mathfrak{u}$ is HUGE. It is **free** and isomorphic to $\mathbb{L}(H_1(\mathfrak{u}))^\wedge$.

Higher relative Albanese manifolds

Suppose that X is a smooth variety and that $(\mathcal{G}, \rho)$ is the relative completion of $\pi_1(X, x)$ relative to a PVHS. Let G be any finite dimensional quotient of $\mathcal{G}$ that has a MHS and let Γ_G be the image of $\pi_1(X, x) \to G(\mathbb{Q})$.

We would like to define

$$\mathrm{Alb}_G(X, x) = \Gamma_G \backslash G(\mathbb{C})/F^0 G.$$

But this is not quite right. To see why, let's provisionally define

$$\mathrm{Alb}^0(X, x) = \Gamma_S \backslash S(\mathbb{C})/F^0 S.$$

To understand the problem, consider the case where X is the modular curve Y_Γ. Here $S = \mathrm{SL}(H)$ and

$$F^0 S = \text{stabilizer of the Hodge filtration of } = \text{a Borel subgroup}$$

So $S(\mathbb{C})/F^0 S = \mathbb{P}^1(H) = \mathfrak{h} \cup \mathbb{P}^1(\mathbb{R}) \cup \overline{\mathfrak{h}}$ and

$$\mathrm{Alb}^0(Y_\Gamma) = \mathrm{SL}_2(\mathbb{Z}) \backslash \mathbb{P}^1(H) = \text{ a mess!}$$

Instead we define

$$\mathrm{Alb}^0(X,x) = \Gamma_S \backslash S(\mathbb{R})/(S(\mathbb{R}) \cap F^0 S).$$

In the case of the modular curve Y_Γ, we have $S(\mathbb{R}) \cap F^0 S = \mathrm{SO}(2)$, so that

$$S(\mathbb{R})/(S(\mathbb{R}) \cap F^0 S) = \mathrm{SL}_2(\mathbb{R})/\mathrm{SO}(2) = \text{the upper half plane } \mathfrak{h}$$

and

$$\mathrm{Alb}^0(Y_\Gamma) = \Gamma \backslash \mathfrak{h} = Y_\Gamma.$$

The 0th relative Albanese mapping is the identity $Y_\Gamma \to Y_\Gamma$.

But we still have the HUGE unipotent radical ...

As above, X is a smooth variety and $(\mathcal{G}, \rho)$ is the relative completion of $\pi_1(X, x)$ relative to a PVHS. We have the diagram

$$\begin{array}{ccc} (\mathcal{G}(\mathbb{C})/F^0\mathcal{G})|_{\mathscr{D}} & \hookrightarrow & \mathcal{G}(\mathbb{C})/F^0\mathcal{G} \\ \downarrow & & \downarrow \\ \mathscr{D} & \hookrightarrow & S(\mathbb{C})/F^0 S \end{array}$$

where $\mathscr{D} = S(\mathbb{R})/(S(\mathbb{R}) \cap F^0 G)$ is the corresponding Griffiths period domain. Now define

$$\mathrm{Alb}(X, x) = \pi_1(X, x)\backslash(\mathcal{G}(\mathbb{C})/F^0\mathcal{G})|_{\mathscr{D}}$$

This is the inverse limit of a family of nilmanifolds over

$$\mathrm{Alb}^0(X, x) := \Gamma_0\backslash\mathscr{D}$$

where Γ_0 is the image of $\pi_1(X, x)$ in $S(\mathbb{Q})$.

Albanese mappings

The (orbifold) fiber of $\mathrm{Alb}(X,x) \to \mathrm{Alb}^0(X,x)$ over $\alpha^0(y) \in \mathrm{Alb}^0(X,x)$ is the (pro) variety $(\ker\rho)\backslash\mathcal{U}_{x,y}(\mathbb{C})/F^0\mathcal{U}_y$.

There is a holomorphic Albanese mapping

$$\alpha : (X,x) \to \mathrm{Alb}(X,x)$$

It is defined using iterated integrals with coefficients in local systems $\mathbb{V}_\alpha$ over X that correspond to the V_α. It lifts the Griffiths period mapping $X \to \Gamma_0\backslash D$ of the original PVHS.

We can truncate α to obtain families of nilmanifolds over $\mathrm{Alb}^0(X,x)$ and families of Albanese mappings.

Back to modular curves

Here $\mathrm{Alb}^0(Y_\Gamma, x) = \Gamma\backslash\mathfrak{h} = Y_\Gamma$. The fiber of $\mathrm{Alb}(Y_\Gamma, x) \to Y_\Gamma$ over y is the (pro) affine space $\mathcal{U}_{x,y}(\mathbb{C})/F^0\mathcal{U}_y$.

