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The Canonical Dual Frame of a Wavelet Frame

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In this paper we show that there exist wavelet frames that have nice dual wavelet frames, but for which the *canonical* dual frame does not consist of wavelets, i.e., cannot be generated by the translates and dilates of a single function.

Key Words: wavelet frame; the canonical dual frame; dual wavelet frame.

1. INTRODUCTION

We start by recalling some notations and definitions. Let $\mathcal{H}$ be a Hilbert space. A set of elements $\{h_k\}_{k\in\mathbb{Z}}$ in $\mathcal{H}$ is said to be a *frame* (see [5]) in $\mathcal{H}$ if there exist two positive constants A and B such that

$$A\|f\|^2 \le \sum_{k\in\mathbb{Z}} |\langle f, h_k\rangle|^2 \le B\|f\|^2, \qquad \forall f \in \mathcal{H}, \tag{1.1}$$

where A and B are called the lower frame bound and upper frame bound, respectively. In particular, when $A = B = 1$, we say that $\{h_k\}_{k\in\mathbb{Z}}$ is a (*normalized*) *tight frame* in $\mathcal{H}$. The

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frame operator S: $\mathcal{H} \to \mathcal{H}$, which is associated with a frame $\{h_k\}_{k\in\mathbb{Z}}$, is defined to be

$$Sf := \sum_{k\in\mathbb{Z}} \langle f, h_k\rangle h_k, \qquad f \in \mathcal{H}. \tag{1.2}$$

It is evident that $\{h_k\}_{k\in\mathbb{Z}}$ is a frame in $\mathcal{H}$ with lower frame bound A and upper frame bound B if and only if S is well defined and $AI \le S \le BI$, where I denotes the identity operator on $\mathcal{H}$.

Let $\{h_k\}_{k\in\mathbb{Z}}$ be a frame in $\mathcal{H}$. If $\{\tilde{h}_k\}_{k\in\mathbb{Z}}$ is another frame in $\mathcal{H}$ such that

$$\langle f, g\rangle = \sum_{k\in\mathbb{Z}} \langle f, \tilde{h}_k\rangle \langle h_k, g\rangle, \qquad \forall f, g \in \mathcal{H}, \tag{1.3}$$

then $\{\tilde{h}_k\}_{k\in\mathbb{Z}}$ is called a *dual frame* of $\{h_k\}_{k\in\mathbb{Z}}$.

Let $\{h_k\}_{k\in\mathbb{Z}}$ be a frame in a Hilbert space $\mathcal{H}$ with lower frame bound A and upper frame bound B. Then its frame operator S is invertible and therefore, $\{S^{-1}h_k\}_{k\in\mathbb{Z}}$ is a dual frame of $\{h_k\}_{k\in\mathbb{Z}}$ since $B^{-1}I \le S^{-1} \le A^{-1}I$ and

$$f = SS^{-1}f = \sum_{k\in\mathbb{Z}} \langle f, S^{-1}h_k\rangle h_k, \qquad \forall f \in \mathcal{H}. \tag{1.4}$$

In the literature, $\{S^{-1}h_k\}_{k\in\mathbb{Z}}$ is called the *canonical dual frame* of the frame $\{h_k\}_{k\in\mathbb{Z}}$ and (1.4) is the canonical representation of any element f in $\mathcal{H}$ using the frame $\{h_k\}_{k\in\mathbb{Z}}$ in $\mathcal{H}$.

Typically, for a given frame $\{h_k\}_{k\in\mathbb{Z}}$ in $\mathcal{H}$, there exist many dual frames other than its canonical dual frame. However, the canonical dual frame $\{S^{-1}h_k\}_{k\in\mathbb{Z}}$ enjoys the following optimal property (see [5]). For any element $f \in \mathcal{H}$,

$$\sum_{k\in\mathbb{Z}} |\langle f, S^{-1}h_k\rangle|^2 \le \sum_{k\in\mathbb{Z}} |c_k|^2, \qquad \text{whenever} \quad f = \sum_{k\in\mathbb{Z}} c_k h_k,\ \{c_k\}_{k\in\mathbb{Z}} \in \ell_2(\mathbb{Z}).$$

In other words, the representation in (1.4) using the canonical dual frame has the smallest ℓ_2 norm of the frame coefficients to represent a given element in $\mathcal{H}$.

Generally speaking, frames with certain structures, such as frames of translates, Gabor frames, and wavelet frames, are very important in both theory and application. The interested reader can consult [2, 3, 5] and references therein on the theory of frames. In the following, let us look at several important families of frames and their canonical dual frames.

A frame of translates is obtained from several functions in $L_2(\mathbb{R})$ by integer shifts. That is, $\{\psi^\ell(\cdot - k)\colon k \in \mathbb{Z},\ \ell = 1, \dots, r\}$ is a frame in the Hilbert space $\mathcal{H}$ defined as the closed linear span of $\psi^\ell(\cdot - k),\ k \in \mathbb{Z},\ \ell = 1, \dots, r$. Then it is well known that its canonical dual frame is given by $\{[S^{-1}\psi^\ell](\cdot - k)\colon k \in \mathbb{Z},\ \ell = 1, \dots, r\}$; like the original frame, it is generated by the integer shifts of r functions since $S[f(\cdot - k)] = [Sf](\cdot - k)$ for all $k \in \mathbb{Z}$ and $f \in \mathcal{H}$.

A Gabor frame is generated from several functions in $L_2(\mathbb{R})$ by modulates and integer shifts. In other words, a Gabor frame is a frame in $L_2(\mathbb{R})$ given by the set $\{e^{i2\pi jb\cdot}\psi^\ell(\cdot - ka)\colon j, k \in \mathbb{Z},\ \ell = 1, \dots, r\}$, where a and b are real positive numbers. A direct calculation shows that $S[e^{i2\pi jb\cdot} f(\cdot - ka)] = e^{i2\pi jb\cdot}[Sf](\cdot - ka)$ for all $j, k \in \mathbb{Z}$ and $f \in L_2(\mathbb{R})$.

Therefore, the canonical dual frame of a Gabor frame generated by $\{\psi^1, \dots, \psi^r\}$ is also generated by modulating and shifting the r functions $\{S^{-1}\psi^1, \dots, S^{-1}\psi^r\}$.

This brings us to wavelets. Throughout this paper, we shall use the following notation

$$f_{j,k} := 2^{j/2} f(2^j \cdot -k), \qquad j, k \in \mathbb{Z},\ f \in L_2(\mathbb{R}).$$

A wavelet frame is generated from several functions in $L_2(\mathbb{R})$ by dilates and integer shifts. We say that $\{\psi^1, \dots, \psi^r\}$ generates *a wavelet frame* in $L_2(\mathbb{R})$ if $\{\psi^\ell_{j,k}\colon\ j, k \in \mathbb{Z},\ \ell = 1, \dots, r\}$ is a frame in $L_2(\mathbb{R})$. We say that $\{\psi^1, \dots, \psi^r\}$ and $\{\tilde\psi^1, \dots, \tilde\psi^r\}$ generate *a pair of dual wavelet frames* if each of them generates a wavelet frame and

$$\langle f, g\rangle = \sum_{\ell=1}^{r} \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} \langle f, \tilde\psi^\ell_{j,k}\rangle \langle \psi^\ell_{j,k}, g\rangle, \qquad \forall f, g \in L_2(\mathbb{R}). \tag{1.5}$$

That is, $\{\psi^1, \dots, \psi^r\}$ and $\{\tilde\psi^1, \dots, \tilde\psi^r\}$ generate a pair of dual wavelet frames if and only if $\{\tilde\psi^\ell_{j,k}\colon\ j, k \in \mathbb{Z},\ \ell = 1, \dots, r\}$ is a dual frame of the frame $\{\psi^\ell_{j,k}\colon\ j, k \in \mathbb{Z},\ \ell = 1, \dots, r\}$.

