What can one say about the loci where the normal function of the Ceresa cycle is constant?

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Introductory comments

- There is much current interest in understanding when the Ceresa (or Gross–Schoen) cycle of a non-hyperelliptic curve is torsion mod algebraic equivalence: Beauville–Schoen, Qiu–Zhang, Laterveer, Laga-Shnidman.
- Here we focus, not on cycles mod algebraic equivalence, but on the torsion locus of the normal function of the Ceresa cycle.
- My goal is to introduce new *global* tools and techniques for understanding the loci in moduli where this normal function is torsion.
- For students and non-experts: this might be a tough talk — but hang in there. I'll do my best to make the material as accessible as possible. There are plenty of open questions, some computational, some foundational.

The Ceresa cycle

Throughout C will be a smooth projective curve of genus $g \geq 2$. Unless otherwise stated, the ground field will be $\mathbb{C}$. For each $x \in C$ (or $x \in \operatorname{Pic}^1 C$) the Abel–Jacobi map

$$\alpha_x : C \to \operatorname{Jac} C$$

takes y to the divisor class of $y - x$. Its image is an algebraic 1-cycle C_x in $\operatorname{Jac} C$. Set $C_x^- = \iota_* C_x$. The *Ceresa cycle* of C is the algebraic 1-cycle

$$Z_{C,x} := C_x - C_x^-$$

in $\operatorname{Jac} C$. It is homologically trivial.

Note that $Z_{C,x}$ and $Z_{C,y}$ are algebraically equivalent.

Detecting cycles via Hodge theory

Suppose that $Z = \sum_j n_j Z_j$ is an algebraic d-cycle on a smooth projective variety Y. When $[Z] = 0$ in $H_{2d}(Y)$ there is an extension

$$0 \longrightarrow H_{2d+1}(Y)(-d) \longrightarrow E_Z \longrightarrow \mathbb{Z}(0) \longrightarrow 0$$

of mixed Hodge structures (and ℓ-adic Galois modules when (Y, Z) is defined over a number field).

Construction: pull the LES of $(Y, |Z|)$ back along $cl_Z : \mathbb{Z} \to H_{2d}(|Z|)$:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H_{2d+1}(Y) & \longrightarrow & H_{2d+1}(Y, |Z|) & \longrightarrow & H_{2d}(|Z|) & \longrightarrow & H_{2d}(Y) \\ & & \| & & \uparrow & & \uparrow{\scriptstyle cl_Z} & & \vdots \\ 0 & \longrightarrow & H_{2d+1}(Y) & \longrightarrow & E_Z(d) & \longrightarrow & \mathbb{Z}(d) & \longrightarrow & 0 \end{array}$$

The top row is exact as $\dim_{\mathbb{R}} Z = 2d$, so that $H_{2d+1}(|Z|) = 0$.

The group of 1-extensions

Suppose that V is a Hodge structure of negative weight, then

$$\mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Z}, V) \cong J(V) := V_{\mathbb{C}}/(V_{\mathbb{Z}} + F^0 V).$$

If V has weight -1, then $V_{\mathbb{C}} = F^0 V \oplus \overline{F^0 V}$, which implies that

$$V_{\mathbb{R}} \to V_{\mathbb{C}}/F^0 V$$

is an $\mathbb{R}$-linear isomorphism. It induces an isomorphism

$$J(V_{\mathbb{R}}) := V_{\mathbb{R}}/V_{\mathbb{Z}} \to J(V)$$

of tori. In particular, $J(V)$ is compact (but typically not algebraic).

The Griffiths invariant

A homologically trivial d-cycle on Y thus determines a point[1]

$$\nu_Z \in J(H_{2d+1}(Y)).$$

This depends only on the rational equivalence class of Z.

The Ceresa cycle $Z_{C,x} = C_x - C_x^-$ determines

$$\nu_{C,x} \in J(H_3(\operatorname{Jac} C)) = J(\Lambda^3 H)$$

where $H = H_1(C)$.

