Mapping class groups of simply connected algebraic manifolds

Richard Hain

Duke University

Princeton Topology Seminar
November 2, 2023

The Mapping class group of a manifold

The *mapping class group* Γ_M of a closed orientable manifold M is the group of *isotopy* classes of orientation preserving diffeomorphisms of M:

$$\Gamma_M := \pi_0 \operatorname{Diff}^+ M.$$

The *Torelli group* T_M of M is the subgroup consisting of the mapping classes that act trivially on the homology of M:

$$T_M := \ker\{\Gamma_M \to \operatorname{Aut} H_\bullet(M;\mathbb{Z})\}.$$

Denote the image of $\Gamma_M \to \operatorname{Aut} H_\bullet(M;\mathbb{Z})$ by S_M. The mapping class group Γ_M is an extension

$$1 \to T_M \to \Gamma_M \to S_M \to 1.$$

There is are also relative/decorated versions: If N is a subset of M (e.g., ∂M or a point) and $\vec{\mathbf{p}}$ is a collection of cohomology classes (e.g., Pontryagin classes, Kähler class), one can define the mapping class group

$$\Gamma_{M,N,\vec{\mathbf{p}}} := \pi_0(\mathrm{Diff}^+(M, N; \vec{\mathbf{p}})).$$

of (M, N) and its Torelli subgroup

$$T_{M,N} := \ker\{\Gamma_{M,N} \to H_\bullet(M, N; \mathbb{Z})\}.$$

If A is the annulus $S^1 \times [-\pi, \pi]$ one has

$$\Gamma_{A,\partial A} = \{t_A^n : n \in \mathbb{Z}\} \cong \mathbb{Z}.$$

The generator

$$t_A : (\theta, t) \mapsto (\theta + t + \pi, t)$$

is the *Dehn twist* about the curve $S^1 \times \{0\}$.

Monodromy homomorphisms

A locally trivial bundle $X \to T$ with fiber M over a smooth manifold T gives rise to a *monodromy representation*

$$\pi_1(T, t_o) \to \Gamma_M \qquad (*)$$

where we identify the fiber over t_o with M.

A case of interest is where $X \to T$ is the universal family over a moduli space (or stack) of complex projective structures on M. One can then ask how close the monodromy representation $(*)$ is to being an isomorphism. Not much appears to be known, even when M is simply connected. (More precise version later.)

The classical case — complex curves

Suppose that M is a compact oriented surface of genus $g \geq 2$.

- Its MCG Γ_M is generated by a finite number of Dehn twists.
- It is finitely presented (algebraic geometry, Thurston, ...).
- Have $S_M = \mathrm{Sp}(H_1(M;\mathbb{Z})) := \mathrm{Aut}(H_1(M;\mathbb{Z}), \langle\ ,\ \rangle)$.
- Its Torelli group T_M is a tough nut to crack:
  - it is a countably generated free group when $g = 2$ (Mess)
  - it is finitely generated when $g \geq 3$ (Johnson)
  - it is conjectured to be finitely presented when $g \gg 3$, but this is not known for any $g \geq 3$.

Teichmüller space $\mathscr{T}_g$

- A *marked Riemann surface* is an isotopy class of diffeomorphisms $f : M \to X$ of M with a compact Riemann surface, or equivalently, a hyperbolic surface.
- The set of marked Riemann surfaces of genus g is a manifold $\mathscr{T}_g$ that is diffeomorphic to $\mathbb{R}^{6g-6}$.
- The mapping class group Γ_M acts on Teichmüller space $\mathscr{T}_g$:

$$\Gamma_M \quad \circlearrowright M \xrightarrow{f} X \qquad [\phi] : [f] \mapsto [f \circ \phi^{-1}].$$

- This action is properly discontinuous and virtually free.
- The moduli space of compact Riemann surfaces is the orbifold quotient:

$$\mathcal{M}_g = \Gamma_M \backslash \mathscr{T}_g.$$

Moduli of compact Riemann surfaces

- The moduli space of curves is the orbifold classifying space $B\Gamma_g$ of Γ_M. That is, the topology of $\mathcal{M}_g$ is determined by Γ_M.
- One manifestation of this is the isomorphism

$$H^\bullet(\Gamma_M; \mathbb{Q}) \cong H^\bullet(\mathcal{M}_g; \mathbb{Q}).$$

  Much geometry of algebraic curves is encoded in the cohomology and structure of Γ_M.
- In this case, the monodromy homomorphism

$$\pi_1(\mathcal{M}_g, [f]) \to \Gamma_M$$

  is an isomorphism.