In view of our observations for frames of translates and for Gabor frames, a natural question is then whether the canonical dual frame of a wavelet frame still has the wavelet structure. This question was negatively answered in Daubechies [2] and Chui and Shi [1]. For any $0 < \varepsilon < 1$, define

$$\psi = \eta + \varepsilon\sqrt{2}\eta(2\cdot), \qquad \text{where } \eta \text{ generates an orthonormal wavelet basis in } L_2(\mathbb{R}). \tag{1.6}$$

Then it was demonstrated in [1, 2] that ψ generates a wavelet frame, but that its canonical dual frame cannot be generated by a single function. Moreover, it was also demonstrated in [2] that even when ψ belongs to the Schwartz class, $S^{-1}\psi$ does not belong to L_p for sufficiently small $p - 1 > 0$. In fact, as shown in [1], $S^{-1}\psi = \sum_{k=0}^{\infty} \varepsilon^k 2^{-k/2} \psi(2^{-k}\cdot)$, so that $S^{-1}\psi$ is not compactly supported if ψ has compact support.

This example demonstrates that the canonical dual frame of a wavelet frame may not preserve the "good qualities" of a wavelet system, such as the dilation-shift structure, good smoothness, and compact support. In this particular case, the $\psi_{j,k}$ $(j, k \in \mathbb{Z})$ constitute not only a frame but a *Riesz basis* (i.e., $\sum_{j,k\in\mathbb{Z}} c_{j,k}\psi_{j,k} = 0$ together with $\sum_{j,k\in\mathbb{Z}} |c_{j,k}|^2 < \infty$ implies $c_{j,k} = 0$ for all $j, k \in \mathbb{Z}$), which implies that the dual frame is unique. There does not, therefore, exist a function $\tilde\psi$ such that ψ and $\tilde\psi$ generate a pair of dual wavelet frames.

It is not clear for which wavelet frames the canonical dual frame consists of wavelets as well. In the example above, there were no dual wavelet frames at all. This makes one wonder whether the existence of dual wavelet frames for a given wavelet frame would be sufficient to force the canonical dual frame to consist likewise of wavelets. Even though this question has not been stated explicitly in the literature to our knowledge, it certainly is implicit in [3]. We show here that canonical dual frames of wavelet frames can still be "bad" even if there exist dual wavelet frames. More precisely, we have the following

MAIN THEOREM 1.1. *There exists* $\{\psi^1, \dots, \psi^r\}$ *in* $L_2(\mathbb{R})$ *such that*

(1) $\{\psi^1, \dots, \psi^r\}$ *generates a wavelet frame;*

(2) *its canonical dual frame* $\{S^{-1}[\psi^\ell_{j,k}]\colon\ j, k \in \mathbb{Z},\ \ell = 1, \dots, r\}$ *cannot be generated by dilates and integer shifts of r functions;*

(3) *there exist* $\{\tilde{\psi}^1, \ldots, \tilde{\psi}^r\}$ *such that* $\{\psi^1, \ldots, \psi^r\}$ *and* $\{\tilde{\psi}^1, \ldots, \tilde{\psi}^r\}$ *generate a pair of dual wavelet frames in* $L_2(\mathbb{R})$.

Note that a construction for $r = r_0$ automatically gives a construction for any other positive integer $r > r_0$ by taking $\psi^\ell = 0$ for $r_0 + 1 \le \ell \le r$.

The structure of this paper is as follows. In Section 2, we handle the case $r = 2$ without any computation. In Section 3, we shall give an example of $\psi \in L_2(\mathbb{R})$ such that $\hat{\psi} \in C^\infty$ is compactly supported and ψ generates a wavelet frame in $L_2(\mathbb{R})$, but such that its canonical dual frame cannot be generated by a single function using dilates and integer shifts; yet we demonstrate for this example that there exist infinitely many functions $\tilde{\psi}$ such that ψ and $\tilde{\psi}$ generate a pair of dual wavelet frames in $L_2(\mathbb{R})$. In Section 4, we illustrate a similar result for wavelet frames with compact support.

Our explicit constructions have the added property that the wavelets can be derived from a multiresolution hierarchy, as is desirable in applications. They therefore settle another implicit question: it is not because a wavelet frame is derived from a refinable function that its canonical dual frame consists of wavelets as well. For more detail on this, see Section 3. We believe that these examples show that one should not attach too much weight to the "optimality" property of the canonical dual frame.

2. THE CANONICAL DUAL FRAME OF A WAVELET FRAME

In this section, we discuss the canonical dual frame of a wavelet frame in general. Then we establish the claim made in Main Theorem 1.1 in Section 1 for the case $r = 2$.

Define the dilation and shift operators on $L_2(\mathbb{R})$ (see [8]) as follows,

$$D^j f := 2^{j/2} f(2^j \cdot) \qquad \text{and} \qquad E^k f = f(\cdot - k), \qquad j, k \in \mathbb{Z},\ f \in L_2(\mathbb{R}).$$

Assume that $\{\psi^1, \ldots, \psi^r\}$ generates a wavelet frame in $L_2(\mathbb{R})$. Let S denote its frame operator defined as follows:

$$Sf = \sum_{\ell=1}^{r} \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} \langle f, \psi^\ell_{j,k}\rangle \psi^\ell_{j,k}, \qquad f \in L_2(\mathbb{R}). \tag{2.1}$$

It is easy to observe that $SD^j = D^j S$ for all $j \in \mathbb{Z}$ and $D^j E^k = E^{2^{-j}k} D^j$. We define the *period* $P(\{\psi^1, \ldots, \psi^r\}) \in \mathbb{N} \cup \{0\}$ of the wavelet frame generated by $\{\psi^1, \ldots, \psi^r\}$ as follows,

$$\begin{aligned} \langle P(\{\psi^1, \ldots, \psi^r\})\rangle := \{k \in \mathbb{Z}\colon\ & S^{-1} E^{k+n} \psi^\ell = E^k S^{-1} E^n \psi^\ell, \\ & \forall n \in \mathbb{Z},\ \ell = 1, \ldots, r\}, \end{aligned} \tag{2.2}$$

where $\langle d\rangle$ denotes the additive group generated by d. Let W denote the closed linear span of $\psi^\ell(\cdot + k)$, $k \in \mathbb{Z}$, $\ell = 1, \ldots, r$, and $\tilde{W} := S^{-1} W$. Then (2.2) implies

$$\begin{aligned} \langle P(\{\psi^1, \ldots, \psi^r\})\rangle &= \{k \in \mathbb{Z}\colon\ S^{-1} E^k f = E^k S^{-1} f,\ \forall f \in W\} \\ &= \{k \in \mathbb{Z}\colon\ E^k S f = S E^k f,\ \forall f \in \tilde{W}\}. \end{aligned}$$

When $\{\psi^1, \ldots, \psi^r\}$ generates a wavelet frame in $L_2(\mathbb{R})$, then the set $\{\psi^\ell_{J,k}\colon \ell = 1, \ldots, r$ and $k = 0, \ldots, 2^J - 1\}$ also generates the same wavelet frame for any nonnegative integer J since $[\psi^\ell_{J,k}]_{j,m} = \psi^\ell_{J+j,k+2^J m}$.