[1] From now on I will suppress the Tate twist — always twist so that the odd weight Hodge structure V in $J(V)$ has weight -1.

Let $a_1, \dots, a_g, b_1, \dots, b_g$ be a symplectic basis of H. Set

$$\theta := \sum_{j=1}^{g} a_j \wedge b_j \in \Lambda^2 H$$

Multiplication by θ is an injective morphism of Hodge structures

$$H \hookrightarrow \Lambda^3 H(-1)$$

It induces an inclusion

$$\operatorname{Jac} C = J(H) \hookrightarrow J(\Lambda^3 H).$$

Eliminating the base point

Proposition (Pulte)

If $x, y \in C$, then

$$\nu_{C,x} - \nu_{C,y} = \textit{the image of } 2([x]-[y]) \in \operatorname{Jac} C \subset J(\Lambda^3 H)$$

Set

$$\Lambda^3_0 H = (\Lambda^3 H)/(\theta \cdot H).$$

The image of $\nu_{C,x}$ in $J(\Lambda^3_0 H)$ does not depend on $x \in C$. It vanishes when C is hyperelliptic.

Notation: Denote the image of $\nu_{C,x}$ in $J(\Lambda^3_0 H)$ by ν_C.

Families of homologically trivial cycles

Suppose that $f : Y \to X$ is a smooth projective morphism and that Z an algebraic cycle on Y whose restriction to each fiber is homologically trivial and has dimension d and codimension e:

$$\begin{array}{ccccc} Z_x & \hookrightarrow & Z & \longrightarrow & Y \\ \downarrow & & \downarrow & & \downarrow f \\ \{x\} & \longrightarrow & X & = & X \end{array}$$

Set

$$\mathbb{V} = R^{2e-1} f_* \mathbb{Z}_X(e)$$

This has fiber $H_{2d+1}(Y_x)(-d)$ over $x \in X$ and weight -1.

The normal function of a family of cycles

The fiberwise Griffiths construction defines a section $\nu_Z : X \to J(\mathbb{V})$

$$\nu_Z : x \mapsto \nu_{Z_x} \in J(V_x)$$

of the family of intermediate jacobians

$$J(\mathbb{V}) \to X$$

It is called the *normal function* of the family of cycles (Y, Z) over X.

This section is holomorphic and satisfies many technical conditions which I will suppress.

The normal function of the Ceresa cycle

In the case of the Ceresa cycle, $f : Y \to X$ is the universal jacobian

$$\mathcal{J} \to \mathcal{M}_g$$

over the moduli space (better, the moduli stack) of smooth projective curves of genus g, where $g > 2$.

We thus have the *Ceresa* normal function

$$\begin{array}{ccc} \nu(C) \in J(\Lambda^3_0 H_1(C)) & \hookrightarrow & J(\Lambda^3_0 \mathbb{H}) \\ \uparrow\big| & & \big\downarrow\uparrow \nu \\ [C] & \longmapsto & \mathcal{M}_g \end{array}$$

It vanishes on the hyperelliptic locus.

Constant and torsion sections

- Suppose $\mathbb{V}$ is a variation of Hodge structure of weight -1 over X. Assume $\mathbb{V}_{\mathbb{Z}}$ is torsion free.
- A section of $J(\mathbb{V})$ over X is **constant** if it is a *constant* section of the constant sub family

$$J(H^0(X, \mathbb{V})) \subseteq J(\mathbb{V}).$$

- A section of $J(\mathbb{V})$ is **torsion** if a positive multiple of it vanishes.
- A section is **torsion mod constants** if a positive multiple of it is constant.

Theorem (Brosnan–Pearlstein, Kato et al, Schnell)

The locus where a normal function is torsion (or torsion mod constants) is an algebraic subvariety of X.