The simplest truncations of this are where we push out along projections

$$\mathcal{U} \to H_1(\mathcal{U}) = H_1(\mathfrak{u}) = \prod_{m\geq 0} V_f^\vee \otimes S^m H \to V_f^\vee \otimes S^m H.$$

(Here the eigenform f can be a cusp form or Eisenstein series.) So for each equivalence class of weight $(m+2)$ Hecke eigenforms f, we have an Albanese mapping

$$\alpha_f : Y_\Gamma \to \mathrm{Alb}_f(Y_\Gamma)$$

where $\mathrm{Alb}_f(Y_\Gamma)$ is an $(V_f^\vee \otimes S^m H)/F^0$ bundle over Y_Γ.

Understanding α_f

Denote by $\mathbb{H}$ the local system over Y_Γ with fiber $H^1(E)$ over the moduli point $[E]$ of the elliptic curve E. It is a PVHS of weight 1. The pullback of $\mathbb{H}$ to $\mathfrak{h}$ is the trivial bundle

$$\mathfrak{h} \times H \to \mathfrak{h}.$$

Denote the associated holomorphic vector bundle by $\mathcal{H}$.

Suppose that $\tau \in \mathfrak{h}$. Set $E_\tau = \mathbb{C}/\Lambda_\tau$, where $\Lambda_\tau = \mathbb{Z} \oplus \mathbb{Z}\tau$. Let $\mathbf{a}, \mathbf{b}$ be the basis of $H_1(E_\tau; \mathbb{Z})$ that corresponds to the basis $1, \tau$ of Λ_τ. Set

$$\mathbf{w}(\tau) = 2\pi i(\mathbf{a}^\vee + \tau \mathbf{b}^\vee) \in H^0(E_\tau, \Omega^1)$$

be the class of the abelian differential dz on E_τ. This is a holomorphic section of $\mathcal{H}$ over $\mathfrak{h}$.

The Hodge structure V_f has Hodge numbers $(m+1,0)$ and $(0,m+1)$. This implies that

$$(V_f^\vee \otimes S^m\mathcal{H})/F^0 = (V_f^{m+1,0})^\vee \otimes S^m\mathcal{H}.$$

The pullback of $\mathrm{Alb}_f(Y_\Gamma)$ to $\mathfrak{h}$ is the trivial bundle

$$\mathfrak{h} \times (V_f^{m+1,0})^\vee \otimes S^m\mathcal{H} \to \mathfrak{h}.$$

Assume, for simplicity, that the Fourier coefficients of f lie in $\mathbb{Q}$. (This implies $\dim V_f = 2$.) Then the pullback of α_f to a section of this bundle is

$$\alpha_f : \tau \mapsto 2\pi i \int_{\tau_0}^{\tau} f(\tau)\mathbf{w}(\tau)^m d\tau$$

where τ_0 is a lift of the base point of Y_Γ to $\mathfrak{h}$.

Higher relative Albanese maps

For $0 \leq j \leq m$, set

$$\omega_{f,j} = 2\pi i\, f(\tau)\tau^j d\tau.$$

At least some of the coefficients of higher Albanese mappings of modular curves are given locally by what Manin calls "iterated Shimura integrals"

$$\int_{\tau_0}^{\tau} \omega_{f_1,j_1}\omega_{f_2,j_2}\cdots\omega_{f_r,j_r}$$

where $f_1, \ldots, f_r$ are normalized Hecke eigen forms of weights $m_1 + 2, \ldots, m_r + 2$ and $0 \leq j_k \leq m_k$.

Higher Genus

Denote the moduli space (stack) of smooth complex projective curves of genus $g \geq 2$ by $\mathcal{M}_g$. Denote the moduli point of the curve C by $[C]$. We have

$$\pi_1(\mathcal{M}_g, [C]) = \Gamma_C = \text{ the mapping class group of } C.$$

This is the group of isotopy classes of orientation preserving diffeomorphisms of C:

$$\Gamma_C = \pi_0(\mathrm{Diff}^+ C).$$

Since $H_1(\mathcal{M}_g; \mathbb{Q})$ vanishes for all $g \geq 2$, $\pi_1^{\mathrm{un}}(\mathcal{M}_g, [C])$ is trivial. So the current version of Chabauty–Kim is not a useful tool for understandng $\mathcal{M}_g(K)$, K is a # field.

Relative completion of Γ_C

The action of Γ_C on $H_1(C)$ induces a homomorphism

$$\rho : \Gamma_C \to \mathrm{Sp}(H_1(C, \mathbb{Z})).$$

It is surjective and has Zariski dense image in $\mathrm{Sp}(H_1(C; \mathbb{Q}))$. The completion of Γ_C relative to ρ is an extension

$$1 \to \mathcal{U} \to \mathcal{G} \to \mathrm{Sp}(H) \to 1$$

where $H = H_1(C)$ and $\mathcal{U}$ is prounipotent. Unlike in genus 1, the Lie algebra $\mathfrak{u}$ of $\mathcal{U}$ is finitely generated for all $g \geq 2$.