PROPOSITION 2.1. *Suppose* $\{\psi^1, \ldots, \psi^r\}$ *generates a wavelet frame in* $L_2(\mathbb{R})$. *For any nonnegative integer* J, *the following statements are equivalent:*

(a) *There exist* $2^J r$ *functions* $\tilde{\psi}^1, \ldots, \tilde{\psi}^{2^J r}$ *such that they generate the canonical dual frame of the wavelet frame* $\{\psi^\ell_{j,k}\colon\ j,k \in \mathbb{Z}$ *and* $\ell = 1, \ldots, r\}$;

(b) $\{S^{-1}[\psi^\ell_{J,k}]\colon\ k = 0, \ldots, 2^J - 1$ *and* $\ell = 1, \ldots, r\}$ *and* $\{\psi^\ell_{J,k}\colon\ k = 0, \ldots, 2^J - 1$ *and* $\ell = 1, \ldots, r\}$ *generate a pair of dual wavelet frames in* $L_2(\mathbb{R})$;

(c) $P(\{\psi^1, \ldots, \psi^r\}) \mid 2^J$, *where* $P(\{\psi^1, \ldots, \psi^r\})$ *is defined in* (2.2).

Moreover, when $\{\psi^1, \ldots, \psi^r\}$ *generates a wavelet frame which is a Riesz basis,* (a)–(c) *are also equivalent to*

(d) $V_J(\{\psi^1, \ldots, \psi^r\})$ *is shift-invariant; that is,* $f \in V_J(\{\psi^1, \ldots, \psi^r\}) \Longrightarrow f(\cdot - k) \in V_J(\{\psi^1, \ldots, \psi^r\})$ *for all* $k \in \mathbb{Z}$, *where* $V_J(\{\psi^1, \ldots, \psi^r\})$ *is the closed linear span of* $\psi^\ell_{j,k}$, $k \in \mathbb{Z}$, $j < J$, *and* $\ell = 1, \ldots, r$.

Proof. It is obvious that (a) $\Leftrightarrow$ (b). Note that (b) is equivalent to

$$[S^{-1}\psi^\ell_{J,k}]_{j,m} = S^{-1}\psi^\ell_{J+j,k+2^J m}, \qquad \forall j,k,m \in \mathbb{Z}. \tag{2.3}$$

In particular, taking $j = -J$ in the above equation, we have

$$\begin{aligned} S^{-1}E^{k+2^J m}\psi^\ell &= S^{-1}\psi^\ell_{0,k+2^J m} = [S^{-1}\psi^\ell_{J,k}]_{-J,m} = D^{-J}E^m S^{-1} D^J E^k \psi^\ell \\ &= D^{-J}E^m D^J S^{-1} E^k \psi^\ell = E^{2^J m} S^{-1} E^k \psi^\ell. \end{aligned}$$

Therefore, $P(\{\psi^1, \ldots, \psi^r\}) \mid 2^J$. Conversely, if $P(\{\psi^1, \ldots, \psi^r\}) \mid 2^J$, then

$$\begin{aligned} S^{-1}\psi^\ell_{J+j,k+2^J m} &= D^{J+j} S^{-1} E^{k+2^J m}\psi^\ell = D^{J+j} E^{2^J m} S^{-1} E^k \psi^\ell \\ &= D^j E^m S^{-1} D^J E^k \psi^\ell = [S^{-1}\psi^\ell_{J,k}]_{j,m}. \end{aligned}$$

Therefore, (2.3) holds and (b) $\Leftrightarrow$ (c).

When $\{\psi^1, \ldots, \psi^r\}$ generates a wavelet frame which is also a Riesz basis, it is evident that (a) $\Rightarrow$ (d). To prove the converse, define $\eta^{\ell,j,k}$ as the orthogonal projection of $\psi^\ell_{j,k}$ onto the space which is the closed linear span of $\{\psi^{\ell'}_{j',k'}\colon\ j',k' \in \mathbb{Z},\ \ell' = 1, \ldots, r\} \backslash \{\psi^\ell_{j,k}\}$. Define a function $\tilde{\psi}^{\ell,j,k} = (\psi^\ell_{j,k} - \eta^{\ell,j,k})/c^{\ell,j,k}$, where $c^{\ell,j,k} := \langle \psi^\ell_{j,k} - \eta^{\ell,j,k}, \psi^\ell_{j,k}\rangle \neq 0$. When (d) holds, it is easy to verify that $\{\tilde{\psi}^{\ell,j,k}\colon\ k = 0, \ldots, 2^J - 1,\ \ell = 1, \ldots, r\}$ generates its canonical dual frame. ■

Let us see how Proposition 2.1 explains the example (1.6) of Daubechies [2] and Chui and Shi [1]. Let ψ and η be as in (1.6). Let W denote the closed linear span of $\eta_{0,2k}$, $k \in \mathbb{Z}$. Note that $\eta(\frac{\cdot-1}{2}) \in V_0(\{\eta\})$ and $\eta_{0,1} \perp V_0(\{\eta\}) \oplus W$. It is evident that $\psi(\frac{\cdot-1}{2}) = \eta(\frac{\cdot-1}{2}) + \varepsilon\sqrt{2}\eta_{0,1} \notin V_0(\{\eta\}) \oplus W \supseteq V_0(\{\psi\})$, whereas $\psi(\frac{\cdot}{2})$ obviously lies in $V_0(\{\psi\})$. Hence, (d) in Proposition 2.1 fails for $J = 0$ and therefore, the canonical dual frame of $\{\psi_{j,k}\colon\ j,k \in \mathbb{Z}\}$ cannot be generated by a single function.

In general, it is difficult to compute $P(\{\psi^1, \ldots, \psi^r\})$ because computing the space $\tilde{W}$ and the inverse operator S^{-1} involves inverting an infinite matrix. The situation would be much simpler if, as happens automatically in the case of Gabor frames and frames of translates, we require that S commute with both dilates and integer shifts in the whole space $L_2(\mathbb{R})$. In order to state our result, let us introduce the definition of Fourier transform as follows:

$$\hat{f}(\xi) = \int_{\mathbb{R}} f(t)e^{-it\xi}\,dt, \qquad f \in L_1(\mathbb{R}).$$

PROPOSITION 2.2. *Let S be the frame operator associated with the wavelet frame in $L_2(\mathbb{R})$ generated by $\{\psi^1, \dots, \psi^r\}$. Then the following statements are equivalent:*

(a) *For some nonzero real number x_0, $S[f(\cdot + x_0)] = [Sf](\cdot + x_0)$ for all $f \in L_2(\mathbb{R})$;*
(b) *$S[f(\cdot + x)] = [Sf](\cdot + x)$ for all $f \in L_2(\mathbb{R})$ and for all $x \in \mathbb{R}$;*
(c) *There exists a positive constant C such that*

$$0 < C^{-1} \le \Psi(\xi) := \sum_{\ell=1}^{r} \sum_{j\in\mathbb{Z}} |\widehat{\psi^\ell}(2^j\xi)|^2 \le C < \infty, \qquad a.e.\ \xi \in \mathbb{R}$$

and

$$\sum_{\ell=1}^{r} \sum_{j=0}^{+\infty} \overline{\widehat{\psi^\ell}(2^j\xi)}\, \widehat{\psi^\ell}(2^j(\xi + 2\pi k)) = 0, \qquad \forall \text{ odd integers } k;$$