Question:

1. *What can we say about the components of the locus of points in X where the normal function of a homologically trivial cycle is constant? Or, more generally, torsion mod constants?*
2. *Is there a bound on the order of the restriction of the genus g Ceresa cycle to components of its torsion locus?*
3. *Is the dimension of the components of the Ceresa torsion locus in $\mathcal{M}_g - \mathcal{H}_g$ bounded?*

Level structures

To avoid working with stacks, we impose a level structure:

A level ℓ structure on a genus g curve C is an isomorphism

$$(\mathrm{Jac}\, C)[\ell] \cong (\mathbb{Z}/\ell\mathbb{Z})^{2g}$$

where the Weil pairing on the LHS corresponds to the "standard" symplectic inner product on the RHS. The moduli space $\mathcal{M}_g(\ell)$ of smooth projective curves with a level $\ell \geq 3$ structure is a smooth quasi-projective variety.

Note: The finite symplectic group $\mathrm{Sp}_g(\mathbb{Z}/\ell)$ acts on $\mathcal{M}_g(\ell)$. The quotient stack $\mathrm{Sp}_g(\mathbb{Z}/\ell)\backslash\backslash\mathcal{M}_g(\ell)$ is $\mathcal{M}_g$.

Torsion leaves are affine

Theorem

If the restriction of the Ceresa normal function to the closed subvariety X of $\mathcal{M}_g(\ell)$ is **torsion mod constants**, *then X is affine.*

Trivial example: X is a point.

Example: The Ceresa normal function vanishes on the hyperelliptic locus. It is affine.

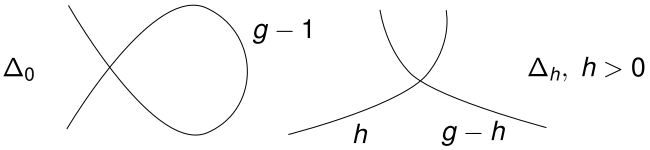
Remark: The converse is not true: if T is an ample curve in $\mathcal{M}_g(\ell)$ whose closure contains at least one boundary point, then T is affine but no multiple of ν is constant as $\pi_1(T, t_0) \to \pi_1(\mathcal{M}_g(\ell), t_0)$ is surjective by Lefschetz.

The Deligne–Mumford compactification

The Deligne–Mumford compactification $\overline{\mathcal{M}}_g$ of $\mathcal{M}_g$ is obtained by adding points corresponding to stable (nodal) curves of genus g to $\mathcal{M}_g$. It has boundary $\Delta := \overline{\mathcal{M}}_g - \mathcal{M}_g$ (a divisor with normal crossings) with irreducible components

$$\Delta = \Delta_0 \cup \Delta_1 \cup \cdots \cup \Delta_{\lfloor g/2 \rfloor}.$$

The generic point of Δ_0 is an irreducible curve with one node. The generic point of Δ_h, when $h > 0$, has one node and two smooth irreducible components, one of genus h and the other of genus $g - h$.



There is a natural compactification $\overline{\mathcal{M}}_g(\ell)$ of $\mathcal{M}_g(\ell)$ where $\overline{\mathcal{M}}_g(\ell) \to \overline{\mathcal{M}}_g$ is ramified over Δ.

The Picard group of $\overline{\mathcal{M}}_g$

The Hodge bundle $\mathcal{E}$ over $\mathcal{M}_g$ has fiber $H^0(\Omega_C^1)$ over $[C]$. It extends to $\overline{\mathcal{M}}_g$. Set $\mathcal{L} := \det \mathcal{E}$.

Denote the class of Δ_h in $\operatorname{Pic} \overline{\mathcal{M}}_g$ by δ_h and the class of $\mathcal{L}$ by λ. When $g \geq 3$

$$\operatorname{Pic} \mathcal{M}_g = \mathbb{Z}\lambda$$

and

$$\operatorname{Pic} \overline{\mathcal{M}}_g = \mathbb{Z}\lambda \oplus \bigoplus_{0 \leq h \leq g/2} \mathbb{Z}\delta_h.$$

Note: These divisor classes pull back to $\overline{\mathcal{M}}_g(\ell)$. The line bundle $\mathcal{L}$ is ample on $\mathcal{M}_g(\ell)$.