Higher dimensions

To what extent does this hold in higher dimensions? The natural setting:

- $\mathscr{M}_M$ is a moduli space that parameterizes a natural family of complex projective structures on M.
- Assume there is a universal family $\mathscr{X} \to \mathscr{M}_M$, where $\mathscr{X} \subset \mathbb{P}^N \times \mathscr{M}_M$.
- Let $\omega_{\mathscr{X}} \in H^2(\mathscr{X})$ be pullback of the hyperplane class along $\mathscr{X} \to \mathbb{P}^N$.
- It restricts to a class $\omega \in H^2(M)$.
- Denote the stabilizer of ω in Γ_M by $\Gamma_{M,\omega}$.

Some basic questions

Suppose that $\phi : (M, \omega) \to (X, \omega_X)$ is a diffeomorphism. We have the monodromy representation

$$\pi_1(\mathscr{M}_M, [\phi]) \to \Gamma_{M,\omega}$$

- Is ρ close to being an isomorphism?
- Does it have finite kernel?
- Does the image have finite index?
- Is $\mathscr{M}_M \to B\Gamma_{M,\omega}$ close to being a homotopy equivalence?

Abelian varieties

Suppose that (A, ω) a principally polarized variety of complex dimension g. Set $H_R = H_1(A; R)$. Hatcher (1978) showed that there is a split surjection exact sequence

$$0 \to \text{(finite abelian group)} \to \Gamma_{(A,0),\omega} \to \mathrm{Sp}(H_{\mathbb{Z}}) \to 1$$

Moduli space $\mathcal{A}_g = \mathrm{Sp}(H_{\mathbb{Z}})\backslash\backslash\mathfrak{h}_g \approx B\mathrm{Sp}(H_{\mathbb{Z}})$. So

$$\pi_1(\mathcal{A}_g, [A]) \to \Gamma_{(A,0),\omega}$$

injective with finite index image and $\mathcal{A}_g \to B\Gamma_{(A,0),\omega}$ is close to being a homotopy equivalence.

Sullivan's first result

Denote the group of self homotopy equivalences of a topological space X by $\operatorname{ho}\operatorname{Aut}(X)$. Denote the localization of X at 0 by $X_{(0)}$.

Theorem (Sullivan, 1977)

If X is a simply connected (or nilpotent) finite complex, then $\operatorname{ho}\operatorname{Aut}(X_{(0)})$ is an affine algebraic $\mathbb{Q}$-group $\mathcal{G}_X^h$ whose reductive quotient is a subquotient of the automorphism group of the rational cohomology ring $H^\bullet(X;\mathbb{Q})$. Moreover the image of $\operatorname{ho}\operatorname{Aut}(X) \to \mathcal{G}_X^h(\mathbb{Q})$ is arithmetic and the kernel is finite.

If, in addition, X is a formal space (e.g., a compact Kähler manifold by DGMS), then the reductive quotient of $\mathcal{G}_X^h$ is the reductive quotient of the group of automorphisms of the cohomology ring $H^\bullet(X;\mathbb{Q})$.