(d) *$\{\eta^1, \dots, \eta^r\}$ generates a tight wavelet frame in $L_2(\mathbb{R})$, where*

$$\widehat{\eta^\ell}(\xi) = \widehat{\psi^\ell}(\xi)/\sqrt{\Psi(\xi)}, \qquad \ell = 1, \dots, r.$$

Proof. Note that $D^jS = SD^j$ for all $j \in \mathbb{Z}$. If $SE^{x_0} = E^{x_0}S$, then

$$\begin{aligned} SE^{2^jx_0} &= SD^{-j}D^jE^{2^jx_0} = D^{-j}SE^{x_0}D^j = D^{-j}E^{x_0}SD^j \\ &= E^{2^jx_0}D^{-j}SD^j = E^{2^jx_0}S, \qquad \forall j \in \mathbb{Z}. \end{aligned}$$

By the assumption in (a), from the above equation, we have $SE^{2^jkx_0} = E^{2^jkx_0}S$ for all $j, k \in \mathbb{Z}$. Since the set $\{2^jkx_0\colon\ j, k \in \mathbb{Z}\}$ is dense in $\mathbb{R}$ and S is a bounded operator, we have $SE^x = E^xS$ for all $x \in \mathbb{R}$. Therefore, (a) $\Longleftrightarrow$ (b).

Now we deduce that (b) implies (c). Since S is a bounded linear operator in $L_2(\mathbb{R})$ and S commutes with all translates, there exists a bounded measurable function m (see [7]) such that

$$\widehat{Sf}(\xi) = m(\xi)\hat{f}(\xi), \qquad \forall f \in L_2(\mathbb{R}). \tag{2.4}$$

By a similar argument as in [6, Theorem 2.2], we deduce from (2.4) that (c) holds with $m(\xi) = \Psi(\xi)$.

Let us now rewrite the frame operator S using the Fourier transform. Since the frame operator S plays a very important role in this paper, let us discuss it in full detail. Following [6], we denote

$$\begin{aligned} L^2_{00}(\mathbb{R}) := \{f \in L_2(\mathbb{R})\colon\ &\hat{f} \in L_\infty(\mathbb{R}) \text{ and } \hat{f}(\xi) = 0 \\ &\text{for all } \xi \notin \{\zeta\colon\ 1/c \le |\zeta| \le c\} \text{ for some constant } c > 1\}. \end{aligned} \tag{2.5}$$

By Plancherel's theorem and the polarization identity, for any $f, g \in L^2_{00}(\mathbb{R})$, we have

$$\begin{aligned} \langle \widehat{Sf}, \hat{g} \rangle &= 2\pi \langle Sf, g \rangle \\ &= 2\pi \sum_{\ell=1}^{r} \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} \langle f, \psi^\ell_{j,k} \rangle \langle \psi^\ell_{j,k}, g \rangle = \frac{1}{2\pi} \sum_{\ell=1}^{r} \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} \langle \hat{f}, \widehat{\psi^\ell_{j,k}} \rangle \langle \widehat{\psi^\ell_{j,k}}, \hat{g} \rangle \end{aligned}$$

$$= \frac{1}{2\pi}\sum_{\ell=1}^{r}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} 2^{-j}\int_{\mathbb{R}} \hat{f}(\xi)\overline{\widehat{\psi^\ell}(2^{-j}\xi)}e^{i2^{-j}k\xi}\,d\xi \int_{\mathbb{R}} \overline{\hat{g}(\xi)}\,\widehat{\psi^\ell}(2^{-j}\xi)e^{-i2^{-j}k\xi}\,d\xi$$

$$= \frac{1}{2\pi}\sum_{\ell=1}^{r}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} 2^{j}\int_{\mathbb{R}} \hat{f}(2^j\xi)\overline{\widehat{\psi^\ell}(\xi)}e^{ik\xi}\,d\xi \int_{\mathbb{R}} \overline{\hat{g}(2^j\xi)}\,\widehat{\psi^\ell}(\xi)e^{-ik\xi}\,d\xi$$

$$= \frac{1}{2\pi}\sum_{\ell=1}^{r}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} 2^{j}\int_0^{2\pi} e^{ik\xi}\sum_{m\in\mathbb{Z}}\hat{f}(2^j(\xi+2\pi m))\overline{\widehat{\psi^\ell}(\xi+2\pi m)}\,d\xi$$

$$\times\int_0^{2\pi} e^{-ik\xi}\sum_{n\in\mathbb{Z}}\overline{\hat{g}(2^j(\xi+2\pi n))}\,\widehat{\psi^\ell}(\xi+2\pi n)\,d\xi.$$

By the Parseval equality, we have

$$\begin{aligned}
\langle\widehat{Sf},\hat{g}\rangle &= \sum_{\ell=1}^{r}\sum_{j\in\mathbb{Z}} 2^j\int_0^{2\pi}\sum_{m\in\mathbb{Z}}\hat{f}(2^j(\xi+2\pi m))\overline{\widehat{\psi^\ell}(\xi+2\pi m)}\\
&\quad\times\sum_{n\in\mathbb{Z}}\overline{\hat{g}(2^j(\xi+2\pi n))}\,\widehat{\psi^\ell}(\xi+2\pi n)\,d\xi\\
&= \sum_{\ell=1}^{r}\sum_{j\in\mathbb{Z}} 2^j\int_{\mathbb{R}}\overline{\hat{g}(2^j\xi)}\,\widehat{\psi^\ell}(\xi)\sum_{m\in\mathbb{Z}}\hat{f}(2^j(\xi+2\pi m))\overline{\widehat{\psi^\ell}(\xi+2\pi m)}\,d\xi\\
&= \sum_{\ell=1}^{r}\sum_{j\in\mathbb{Z}}\int_{\mathbb{R}}\overline{\hat{g}(\xi)}\,\widehat{\psi^\ell}(2^{-j}\xi)\sum_{m\in\mathbb{Z}}\hat{f}(\xi+2\pi 2^j m)\overline{\widehat{\psi^\ell}(2^{-j}\xi+2\pi m)}\,d\xi\\
&= \sum_{\ell=1}^{r}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}\int_{\mathbb{R}}\overline{\hat{g}(\xi)}\,\widehat{\psi^\ell}(2^{j}\xi)\hat{f}(\xi+2\pi 2^{-j}k)\overline{\widehat{\psi^\ell}(2^{j}\xi+2\pi k)}\,d\xi.
\end{aligned}$$

As in the proof of [6, Lemma 2.3], we denote

$$h(\xi) := \sum_{\ell=1}^{r}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}|F_{\ell,j,k}(\xi)|,$$

with

$$F_{\ell,j,k}(\xi) := \overline{\hat{g}(\xi)}\,\widehat{\psi^\ell}(2^j\xi)\overline{\widehat{\psi^\ell}(2^j\xi+2\pi k)}\hat{f}(\xi+2\pi 2^{-j}k).$$