Now suppose that $\overline{X}$ is a smooth projective variety and that $f : X \to \overline{\mathcal{M}}_g$ is a morphism. Set $\Delta_X = f^{-1}(\Delta)$ and $X = \overline{X} - \Delta_X$.

Theorem

There is a computable effective $\mathbb{Q}$-divisor $j_{\overline{X}}$ on $\overline{X}$, supported on Δ_X, with the following properties:

1. *The divisor*

$$(8g+4)\lambda - \left(g\delta_0 + 4\sum_{h=1}^{\lfloor g/2 \rfloor} h(g-h)\delta_h + j_{\overline{X}}\right)$$

   has non-negative degree on all complete curves T in $\overline{X}$ with $f(T) \not\subset \Delta_0$.

2. *The equality*

$$(8g+4)\lambda = g\delta_0 + 4\sum_{h=1}^{\lfloor g/2 \rfloor} h(g-h)\delta_h + j_{\overline{X}}$$

   holds in $(\operatorname{Pic}\overline{X}) \otimes \mathbb{Q}$ if and only if ν is constant mod torsion on X.

Remarks

- $j_{\overline{X}}$ is the *jumping divisor*, which I will define shortly.
- It is a non-negative $\mathbb{Q}$-linear combination of codimension 1 boundary components of X. (Brosnan–Pearlstein, Burgos–Holmes–de Jong)
- The jumping divisor can be computed using the work of Brosnan–Pearlstein and/or de Jong–Shokrieh.
- The divisor class

$$M := (8g+4)\lambda - g\delta_0 - 4\sum_{h=1}^{\lfloor g/2 \rfloor} h(g-h)\delta_h \in \operatorname{Pic}\overline{\mathcal{M}}_g$$

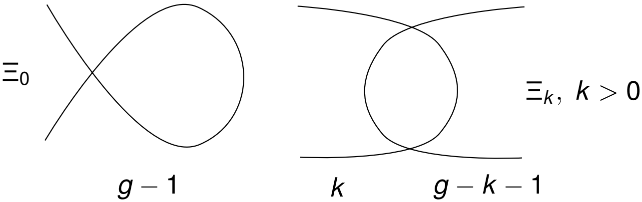
  is the *Moriwaki divisor* (class). It plays a role in the Arakelov geometry of $\mathcal{M}_g$.
- The first statement is a strengthened version of Moriwaki's inequality.

Example: hyperelliptic curves

The boundary $\overline{\mathcal{H}}_g - \mathcal{H}_g$ (a normal crossing divisor) has components

$$\Delta_h,\ 0 < h \leq g/2 \text{ and } \Xi_k,\ 0 \leq k \leq (g-1)/2.$$

The generic point of Ξ_k is an irreducible hyperelliptic curve with one node when $k = 0$ and the union of two hyperelliptic curves of genera k and $g - k - 1$ when $k > 0$.



$$\operatorname{Pic}\overline{\mathcal{H}}_g = \mathbb{Z}\xi_0 \oplus \bigoplus_{0<k<(g-1)/2} \mathbb{Z}\xi_k \oplus \bigoplus_{0<h\leq g/2} \mathbb{Z}\delta_h$$

- The restriction mapping $\operatorname{Pic} \overline{\mathcal{M}}_g \to \operatorname{Pic} \overline{\mathcal{H}}_g$ is

$$\delta_0 \mapsto \xi_0 + 2 \sum_{k>0} \xi_k \text{ and } \delta_h \mapsto \delta_h \text{ when } h > 0.$$