Examples and comments

1. When $M = (S^1)^n$, $\mathcal{G}_A^h = \mathrm{GL}_n/\mathbb{Q}$.
2. When $M = \mathbb{P}^n_{\mathbb{C}}$,

$$\mathcal{G}^h_{\mathbb{P}^n} \cong \mathrm{Aut}\, H^\bullet(\mathbb{P}^n; \mathbb{Q}) = \mathbb{G}_m/\mathbb{Q}.$$

3. When $M = U(9)$, the Sullivan minimal model is

$$H^\bullet(U(9); \mathbb{Q}) \cong \Lambda^\bullet(y_1, y_3, y_5, y_7, y_9)$$

where $|y_j| = j$. Its automorphism group an extension of $(\mathbb{G}_m)^5$ by the unipotent group $\mathbb{G}_a$:

$$y_9 \mapsto y_9 + t\, y_1 y_3 y_5, \quad y_j \mapsto y_j \text{ when } j \neq 9, \quad t \in \mathbb{Q}.$$

In this case, $\mathrm{Aut}\, H^\bullet(M)$ is not reductive.

A Johnson homomorphisms for simply connected manifolds

If M is simply connected, $\pi_3(M) \otimes \mathbb{Q}$ is an extension.

$$0 \to \operatorname{Sym}^2 H_2(M;\mathbb{Q})/\operatorname{im}\Delta \to \pi_3(M, x_o) \otimes \mathbb{Q} \to H_3(M;\mathbb{Q}) \to 0,$$

where $\Delta : H_4(M;\mathbb{Q}) \to S^2 H_2(M;\mathbb{Q})$ is the dual of the cup product. The action of T_M on this gives rise to Johnson homomorphism

$$\tau_M : H_1(T_M;\mathbb{Q}) \to \operatorname{Hom}(H_3(M;\mathbb{Q}), \operatorname{Sym}^2 H_2(M;\mathbb{Q})/\operatorname{im}\Delta).$$

This is a higher dimensional analogue of the Johnson homomorphism in the surface case. It is trivial when $b_2 = 1$, such as when M is a complete intersection.

Generalized Johnson homomorphism

Denote the kernel of $\mathrm{ho}\,\mathrm{Aut}(M) \to \mathrm{Aut}\, H^\bullet(X)$ by $\mathrm{ho}\, T_M$.

Theorem (H, 2023)

If M is a simply connected Kähler 3-fold, then the Johnson homomorphism induces an S_M-invariant surjection

$$\tau_M : H_1(\mathrm{ho}\, T_M; \mathbb{Q}) \to \mathrm{Hom}(H_3(M;\mathbb{Q}), \mathrm{Sym}^2 H_2(M;\mathbb{Q}) / \operatorname{im} \Delta).$$

Question

Is this an isomorphism? I do not know if

$$\mathrm{ho}\, T_M \to \mathrm{Aut}\, \pi_\bullet(M) \otimes \mathbb{Q}$$

is close to being injective (e.g., finite kernel) or if the image $\otimes \mathbb{Q}$ is isomorphic to

$$\mathrm{Hom}(H_3(M;\mathbb{Q}), \mathrm{Sym}^2 H_2(M;\mathbb{Q}) / \operatorname{im} \Delta).$$

Pontryagin Distortion

The *distortion of the Pontryagin classes* is used to detect elements of $\ker\{\Gamma_M \to \mathrm{ho}\,\mathrm{Aut}(M)\}$.

- For $\varphi \in T_M$, the Wang sequence splits into SESs

$$0 \to H^{j-1}(M) \to H^j(M_\varphi) \to H^j(M) \to 0.$$

- A homotopy $F : M \times I \to M$ from φ to the identity induces a smooth homotopy equivalence $\widehat{F} : M \times S^1 \to M_\varphi$.
- The kth *distortion* of F is

$$\delta_k(F) = \widehat{F}^*(p_k(M_\varphi)) - p_k(M) \times 1 \in H^{4k-1}(M).$$

- The *distortion* of φ is

$$\delta(\varphi) := \big(\delta_k(F)\big)_k \in \big[\textstyle\bigoplus_{4k \leq \dim_R M} H^{4k-1}(M;\mathbb{Q})\big]/I =: \mathcal{D}_M,$$

where I is the distortion of homotopies from id_M to itself.

Sullivan's result for MCGs

Set $\vec{\mathbf{p}} = (p_1, p_2, \dots)$ and let $\mathcal{G}^{h+}_{M,\vec{\mathbf{p}}}$ be the stabilizer of $\vec{\mathbf{p}}$ and μ_M.