For $f,g\in L_{00}^2(\mathbb{R})$, when $\sum_{\ell=1}^{r}\sum_{j\in\mathbb{Z}}|\widehat{\psi^\ell}(2^j\cdot)|^2\in L_\infty(\mathbb{R})$, under the decay condition that $\|\cdot\|^{\delta_1}\widehat{\psi^\ell}(\cdot)\in L_\infty$ and $\widehat{\psi^\ell}\in L_{2-\delta_2}$ for all $\ell=1,\dots,r$ for some $\delta_1>0$ and $\delta_2>0$, it was proved in the proof of [6, Lemma 2.3] that $h\in L_1(\mathbb{R})$. In fact, we found that such decay condition can be easily removed. For completeness, let us present a proof here. Since $f,g\in L_{00}^2(\mathbb{R})$, there exists a constant $c>1$ such that both $\hat{f}$ and $\hat{g}$ vanish outside $K:=[-c,-1/c]\cup[1/c,c]$. By the Cauchy–Schwartz inequality,

$$\begin{aligned}
&\int_{\mathbb{R}}|F_{\ell,j,k}(\xi)|\,d\xi\\
&\le\|\hat{f}\|_\infty\|\hat{g}\|_\infty\int_K|\widehat{\psi^\ell}(2^j\xi)\widehat{\psi^\ell}(2^j\xi+2\pi k)|\chi_K(\xi+2\pi 2^{-j}k)\,d\xi\\
&=\|\hat{f}\|_\infty\|\hat{g}\|_\infty\int_{K\cap(K-2\pi 2^{-j}k)}|\widehat{\psi^\ell}(2^j\xi)\widehat{\psi^\ell}(2^j\xi+2\pi k)|\,d\xi
\end{aligned}$$

$$\leq \|\hat{f}\|_\infty \|\hat{g}\|_\infty \left(\int_K |\widehat{\psi^\ell}(2^j\xi)|^2\, d\xi\right)^{1/2} \left(\int_{K-2\pi 2^{-j}k} |\widehat{\psi^\ell}(2^j\xi+2\pi k)|^2\, d\xi\right)^{1/2}$$
$$= \|\hat{f}\|_\infty \|\hat{g}\|_\infty \int_K |\widehat{\psi^\ell}(2^j\xi)|^2\, d\xi$$
$$= 2^{-j}\|\hat{f}\|_\infty \|\hat{g}\|_\infty \int_{2^j K} |\widehat{\psi^\ell}(\xi)|^2\, d\xi.$$

When $\sum_{\ell=1}^r \sum_{j\leq 0} |\widehat{\psi^\ell}(2^j\cdot)|^2$ is locally integrable, we have

$$\int_{\mathbb{R}} \sum_{\ell=1}^r \sum_{j\leq 0} \sum_{k\in\mathbb{Z}} |F_{\ell,j,k}(\xi)|\, d\xi = \sum_{\ell=1}^r \sum_{j\leq 0} \sum_{\{k\in\mathbb{Z}:\ |k|\leq c/\pi\}} \int_{\mathbb{R}} |F_{\ell,j,k}(\xi)|\, d\xi$$
$$\leq 2c/\pi \|\hat{f}\|_\infty \|\hat{g}\|_\infty \int_K \sum_{\ell=1}^r \sum_{j\leq 0} |\widehat{\psi^\ell}(2^j\xi)|^2\, d\xi < \infty$$

and

$$\int_{\mathbb{R}} \sum_{\ell=1}^r \sum_{j>0} \sum_{k\in\mathbb{Z}} |F_{\ell,j,k}(\xi)|\, d\xi$$
$$= \sum_{\ell=1}^r \sum_{j>0} \sum_{\{k\in\mathbb{Z}:\ |k|\leq 2^j c/\pi\}} \int_{\mathbb{R}} |F_{\ell,j,k}(\xi)|\, d\xi$$
$$\leq \sum_{\ell=1}^r \sum_{j>0} \sum_{\{k\in\mathbb{Z}:\ |k|\leq 2^j c/\pi\}} \|\hat{f}\|_\infty \|\hat{g}\|_\infty 2^{-j} \int_{2^j K} |\widehat{\psi^\ell}(\xi)|^2\, d\xi$$
$$\leq 2c/\pi \|\hat{f}\|_\infty \|\hat{g}\|_\infty \sum_{\ell=1}^r \int_{\mathbb{R}} |\widehat{\psi^\ell}(\xi)|^2 \sum_{j>0} \chi_{2^j K}(\xi)\, d\xi$$
$$\leq 2c/\pi \|\hat{f}\|_\infty \|\hat{g}\|_\infty \left\| \sum_{j\in\mathbb{Z}} \chi_{2^j K} \right\|_\infty \sum_{\ell=1}^r \|\widehat{\psi^\ell}\|_2^2 < \infty.$$

Hence, when $\sum_{\ell=1}^r \sum_{j\leq 0} |\widehat{\psi^\ell}(2^j\cdot)|^2$ is locally integrable and $\psi^\ell \in L_2(\mathbb{R})$ for all $\ell = 1,\dots,r$, then for all $f, g \in L_{00}^2(\mathbb{R})$, we have $h \in L_1(\mathbb{R})$. Therefore, we can rearrange the order of terms in the sum in the expression for $\langle \widehat{Sf}, \hat{g}\rangle$ to get

$$\langle \widehat{Sf}, \hat{g}\rangle = \int_{\mathbb{R}} \overline{\hat{g}(\xi)} \Bigg[\hat{f}(\xi) \sum_{j\in\mathbb{Z}} |\widehat{\psi^\ell}(2^j\xi)|^2 + \sum_{j_0\in\mathbb{Z}} \sum_{k\in\mathbb{Z}\setminus 2\mathbb{Z}} \hat{f}(\xi + 2\pi 2^{j_0} k)$$
$$\times \sum_{\ell=1}^r \sum_{j=0}^{\infty} \overline{\widehat{\psi^\ell}(2^{j+j_0}\xi)}\, \widehat{\psi^\ell}(2^j(2^{j_0}\xi + 2\pi k)) \Bigg] d\xi.$$

It follows that for any $f \in L_{00}^2(\mathbb{R})$,

$$\widehat{Sf}(\xi) = \hat{f}(\xi) \sum_{\ell=1}^r \sum_{j\in\mathbb{Z}} |\widehat{\psi^\ell}(2^j\xi)|^2$$
$$+ \sum_{j_0\in\mathbb{Z}} \sum_{k\in\mathbb{Z}\setminus 2\mathbb{Z}} \hat{f}(\xi + 2\pi 2^{j_0} k) \sum_{\ell=1}^r \sum_{j=0}^{+\infty} \overline{\widehat{\psi^\ell}(2^{j+j_0}\xi)}\, \widehat{\psi^\ell}(2^j(2^{j_0}\xi + 2\pi k)), \tag{2.6}$$

with the series converging absolutely. If (c) holds, then we have

$$\widehat{Sf}(\xi)=\hat{f}(\xi)\sum_{\ell=1}^{r}\sum_{j\in\mathbb{Z}}|\widehat{\psi^{\ell}}(2^{j}\xi)|^{2},\qquad \forall f\in L_{00}^{2}(\mathbb{R}).$$

Since $L_{00}^{2}(\mathbb{R})$ is dense in $L_2(\mathbb{R})$ and by assumption $\sum_{\ell=1}^{r}\sum_{j\in\mathbb{Z}}|\widehat{\psi^{\ell}}(2^{j}\cdot)|^{2}\in L_\infty(\mathbb{R})$, it is easy to see that (2.4) holds with $m(\xi)=\Psi(\xi)=\sum_{\ell=1}^{r}\sum_{j\in\mathbb{Z}}|\widehat{\psi^{\ell}}(2^{j}\xi)|^{2}$. Therefore, (b) holds. The equivalence between (c) and (d) is a standard result in the literature; for example, see [6, 8]. In fact, the equivalence relation between (c) and (d) can be easily proved using (2.6) and [6, Theorem 2.2]. ■

We shall also need the following results:

LEMMA 2.3 [6, Theorem 4.1]. *Let K be a Lebesgue measurable set of $\mathbb{R}$ and let $\hat{\psi}=\chi_K$, where χ_K denotes the characteristic function of K. Then ψ generates a normalized tight wavelet frame in $L_2(\mathbb{R})$ if and only if*

$$\sum_{j\in\mathbb{Z}}\chi_{2^{j}K}(\xi)=1,\qquad \sum_{k\in\mathbb{Z}}\chi_K(\xi+2\pi k)\le 1,\qquad a.e.\ \xi\in\mathbb{R}.$$

LEMMA 2.4. *Let U denote the set of functions ψ such that ψ generates a normalized tight wavelet frame in $L_2(\mathbb{R})$. Then the linear span of U is dense in $L_2(\mathbb{R})$.*

Proof. It suffices to prove that $f\perp U$ implies $f\equiv 0$. For any fixed $\xi_0>0$ such that $\xi_0\notin\{2^{j}\pi,\pi j\colon\ j\in\mathbb{Z}\}$, we shall demonstrate that $\hat{f}$ vanishes in a neighborhood of both ξ_0 and $-\xi_0$ by constructing an appropriate function $\psi\in U$.