- Since $\nu \equiv 0$ on $\mathcal{H}_g$, the second theorem + Cornalba–Harris imply that in $\operatorname{Pic} \overline{\mathcal{H}}_g$

$$M - j_{\overline{\mathcal{H}}_g} = 0 = (8g+4)\lambda - g\xi_0 - 2 \sum_{k=1}^{\lfloor (g-1)/2 \rfloor} (k+1)(g-k)\xi_k - 4 \sum_{h=1}^{\lfloor g/2 \rfloor} h(g-h)\delta_h.$$

- So the jumping divisor of $\overline{\mathcal{H}}_g$ is

$$j_{\overline{\mathcal{H}}_g} = 2 \sum_{k=1}^{\lfloor (g-1)/2 \rfloor} k(g-k-1)\xi_k \in \operatorname{Pic} \overline{\mathcal{H}}_g$$

I'll discuss the elements of the second theorem first, then sketch the proof of the first.

Biextensions

Suppose V is a Hodge structure of weight -1. Recall

$$\begin{aligned} J(V) &= \operatorname{Ext}^1(\mathbb{Z}, V) \\ &= \{\text{MHSs with weight graded quotients } \mathbb{Z}, V\} \end{aligned}$$

The dual torus is

$$J(V)^\vee := \operatorname{Pic}^0 J(V) = \operatorname{Ext}^1(V, \mathbb{Z}(1)).$$

A polarization $\phi : V \otimes V \to \mathbb{Z}(1)$ induces an isogeny

$$J(V) \to J(V)^\vee.$$

Set

$$B(V) = \{\text{MHSs with weight graded quotients } \mathbb{Z}, V, \mathbb{Z}(1)\}$$

This is the set of biextensions E with $\mathrm{Gr}^W_{-1} E = V$. Have

$$B(V) \to J(V) \times J(V)^\vee, \quad E \mapsto (E/W_{-2}, W_{-1}E)$$

This is a torsor under $\mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Z}, \mathbb{Z}(1)) = \mathbb{C}^\times$.

Example: If p, q, r, s are distinct points on the curve C, then

$$H_1(C - \{p, q\}, \{r, s\}) \in B(H_1(C)).$$

Have

$$H_1(C, \{r, s\}) \in \mathrm{Ext}^1(\mathbb{Z}, H)$$

$$H_1(C - \{p, q\}) = \mathrm{Hom}(H_1(C, \{p, q\}), \mathbb{Z}(1)) \in \mathrm{Ext}^1(H, \mathbb{Z}(1)).$$

The biextension metric

The $\mathbb{C}^\times$ bundle $B(V) \to J(V) \times J(V)^\vee$ has a canonical metric. Set

$$B_{\mathbb{R}}(V) = \{\mathbb{R}\text{-MHSs with weight graded quotients } \mathbb{R}, V_{\mathbb{R}}, \mathbb{R}(1)\}$$

Since $\mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{R}, V) = V_{\mathbb{C}}/(V_{\mathbb{R}} + F^0 V) = 0$, there are canonical isomorphisms

$$h : B_{\mathbb{R}}(V) \xrightarrow{\simeq} \mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{R}, \mathbb{R}(1)) \xrightarrow{\simeq} \mathbb{R}.$$

Define

$$\|E\| = \exp h(E_{\mathbb{R}}) \in \mathbb{R}^\times$$

Relative version

When $\mathbb{V}$ is a polarized variation of Hodge structure of weight -1 over X, we have bundles

$$B_X \longrightarrow J(\mathbb{V}) \times_X J(\mathbb{V})^\vee \longrightarrow X$$