Theorem (Sullivan, 1977)

If M is a simply connected closed manifold of (real) dimension ≥ 5, there is an affine algebraic group $\mathcal{G}_M$, defined over $\mathbb{Q}$, that is an extension

$$1 \to \mathcal{D}_M \to \mathcal{G}_M \to \mathcal{G}^{h+}_{M,\vec{\mathbf{p}}} \to 1$$

and a homomorphism $\Gamma_M \to \mathcal{G}_M(\mathbb{Q})$ with arithmetic image and finite kernel.

When M is formal, the reductive quotient of $\mathcal{G}_M$ is the reductive quotient of the group of automorphisms of the ring $H^\bullet(M; \mathbb{Q})$ that fixes $\vec{\mathbf{p}}$ and μ_M.

Corollary

If $\dim_{\mathbb{R}} M \geq 5$, then Γ_M is finitely presented.

The result of Kreck and Su

Kreck and Su gave a complete computation of the mapping class groups of simply connected 3-folds with $b_2 = 1$. Below is a rational (and much simplified) version of their main result.

Theorem (Kreck–Su, 2022)

If M is a simply connected compact Kähler 3-fold with $b_2 = 1$, then the distortion homomorphism

$$\delta_M : T_M \to H^3(M; \mathbb{Q})$$

has finite kernel and image a full lattice.

Automorphisms of $H^\bullet$ of a compact Kähler manifold

We've already seen that the automorphism group of the cohomology ring of $U(9)$ (a formal space) is not reductive.

Theorem (H, 2023)

Suppose that $\mathbb{k}$ is a subfield of $\mathbb{R}$. If M is a compact Kähler manifold with Kähler class $\omega \in H^2(M; \mathbb{k})$, then the automorphism group of its cohomology ring that fixes ω is a reductive $\mathbb{k}$ group.

This is proved using the Hard Lefschetz Theorem.

Smooth hypersurfaces

Projective space:

$$\mathbb{P}^{n+1} = (\mathbb{C}^{n+2} - \{0\})/\mathbb{C}^{\times}.$$

Coordinates $\mathbf{x} = (x_0, \ldots, x_{n+1}) \in \mathbb{C}^{n+2}$ and $[\mathbf{x}] \in \mathbb{P}^{n+1}$.

- A non-zero polynomial $f(\mathbf{x}) \in \mathrm{Sym}^d\, \mathbb{C}^{n+2}$ defines a hypersurface

  $$X_f := \{[\mathbf{x}] \in \mathbb{P}^{n+1} : f(\mathbf{x}) = 0\}.$$

  of degree d in $\mathbb{P}^{n+1}$.
- It is smooth when $f(\mathbf{x})$ has nowhere vanishing discriminant.

Moduli of hypersurfaces

- Let $\mathscr{U}_{n,d}$ be the space of homogeneous polynomials of degree d in $n+2$ variables with non-vanishing discriminant.
- The group $\mathrm{GL}_{n+2}(\mathbb{C})$ acts on it. The (stack) quotient is the moduli space $\mathscr{H}_{n,d}$ of hypersurfaces in $\mathbb{P}^{n+1}$ of degree d.
- The map $\mathscr{U}_{n,d} \to \mathscr{H}_{n,d}$ is a principal $\mathrm{GL}_{n+2}(\mathbb{C})$ bundle, so we have a central extension

$$0 \to \mathbb{Z} \to \pi_1(\mathscr{U}_{n,d}, f) \to \pi_1(\mathscr{H}_{n,d}, [X_f]) \to 1$$

where X_f denotes the hypersurface in $\mathbb{P}^{n+1}$ defined by the homogeneous polynomial f.