Let $\delta=\operatorname{dist}(\xi_0,\pi\mathbb{Z})$, where $\operatorname{dist}(A,B):=\inf\{|x-y|\colon\ x\in A, y\in B\}$. By the assumption, we have $\delta>0$. Since $0<\xi_0\in\bigcup_{j\in\mathbb{Z}}(2\pi 2^{j-1},2\pi 2^{j}]$, there exists a unique integer j_0 such that $\xi_0\in(2\pi 2^{j_0-1},2\pi 2^{j_0}]$. Take $K_1:=(2\pi 2^{j_0-1},2\pi 2^{j_0}]\cap(\xi_0-\delta/3,\xi_0+\delta/3)$. Then K_1 is a neighborhood of ξ_0 since $\xi_0\notin\bigcup_{j\in\mathbb{Z}}2^{j}\pi$. Clearly, for large enough $j_1\in\mathbb{N}$, we have $K_2:=2^{-j_1}[(2\pi 2^{j_0-1},2\pi 2^{j_0}]\backslash K_1]\subset(-\delta/3,\delta/3)$. Let $K=K_1\cup(-K_1)\cup K_2\cup(-K_2)$ and $\hat{\psi}=\chi_K$. We claim that $\psi\in U$. That is, ψ generates a tight wavelet frame in $L_2(\mathbb{R})$.

Since $2^{j_1}K_2\cup K_1=(2\pi 2^{j_0-1},2\pi 2^{j_0}]$, we have $\sum_{j\in\mathbb{Z}}\chi_{2^{j}K}(\xi)=1$ for all $\xi\ne 0$. Note that $(-K_2)\cup K_2\subset(-\delta/3,\delta/3)$ and $K_1-K_1\subseteq(\xi_0-\delta/3,\xi_0+\delta/3)-(\xi_0-\delta/3,\xi_0+\delta/3)=(-2\delta/3,2\delta/3)$. By

$$\operatorname{dist}(K_1\cup(-K_1),2\pi\mathbb{Z})=\operatorname{dist}(K_1,2\pi\mathbb{Z})\ge\operatorname{dist}(\xi_0,\pi\mathbb{Z})-\delta/3\ge 2\delta/3,$$

we have $K\cap(K+2\pi k)=\emptyset$ for all $k\in\mathbb{Z}\backslash\{0\}$. Hence, $\sum_{k\in\mathbb{Z}}\chi_K(\xi+2\pi k)\le 1$. By Lemma 2.3, ψ generates a tight wavelet frame and therefore, $\psi\in U$.

Since $f\perp U$ and $\psi(\cdot-k)\in U$ for all $k\in\mathbb{Z}$, we have $\langle f,\psi(\cdot-k)\rangle=0$ for all $k\in\mathbb{Z}$. Or equivalently,

$$\int_{\mathbb{R}}\hat{f}(\xi)\overline{\hat{\psi}(\xi)}e^{ik\xi}\,d\xi=\int_K\hat{f}(\xi)e^{ik\xi}\,d\xi=0,\qquad \forall k\in\mathbb{Z}.$$

Since $K\cap(K+2\pi k)=\emptyset$ for all $k\in\mathbb{Z}\backslash\{0\}$, we deduce that $\hat{f}(\xi)=0$ a.e. $\xi\in K$, which completes the proof. ■

Take now ψ^1 and $\tilde{\psi}^1$ such that ψ^1 and $\tilde{\psi}^1$ generate a pair of dual wavelet frames, but ψ^1 does not satisfy the condition (c) in Proposition 2.2. For example, we can take ψ^1 and $\tilde{\psi}^1$ to be wavelets generating biorthogonal wavelet bases.

The following theorem establishes Main Theorem 1.1 for the case $r = 2$.

THEOREM 2.5. *Let ψ^1 and $\tilde{\psi}^1$ be given as above. For any positive integer J, there exists $\psi^2 \in L_2(\mathbb{R})$ such that ψ^2 generates a tight wavelet frame and either $P(\{\psi^1, \psi^2\}) = 0$ or $P(\{\psi^1, \psi^2\}) > J$. Consequently, the canonical dual frame of the wavelet frame generated by $\{\psi^1, \psi^2\}$ cannot be generated by two functions, though $\{\psi^1, \psi^2\}$ and $\{\tilde{\psi}^1/2, \psi^2/2\}$ do generate a pair of dual wavelet frames in $L_2(\mathbb{R})$.*

Proof. Let U denote the set of functions ψ that generate a tight wavelet frame in $L_2(\mathbb{R})$. For any $\psi^2 \in U$, we denote by S_1 and S the frame operators associated with the wavelet frames generated by ψ^1 and $\{\psi^1, \psi^2\}$, respectively. Then

$$Sf = \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} \langle f, \psi^1_{j,k}\rangle \psi^1_{j,k} + \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} \langle f, \psi^2_{j,k}\rangle \psi^2_{j,k} = S_1 f + f = (S_1 + I) f.$$

Thus, S is independent of the choice of ψ^2 in U.

Now we prove the claim by contradiction. Suppose there exists a positive integer J such that $0 < P(\{\psi^1, \psi^2\}) \le J$ for all $\psi^2 \in U$. By the definition of the period $P(\{\psi^1, \psi^2\})$ of a wavelet frame, we have

$$S^{-1} E^{J!} \psi^2 = E^{J!} S^{-1} \psi^2, \qquad \forall \psi^2 \in U.$$

By Lemma 2.4, the linear span of U is dense in $L_2(\mathbb{R})$. Therefore, $S^{-1} E^{J!} f = E^{J!} S^{-1} f$ for all $f \in L_2(\mathbb{R})$. In other words, $S E^{J!} = E^{J!} S$. Hence, $S = S_1 + I$ implies that $S_1 E^{J!} = E^{J!} S_1$; by Proposition 2.2, this implies that ψ^1 should satisfy condition (c) in that Proposition, which is a contradiction. ■

Let U_0 be a subset of U in Lemma 2.4 such that all the elements in U_0 have some desirable properties such as having compact support, belonging to $C^\alpha(\mathbb{R})$ for some large number $\alpha > 0$, and/or having certain order of vanishing moments. As long as the linear span of U_0 is dense in $L_2(\mathbb{R})$, by the same argument as in the proof of Theorem 2.5, all the claims in Theorem 2.5 hold (therefore, all the claims in Main Theorem 1.1 hold for the case $r = 2$) with the functions ψ^1, ψ^2, and $\tilde{\psi}^1$ having the same desirable properties as the elements in U_0 have.