Its restriction to $x \in X$ is

$$B(V_x) \to J(V_x) \times J(V_x)^\vee \to \{x\}.$$

It is naturally metrized. We can pull this back along

$$J(\mathbb{V}) \to J(\mathbb{V}) \times_X J(\mathbb{V})^\vee$$

to obtain a metrized $\mathbb{C}^\times$ bundle over $J(\mathbb{V})$. Denote the corresponding metrized line bundle by

$$\mathscr{B}(\mathbb{V}) \to J(\mathbb{V}).$$

Biextension line bundles

This can be pulled back along a normal function $\nu : X \to J(\mathbb{V})$ to obtain a metrized line bundle

$$\mathscr{B}_X := \nu^* \mathscr{B}(\mathbb{V})$$

over X. Suppose that $\overline{X}$ is a smooth projective completion of X

Theorem (Lear, 1990; B–P, 2019; B–H–deJ, 2019)

A positive power $\mathscr{B}_X^{\otimes r}$ of the biextension line bundle $\mathscr{B}_X$ extends to a holomorphic line bundle $\mathscr{B}^r_{\overline{X}}$ over $\overline{X}$ with the property that the metric on $\mathscr{B}_X^{\otimes r}$ extends to a continuous metric on $\overline{X} - \Delta_X^{\mathrm{sing}}$.

Remark: (Brosnan–Pearlstein) If ν is admissible (and $r = 1$), then $H^0(X, \mathscr{B}^\times_{\overline{X}})$ is the space of admissible biextensions over X with extensions $\nu \in \mathrm{Ext}^1_{\mathrm{MHS}(X)}(\mathbb{Z}, \mathbb{V})$ and $\nu^\vee \in \mathrm{Ext}^1_{\mathrm{MHS}(X)}(\mathbb{V}, \mathbb{Z}(1))$.

Positivity

There is a unique 2-form ω_ϕ on $J(\mathbb{V})$ that is translation invariant on every fiber, corresponds to the polarization $\phi \in H^2(J(V_x))$ and is locally constant with respect to the isomorphism

$$J(\mathbb{V}) \cong \mathbb{V}_{\mathbb{R}}/\mathbb{V}_{\mathbb{Z}}.$$

Proposition (H, 2014; Pearlstein–Peters, 2019)

The curvature of $\mathscr{B}_X$ is $2\nu^*\omega_\phi$. It is a semi-positive $(1,1)$-form on X that extends to a locally L_1 form (current) on $\overline{X} - S$, where $S \subset \Delta_X^{\mathrm{sing}}$. It vanishes if and only if ν is constant.

Now take $X = \mathcal{M}_g$, $\overline{X} = \overline{\mathcal{M}}_g$.

Theorem (H-Reed, 2004)

The biextension line bundle of the normal function of the Ceresa cycle is the Moriwaki line bundle:

$$c_1(\mathscr{B}_{\overline{\mathcal{M}}_g}) = (8g+4)\lambda - g\delta_0 - 4\sum_{h>0} h(g-h)\delta_h \in \operatorname{Pic}\overline{\mathcal{M}}_g.$$

Theorem

A normal function ν is constant if and only if $\mathscr{B}_{\overline{X}}$ is trivial on $\overline{X}$.

Corollary

$k\nu$ is constant on X if and only if $\mathscr{B}_{\overline{X}}^{\otimes k^2}$ is trivial on $\overline{X}$.

This establishes the second part of the second theorem.

For this result to be useful, we need to be able to compute the Chern class of $\mathscr{B}_{\overline{X}}$. This leads us to *height jumping* . . .

Height jumping

Consider $f : (\overline{X}, X) \to (\overline{Y}, Y)$ with $\overline{X}$ and $\overline{Y}$ smooth. Have PVHS $\mathbb{V}$ over Y and normal function $\nu : Y \to J(\mathbb{V})$. Have $\mathscr{B}_X = f^*\mathscr{B}_Y$, so

$$f^*\mathscr{B}_{\overline{Y}} = \mathscr{B}_{\overline{X}}(j_{\overline{X}/\overline{Y}}).$$

where $j_{\overline{X}/\overline{Y}}$ is supported on Δ_Y. The following result is key.

Theorem (Brosnan–Pearlstein, Burgos–Holmes–de Jong; 2019)

The jumping divisor $j_{\overline{X}/\overline{Y}}$ is effective.