Lefschetz hyperplane theorem

Theorem (Lefschetz, special case)

If $n \geq 2$ and X is a smooth hypersurface in $\mathbb{P}^{n+1}$, then

1. *X is simply connected,*
2. *the restriction map*

$$H^j(\mathbb{P}^{n+1};\mathbb{Q}) \to H^j(X;\mathbb{Q})$$

   is an isomorphism when $j \neq n$,

3. *in degree n we have an exact sequence*

$$0 \to H^n(\mathbb{P}^{n+1};\mathbb{Q}) \to H^n(X;\mathbb{Q}) \to H^n_o(X;\mathbb{Q}) \to 0$$

The cokernel is the *primitive cohomology* of X. It has a non-degenerate $(-1)^n$ symmetric bilinear form $\langle\ ,\ \rangle$.

- The monodromy homomorphisms are:

$$\pi_1(\mathscr{U}_{n,d}, f) \to \pi_1(\mathscr{H}_{n,d}, [X_f]) \to \Gamma_M \to \mathrm{Aut}(H^n_o(X_f; \mathbb{Z}); \langle\ ,\ \rangle).$$

- Beauville (1986) computed the images of $\pi_1(\mathscr{M}, [X])$ and Γ_M. Both have finite index in $\mathrm{Aut}(H^n_o(X; \mathbb{Z}); \langle\ ,\ \rangle)$.
- When $n = 3$, we have

$$\dim H^n_o(X; \mathbb{Q}) = \frac{(d-1)^5 + 1}{d} - 1$$

which is positive for all $d \geq 3$. The only interesting part of the monodromy representation is

$$\pi_1(\mathscr{U}_{3,d}, f) \to \mathrm{Sp}(H^3_o(X_f); \mathbb{Z}).$$

- Since Beauville computed S_M, the problem is to understand or compute the Torelli group T_M.
- We will skip $\dim_{\mathbb{C}} X = 2$ as 4-manifold topology is harder. There are recent results in dimension 4 by Konno–Lin, Konno–Mallick–Taniguchi and Baraglia.
- Not much is known, apart from the results of Kreck and Su in complex dimension 3.

Distortion for complete intersections

In the case of a complete intersection M, we can extend the distortion homomorphism

$$\delta : \ker\{\Gamma_M \to \mathcal{G}^h_M(\mathbb{Q})\} \to \mathcal{D}_M$$

to its Torelli group. In this case, the Pontryagin classes are multiplies $p_k(M) = a_k \omega^{2k}$ of powers of the hyperplane class.

Proposition (H, 2023)

Suppose that M is a smooth manifold with $b_1 = 0$. If there is $\omega \in H^2(M; \mathbb{Q})$ such that $p_k(M) = a_k \omega^{2k}$, then

$$\mathcal{D}_M = \bigoplus_k H^{4k-1}(M; \mathbb{Q})$$

(no indeterminacies) and the distortion homomorphism extends naturally to a homomorphism $\tilde{\delta} : T_M \to \mathcal{D}_M$.

For a smooth hypersurface in $\mathbb{P}^{n+1}$ and degree d, set

$$K_M = \ker\{\rho_M : \pi_1(\mathscr{H}_{n,d}[M]) \longrightarrow \operatorname{Aut} H^n_o(M;\mathbb{Q})\}.$$

This maps to T_M.

Theorem (H, 2023)

Suppose that $n \geq 3$ and $n \equiv 3 \bmod 4$. If M is a smooth hypersurface in $\mathbb{P}^{n+1}$ of degree $d > 1$, then

$$K_M \xrightarrow{\lambda_M} T_M \xrightarrow{\delta} H^n(M;\mathbb{Q})$$

is trivial. If $d \geq 3$, then the image of λ_M has infinite index in T_M.

The following is a simplified version of a result of Carlson and Toledo.

Theorem (Carlson–Toledo, 1999)

Suppose that M is a smooth hypersurface of degree d in $\mathbb{P}^{n+1}$. If $d \geq 3$ and $n > 1$, the kernel of the representation $\pi_1(\mathscr{H}_{n,d}, [M]) \to \operatorname{Aut} H^n(M)$ surjects onto a lattice in a non compact, almost simple $\mathbb{R}$-group of rank ≥ 2. In particular, it contains a non-abelian free subgroup.