3. CANONICAL DUAL FRAMES WITHOUT THE WAVELET STRUCTURE

In this section, we shall present examples that settle the claim in Main Theorem 1.1 for $r = 1$. More precisely, we shall construct a function ψ such that $\hat{\psi} \in C^\infty$ is compactly supported and ψ generates a wavelet frame in $L_2(\mathbb{R})$, and such that its canonical dual frame cannot be generated by a single function, even though there are infinitely many functions $\tilde{\psi}$ such that ψ and $\tilde{\psi}$ generate a pair of dual wavelet frames in $L_2(\mathbb{R})$.

Before presenting the result, we prove a few lemmas that will be needed later on. For a measurable set K, $|K|$ denotes its Lebesgue measure.

LEMMA 3.1. *Let K be a compact subset of $\mathbb{R}\backslash\{0\}$ such that $|K\cap(K+2\pi k)|=0$ for all $k\in\mathbb{Z}\backslash\{-1,0,1\}$. Suppose that ψ, defined by $\hat{\psi}:=\chi_K$, generates a wavelet frame in $L_2(\mathbb{R})$. Let S denote its frame operator defined in* (2.1). (*Note that we do not assume here that ψ generates a wavelet frame.*) *Then S is a bounded linear operator on $L_2(\mathbb{R})$ and for any $f\in L_2(\mathbb{R})$,*

$$\widehat{Sf}(\xi)=\hat{f}(\xi)\sum_{j\in\mathbb{Z}}\chi_{2^jK}(\xi)+\sum_{j\in\mathbb{Z}}\hat{f}(\xi+2\pi 2^j)\chi_{2^{-j}K_1}(\xi)$$
$$+\sum_{j\in\mathbb{Z}}\hat{f}(\xi-2\pi 2^j)\chi_{2^{-j}K_2}(\xi) \tag{3.1}$$

with the series converging absolutely, where

$$K_1=K\cap(K-2\pi)\qquad and\qquad K_2=K\cap(K+2\pi).$$

Proof. Since $\hat{\psi}=\chi_K$ and K is a compact subset of $\mathbb{R}\backslash\{0\}$, it is evident that $\sum_{j\in\mathbb{Z}}|\hat{\psi}(2^j\cdot)|^2\in L_\infty$. Therefore, the identity (2.6) holds for all $f\in L_{00}^2(\mathbb{R})$.

By the assumption on K, we have

$$\begin{aligned}
&|\operatorname{supp}\hat{\psi}(2^{j+j_0}\cdot)\cap\operatorname{supp}\hat{\psi}(2^j(2^{j_0}\cdot+2\pi k))|\\
&\quad=|2^{-j-j_0}K\cap 2^{-j-j_0}(K-2\pi 2^jk)|\\
&\quad=2^{-j-j_0}|K\cap(K-2\pi 2^jk)|=0\\
&\qquad\forall j>0,\ k\in\mathbb{Z}\text{ or } j=0,k\neq -1,0,1.
\end{aligned}$$

Thus (2.6) reduces to (3.1).

We now show that S is a bounded operator and (3.1) holds for all $f\in L_2(\mathbb{R})$.

Define a linear operator S_1 on $L_2(\mathbb{R})$ as follows:

$$\widehat{S_1f}(\xi):=\hat{f}(\xi)\sum_{j\in\mathbb{Z}}\chi_{2^jK}(\xi)+\sum_{j\in\mathbb{Z}}\hat{f}(\xi+2\pi 2^j)\chi_{2^{-j}K_1}(\xi)$$
$$+\sum_{j\in\mathbb{Z}}\hat{f}(\xi-2\pi 2^j)\chi_{2^{-j}K_2}(\xi). \tag{3.2}$$

Since K is a compact subset of $\mathbb{R}\backslash\{0\}$, by $K_1\cup K_2\subseteq K$, it is easy to observe that the infinity sum in the definition of the operator S_1 is in fact a finite sum due to the fact that $\sum_{j\in\mathbb{Z}}\chi_{2^jK}\in L_\infty$. So $\widehat{S_1f}$ is well defined for every $f\in L_2(\mathbb{R})$. We now show that S_1 is a bounded linear operator on $L_2(\mathbb{R})$. From (3.2), we have

$$\|\widehat{S_1f}\|_2\le\left\|\sum_{j\in\mathbb{Z}}\chi_{2^jK}\right\|_\infty\|\hat{f}\|_2+\left\|\sum_{j\in\mathbb{Z}}\hat{f}(\cdot+2\pi 2^j)\chi_{2^{-j}K_1}\right\|_2+\left\|\sum_{j\in\mathbb{Z}}\hat{f}(\cdot-2\pi 2^j)\chi_{2^{-j}K_2}\right\|_2.$$

Since $\sum_{j\in\mathbb{Z}}\chi_{2^jK_1}\in L_\infty$, we have

$$\begin{aligned}
&\int_{\mathbb{R}}\left|\sum_{j\in\mathbb{Z}}\hat{f}(\xi+2\pi 2^j)\chi_{2^{-j}K_1}(\xi)\right|^2d\xi\\
&\qquad\le\int_{\mathbb{R}}\sum_{j\in\mathbb{Z}}|\hat{f}(\xi+2\pi 2^j)|^2\chi_{2^{-j}K_1}(\xi)\sum_{j\in\mathbb{Z}}\chi_{2^{-j}K_1}(\xi)\,d\xi
\end{aligned}$$

$$\begin{aligned}
&\le \Big\|\sum_{j\in\mathbb{Z}} \chi_{2^{-j}K_1}\Big\|_\infty \sum_{j\in\mathbb{Z}} \int_{\mathbb{R}} |\hat f(\xi+2\pi 2^j)|^2 \chi_{2^{-j}K_1}(\xi)\,d\xi\\
&= \Big\|\sum_{j\in\mathbb{Z}} \chi_{2^{-j}K_1}\Big\|_\infty \sum_{j\in\mathbb{Z}} \int_{\mathbb{R}} |\hat f(\xi)|^2 \chi_{2^{-j}K_1+2\pi 2^j}(\xi)\,d\xi\\
&= \Big\|\sum_{j\in\mathbb{Z}} \chi_{2^{-j}K_1}\Big\|_\infty \int_{\mathbb{R}} |\hat f(\xi)|^2 \sum_{j\in\mathbb{Z}} \chi_{2^{-j}K_1+2\pi 2^j}(\xi)\,d\xi\\
&\le \Big\|\sum_{j\in\mathbb{Z}} \chi_{2^{-j}K_1}\Big\|_\infty \|\hat f\|_2^2 \Big\|\sum_{j\in\mathbb{Z}} \chi_{2^{-j}K_1+2\pi 2^j}\Big\|_\infty.
\end{aligned}$$

In the following, we demonstrate that $\|\sum_{j\in\mathbb{Z}} \chi_{2^{-j}K_1+2\pi 2^j}\|_\infty < \infty$. Since K_1 is a compact set, it is easy to see that for a large enough integer j_0,

$$\Big\|\sum_{j\ge j_0} \chi_{2^{-j}K_1+2\pi 2^j}\Big\|_\infty \le 1.$$

On the other hand, since K_1 is a compact subset of $\mathbb{R}\backslash\{0\}$, there exists a positive number $\delta > 0$ such that $K_1 \subset F := (-1/\delta, -\delta) \cup (\delta, 1/\delta) \subseteq \mathbb{R}\backslash\{0\}$. Consequently, for a large enough integer j_0, we have

$$\Big\|\sum_{j\le -j_0} \chi_{2^{-j}K_1+2\pi 2^j}\Big\|_\infty \le \Big\|\sum_{j\le -j_0} \chi_{2^{-j}F}\Big\|_\infty < \infty.$$