Corollary (strengthened Moriwaki inequality)

The Moriwaki divisor has non-negative degree on all curves in $\overline{\mathcal{M}}_g$ that do not lie in Δ_0.

This establishes part (1) of the second theorem.

Why height jumping?

Toy example: Consider a biextension defined over $\mathbb{D}^* \times \mathbb{D}^*$ with coordinates (z_1, z_2). The biextension bundle $\mathscr{B}$ extends to (as a necessarily trivial) line bundle over $\mathbb{D}^2$. In general, the metric is defined only over $\mathbb{D}^2 - \{0\}$. When there is height jumping, the metric has the form

$$\log \|\sigma(z_1, z_2)\|_{\mathscr{B}} \sim -\log|z_1| \log|z_2| / (\log|z_1| + \log|z_2|)$$

where σ is a trivializing section and $\sim$ means that the difference is a bounded function, smooth on $(\mathbb{D}^*)^2$.

If $0 < |a| < 1$, then

$$\log \|\sigma(t, a)\|_{\mathscr{B}} \sim -\log|a| / (1 + \log|a| / \log|t|)$$

which is continuous in t and bounded near $t = 0$. So the metric is continuous over $\mathbb{D}^2 - \{0\}$. However ...

If we restrict to the curve $f : \mathbb{D} \to \mathbb{D}^2$ given by $t \mapsto (t^{n_1}, t^{n_2})$ with $n_1, n_2 > 0$, then

$$\log \|\sigma(t)\|_{\mathscr{B}} \sim -\frac{n_1 n_2}{n_1 + n_2} \log |t|$$

so that $\|\sigma(t)/t^{n_1 n_2/(n_1+n_2)}\| \sim 1$. This implies that

$$j_f = \frac{n_1 n_2}{n_1 + n_2}[0].$$

Note: In the normal crossing case, the local behaviour at a boundary point p of the biextension metric is determined by a local monodromy representation

$$\rho_p : \pi_1((\mathbb{D}^*)^d, *) \to \mathrm{Sp}(V) \ltimes V.$$

Meme: $\log \|\sigma\|$ is a period of a real variation of mixed Hodge structure. Its growth as one approaches the boundary point p along a curve is a rational multiple of $\log |t|$ determined by the local monodromy ρ_p.

Sketch of proof of the affine result

Suppose that X is a closed subvariety of $\mathcal{M}_g(\ell)$ on which ν is torsion mod constants.

- Extend to a smooth compactification $\overline{X} \to \overline{\mathcal{M}}_g(\ell)$.
- Have $f^*M = j_{\overline{X}}$ mod torsion in $\operatorname{Pic}\overline{X}$, where $j_{\overline{X}} := j_{\overline{X}/\overline{\mathcal{M}}_g}$.
- Rearrange this to see that a positive multiple of λ is supported on an effective divisor with support $|\Delta_{\overline{X}}|$.
- But λ is ample on $\mathcal{M}_g(\ell)$ (Baily), and therefore on X. So the pullback of an ample divisor on the closure of the image of X in $\mathbb{P}^N$ is effective divisor with support $|\Delta_{\overline{X}}|$.
- Conclude that $X = \overline{X} - \Delta_{\overline{X}}$ is affine.

Question and/or (wild) speculation

Question:

Is there an arithmetic or Arakelov version of of the main theorem?