In summary:

Corollary

If $d \geq 3$ and $n \geq 3$, then the kernel of the monodromy representation

$$\lambda_M : \pi_1(\mathscr{H}_{n,d}, [M]) \to \Gamma_M$$

contains a non-abelian free group. When $n \equiv 3 \mod 4$, its image has infinite index.

Where does the kernel come from?

Consider

$$\begin{array}{ccccc} \mathscr{U}_{n,d} & \xrightarrow{\phi} & \mathscr{U}_{n+1,d} & \longrightarrow & \mathscr{H}_{n+1,d} \\ \downarrow{\scriptstyle \pi} & & & & \\ \mathscr{H}_{n,d} & & & & \end{array}$$

where ϕ takes $f(x)$ to $y^d + f(x)$. If M is the hypersurface $f(x) = 0$, then the hypersurface $\widehat{M}$ corresponding to $\phi([M])$ is the cyclic cover of $\mathbb{P}^{n+1}$ of degree d branched along M.
This provides a second monodromy representation

$$\hat{\rho}_M : \pi_1(\mathscr{U}_{n,d}) \to (\text{Aut } H_o^{n+1}(\widehat{M}; \mathbb{Q}))/\text{scalars}.$$

Carlson and Toledo show the image is a lattice in the Zariski closure of the image of $\hat{\rho}_M$ (a reductive group), which they show has a non-compact factor of real rank ≥ 2.

The wrong problem?

A hypersurface M in $\mathbb{P}^{n+1}$ can (and should?) be regarded as a pair $(\mathbb{P}^{n+1}, M)$. Perhaps instead we should consider the MCG

$$\Gamma_{(\mathbb{P}^{n+1},M)} := \pi_0 \operatorname{Diff}^+(\mathbb{P}^{n+1}, M)$$

where $\operatorname{Diff}^+(\mathbb{P}^{n+1}, M)$ denotes the group of orientation preserving diffeomorphisms of $\mathbb{P}^{n+1}$ that restrict to a diffeomorphism of M. The geometric monodromy of the universal hypersurface $\mathscr{X} \subset \mathscr{H}_{n,d} \times \mathbb{P}^{n+1}$ is a homomorphism

$$\lambda_{(\mathbb{P}^{n+1},M)} : \pi_1(\mathscr{H}_{n,d}, [M]) \to \Gamma_{(\mathbb{P}^{n+1},M)}$$

Question

How close is $\lambda_{(\mathbb{P}^{n+1},M)}$ to being an isomorphism?

Future directions?

- Try to understand the problem for Hyper Kähler manifolds or, more generally, Calabi–Yau manifolds. (They are simply connected and have reasonably well understood moduli spaces via work of Verbitzky, Looijenga, ...) Kreck and Su in an earlier paper (2019) show that the Torelli group of the HK-manifold $K^2(T)$ surjects onto a lattice.
- One should be able to use (mixed) Hodge theory to study the groups $\pi_1(\mathscr{M}_M)$ and Γ_M and the monodromy homomorphism $\pi_1(\mathscr{M}_M) \to \Gamma_M$.
- What can one say about mapping class groups of algebraic surfaces? E.g., for K3 surfaces or hypersurfaces in $\mathbb{P}^3$?

- A. Beauville: *Le groupe de monodromie des familles universelles d'hypersurfaces et d'intersections complètes*, Springer LNM 1194, 1986, 8–18.
- J. Carlson, D. Toledo: *Discriminant complements and kernels of monodromy representations*, Duke Math. J. 97 (1999), 621–648.
- R. Hain: *Mapping Class Groups of Simply Connected Kähler Manifolds*, [arXiv:2304.01410]
- A. Hatcher: *Concordance spaces, higher simple-homotopy theory, and applications*, Proc. Symp. Pure Math., 1978.
- M. Kreck, Y. Su: *Mapping class group of manifolds which look like 3-dimensional complete intersections*, [arXiv:2009.08054]
- D. Sullivan: *Infinitesimal computations in topology*, Publ. Math. IHES, No. 47 (1977), 269–331.