A celebrated result of Dennis Johnson implies that when $g \geq 3$

$$H_1(\mathfrak{u}) = (\Lambda^3 H)/H =: V$$

as an $\mathrm{Sp}(H)$ module, so that $\mathfrak{u}$ is a quotient of $\mathbb{L}(V)^\wedge$.

Observe that

$$\mathrm{Sp}(H_{\mathbb{C}})/F^0\mathrm{Sp}(H_{\mathbb{C}}) = \text{grassmanian of } g\text{-planes in } H^1(C,\mathbb{C})$$

and that

$$\mathrm{Sp}(H_{\mathbb{R}})/(\mathrm{Sp}(H_{\mathbb{R}}) \cap F^0\mathrm{Sp}(H_{\mathbb{C}})) \cong \mathrm{Sp}_g(\mathbb{R})/U(g)$$

is the Siegel upper half plane

$$\mathfrak{h}_g = \{Z \in \mathbb{M}_g(\mathbb{C}) : Z = Z^T \text{ and } \mathrm{Im}(Z) > 0\}$$

Thus

$$\mathrm{Alb}^0(\mathcal{M}_g) = \mathrm{Sp}(H_{\mathbb{R}})/(\mathrm{Sp}(H_{\mathbb{R}}) \cap F^0\mathrm{Sp}(H_{\mathbb{C}})) = \mathrm{Sp}_g(\mathbb{Z})\backslash\mathfrak{h}_g = \mathcal{A}_g,$$

the moduli space of principally polarized abelian varieties (ppav's) of dimension g, and

$$\alpha^0 : \mathcal{M}_g \to \mathcal{A}_g = \mathrm{Alb}^0(\mathcal{M}_g)$$

is the period mapping.

We can extend the definition of V to all ppav's A by setting

$$V_A = (\Lambda^3 H_1(A))/H_1(A) = PH_3(A).$$

Put it in weight -1, so that its Hodge numbers are

$$(-2,1),\ (-1,0),\ (0,-1),\ (1,-2).$$

Set

$$J(V_A) = V_{A,\mathbb{Z}} \backslash V_{A,\mathbb{C}} / F^0 V_A$$

We can assemble these into the family

$$J(\mathbb{V}) \to \mathcal{A}_g$$

of compact complex tori over $\mathcal{A}_g$. This is $\mathrm{Alb}^1(\mathcal{M}_g)$.

When $g \geq 3$, we have the diagram

$$\begin{array}{ccccc} & & \mathrm{Alb}^1(\mathcal{M}_g) & \xrightarrow{\cong} & J(\mathbb{V}) \\ & \nearrow^{\alpha^1} & \downarrow & & \downarrow \\ \mathcal{M}_g & \xrightarrow[\alpha^0]{} & \mathrm{Alb}^0(\mathcal{M}_g) & \xrightarrow{\cong} & \mathcal{A}_g \end{array}$$

The Albanese mapping α^1 is the normal function of the Ceresa cycle. (More about this in Padma Srinivasan's talk.)

The normal function associated to a cusp form

Set

$$\mathbb{U}_f := V_f^\vee \otimes S^m \mathbb{H}$$

This is a PVHS of weight $-(m+1)+m=-1$ over Y_Γ.

We can quotient it out by $\mathbb{U}_\mathbb{Z}$ to obtain a family of compact complex tori $J(\mathbb{U}_f)$ over Y_Γ with fiber

$$J(U_y) := U_{y,\mathbb{Z}} \backslash \left[V_f^\vee \otimes S^m H^1(E;\mathbb{C}) \right] / F^0 \left[V_f^\vee \otimes S^m H^1(E) \right]$$

over $y=[E]$. This is the family of Griffiths intermediate jacobians associated to $\mathbb{U}_f$. The Albanese mapping α_f descends to a section ν_f of $J(\mathbb{U}_f)$. It is a (admissible) normal function

$$\begin{array}{ccc} \mathrm{Alb}_f(Y_\Gamma) & \longrightarrow & J(\mathbb{U}_f) \\ {\scriptstyle \alpha_f}\uparrow\downarrow & \overset{\nu_f}{\nearrow} & \downarrow \\ Y_\Gamma & = & Y_\Gamma \end{array}$$

This should be the normal function associated to certain algebraic cycles associated to Hecke operators.

Questions

- Is there a *p*-adic version of relative Albanese varieties and maps? Can Chabauty–Kim be extended to this situation?[2] Might it be useful?
- Can modular forms of weight > 2 be used to bound K rational points on modular curves?
- Can the *p*-adic Ceresa normal function) be used to bound the K-rational points of $\mathcal{M}_g$, K a number field, especially when $g \gg 0$. Note that $\mathcal{M}_g$ is unirational when $g \geq 14$.
- Might this shed any light on Lang's Conjecture when $g \geq 24$, where $\mathcal{M}_g$ is of general type?

[2] After giving this talk I learned that Noam Kantor developed such a *p*-adic verion in his PhD thesis. See references.

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