One possible version:

- Suppose C is a smooth curve over a number field K ...
- and that $\mathscr{C}$ is a (flat, regular, ...) model of C over $\mathcal{O}_K$.
- Define the arithmetic Moriwaki divisor $\widehat{M} \in \widehat{\mathrm{Pic}}\, \mathcal{M}_{g/\mathbb{Z}}$.
- Define $\widehat{M}_{\mathscr{C}} \in \widehat{\mathrm{Pic}}(\mathrm{Spec}\, \mathcal{O}_K)$ to be the pullback of $\widehat{M}$ along $\mathrm{Spec}\, \mathcal{O}_K \to \overline{\mathcal{M}}_{g/\mathbb{Z}}$.
- Define the arithmetic jumping divisor $\widehat{\mathscr{J}}_{\mathscr{C}} \in \widehat{\mathrm{Pic}}(\mathrm{Spec}\, \mathcal{O}_K)$ as a sum over the singular fibers of $\mathscr{C} \to \mathrm{Spec}\, \mathcal{O}_K$ using the local Galois representations $G_{K_{\mathfrak{P}}} \to \mathrm{GSp}(H_{\mathbb{Z}_\ell}) \ltimes V_{\mathbb{Z}_\ell}$, where $\mathfrak{P} \nmid \ell$. (Assume $\widehat{\mathscr{J}}_{\mathscr{C}}$ is integral.)
- Is the class of the Ceresa (or GS) cycle in $\mathrm{CH}^1(\mathrm{Jac}\, C)$ trivial mod translations if and only if $\widehat{M}_{\mathscr{C}} = \widehat{\mathscr{J}}_{\mathscr{C}}$?

References

[1] A. Beauville, C. Schoen: *A non-hyperelliptic curve with torsion Ceresa cycle modulo algebraic equivalence*, IMRN 2023, 3671–3675.

[2] P. Brosnan, G. Pearlstein: *On the algebraicity of the zero locus of an admissible normal function*, Compos. Math. 149 (2013), 1913–1962.

[3] P. Brosnan, G. Pearlstein: *Jumps in the Archimedean height*, Duke Math. J. 168 (2019), 1737–1842.

[4] J. Burgos Gil, D. Holmes, R. de Jong: *Positivity of the height jump divisor*, IMRN 2019, 2044–2068.

[5] M. Cornalba, J. Harris: *Divisor classes associated to families of stable varieties, with applications to the moduli space of curves*, Ann. Sci. École Norm. Sup. (4) 21 (1988), 455–475.

[6] R. Hain: *Biextensions and heights associated to curves of odd genus*, Duke Math. J. 61 (1990), 859–898.

[7] R. Hain: *Normal functions and the geometry of moduli spaces of curves*, Handbook of moduli. Vol. I, 527–578, Adv. Lect. Math. 24, 2013.

[8] R. Hain, D. Reed: *On the Arakelov geometry of moduli spaces of curves*, J. Diff. Geom. 67 (2004), 195–228.

[9] R. de Jong, F. Shokrieh: *Jumps in the height of the Ceresa cycle*, J. Diff. Geom. 126 (2024), 169–214.

[10] K. Kato, C. Nakayama, S. Usui: *Analyticity of the closures of some Hodge theoretic subspaces*, Proc. Japan Acad. Ser. A Math. Sci. 87 (2011), 167–172.

[11] M. Kerr, S. Tayou: *On the torsion locus of the Ceresa normal function*, [arXiv:2406.19366]

[12] J. Laga, A. Shnidman: *Ceresa cycles of bielliptic Picard curves*, [arXiv:2312.12965]

[13] R. Laterveer: *On the tautological ring of Humbert curves*, [arXiv:2211.15291]

[14] D. Lear: *Extensions of normal functions and asymptotics of the height pairing*. Ph.D. Thesis, University of Washington, 1990, 90 pp.

[15] A. Moriwaki: *A sharp slope inequality for general stable fibrations of curves*, J. Reine Angew. Math. 480 (1996), 177–195.

[16] G. Pearlstein: *SL_2-orbits and degenerations of mixed Hodge structure*, J. Diff. Geom. 74 (2006), 1–67.

[17] Congling Qiu, Wei Zhang: *Vanishing results in Chow groups for the modified diagonal cycles*, [arXiv:2209.09736]

[18] C. Schnell: *Complex analytic Néron models for arbitrary families of intermediate Jacobians*, Invent. Math. 188 (2012), 1–81.