Therefore, there exists a positive constant C such that

$$\Big\|\sum_{j\in\mathbb{Z}} \hat f(\cdot+2\pi 2^j)\chi_{2^{-j}K_1}(\cdot)\Big\|_2 \le C\|\hat f\|_2, \qquad \forall f \in L_2(\mathbb{R}).$$

A similar argument holds for K_2. So S_1 is a bounded linear operator. Note that S is a nonnegative linear operator. Since S_1 and S agree in the dense subset $L^2_{00}(\mathbb{R})$ of $L_2(\mathbb{R})$, $S = S_1$ and therefore, (3.1) holds for every $f \in L_2(\mathbb{R})$. ■

LEMMA 3.2. *Let $a_1, \ldots, a_N$ be complex-valued numbers. If $\sum_{j=1}^N a_j(e^{i2\pi 2^{-j}k} - 1) = 0$ for all $k = 2^\ell$, $\ell = 0, 1, \ldots, N-1$, then $a_1 = a_2 = \cdots = a_N = 0$.*

Proof. Suppose not. Then without loss of generality, we assume $a_N \ne 0$. Set $k = 2^{N-1}$ in the above equation; we have $\sum_{j=1}^{N-1} a_j(1-1) + a_N(1+1) = 0$. Therefore, $a_N = 0$, which is a contradiction. ■

We are now ready for our construction:

THEOREM 3.3. *Let $0 < \delta_0 \le \delta_1 \le \delta_2 < \pi/2$ and pick an integer $J \ge 1 + \log_2(2\pi/\delta_0)$. Define*

$$\begin{aligned}
K = {}&[-2\pi + 2\pi 2^{-2J}, -2\pi + \delta_0] \cup [-2\delta_2, -\delta_1] \cup [-\delta_0, -2\pi 2^{-2J}]\\
&\cup [2\pi 2^{-2J}, \delta_0] \cup [\delta_1, 2\delta_2] \cup [2\pi - \delta_0, 2\pi - 2\pi 2^{-2J}]
\end{aligned}$$

and set $\hat\psi = \chi_K$. For any $\delta_1 \le \delta_3 \le \delta_2$, define $\hat{\tilde\psi} = \chi_{[-2\delta_3,-\delta_3]\cup[\delta_3,2\delta_3]}$. Then ψ and $\tilde\psi$ generate a pair of dual wavelet frames in $L_2(\mathbb{R})$. Moreover, $P(\{\psi\}) \ne 1$ and consequently, *the canonical dual frame of the wavelet frame* $\{\psi_{j,k}\colon\ j,k\in\mathbb{Z}\}$ *cannot be generated by a single function.*

Proof. Since both $\operatorname{supp}\hat{\psi}$ and $\operatorname{supp}\hat{\tilde{\psi}}$ are contained in the set $[-2\pi,-2\pi 2^{-2J}]\cup[2\pi 2^{-2J},2\pi]$, by [6, Proposition 2.6], there exists a positive constant C such that

$$\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}\left(|\langle f,\psi_{j,k}\rangle|^2+|\langle f,\tilde{\psi}_{j,k}\rangle|^2\right)\le C\|f\|^2,\qquad \forall f\in L_2(\mathbb{R}).$$

Since $\bar{\hat{\psi}}\hat{\tilde{\psi}}=\chi_{[-2\delta_3,-\delta_3]\cup[-\delta_3,2\delta_3]}$, we have $\sum_{j\in\mathbb{Z}}\overline{\hat{\psi}(2^j\xi)}\hat{\tilde{\psi}}(2^j\xi)=1$, a.e. $\xi\in\mathbb{R}$. Observe that $\operatorname{supp}\hat{\psi}\cap(\operatorname{supp}\hat{\tilde{\psi}}+2\pi k)=\emptyset$ for all $k\in\mathbb{Z}\backslash\{0\}$ since $\delta_2<\pi/2$. Thus,

$$\sum_{j=0}^{+\infty}\overline{\hat{\psi}(2^j\xi)}\hat{\tilde{\psi}}(2^j(\xi+2\pi k))=0,\qquad \forall\text{ odd integers }k.$$

It follows from [6, Theorem 2.7] that ψ and $\tilde{\psi}$ generate a pair of dual wavelet frames.

Let S be the frame operator associated with ψ. Note that K satisfies the conditions in Lemma 3.1. Let $f=S^{-1}\psi$. Then we have

$$\begin{aligned}\hat{\psi}(\xi)=\widehat{Sf}(\xi)=\hat{f}(\xi)\sum_{j\in\mathbb{Z}}\chi_{2^jK}(\xi)&+\sum_{j\in\mathbb{Z}}\hat{f}(\xi+2\pi 2^j)\chi_{2^{-j}K_1}(\xi)\\&+\sum_{j\in\mathbb{Z}}\hat{f}(\xi-2\pi 2^j)\chi_{2^{-j}K_2}(\xi),\end{aligned}\tag{3.3}$$

where $K_1=K\cap(K-2\pi)=[-2\pi+2\pi 2^{-2J},-2\pi+\delta_0]\cup[-\delta_0,-2\pi 2^{-2J}]$ and $K_2=K\cap(K+2\pi)=[2\pi 2^{-2J},\delta_0]\cup[2\pi-\delta_0,2\pi-2\pi 2^{-2J}]$.

Suppose that $P(\{\psi\})=1$. Then we must have $S[f(\cdot+k)]=\psi(\cdot+k)$ for all $k\in\mathbb{Z}$, or equivalently, from Lemma 3.1, for all $k\in\mathbb{Z}$,

$$\begin{aligned}\hat{\psi}(\xi)e^{ik\xi}=\widehat{Sf(\cdot+k)}(\xi)=\hat{f}(\xi)e^{ik\xi}&\sum_{j\in\mathbb{Z}}\chi_{2^jK}(\xi)\\&+\sum_{j\in\mathbb{Z}}\hat{f}(\xi+2\pi 2^j)e^{ik(\xi+2\pi 2^j)}\chi_{2^{-j}K_1}(\xi)\\&+\sum_{j\in\mathbb{Z}}\hat{f}(\xi-2\pi 2^j)e^{ik(\xi-2\pi 2^j)}\chi_{2^{-j}K_2}(\xi).\end{aligned}$$

Multiplying both sides of the above equation with $e^{-ik\xi}$, we deduce that

$$\begin{aligned}\hat{\psi}(\xi)=\hat{f}(\xi)\sum_{j\in\mathbb{Z}}\chi_{2^jK}(\xi)&+\sum_{j\in\mathbb{Z}}\hat{f}(\xi+2\pi 2^j)e^{i2\pi 2^jk}\chi_{2^{-j}K_1}(\xi)\\&+\sum_{j\in\mathbb{Z}}\hat{f}(\xi-2\pi 2^j)e^{-i2\pi 2^jk}\chi_{2^{-j}K_2}(\xi).\end{aligned}$$

Subtracting (3.3) from the above equation, we have

$$\begin{aligned}&\sum_{j=-\infty}^{-1}\hat{f}(\xi+2\pi 2^j)\chi_{2^{-j}K_1}(\xi)[e^{i2\pi 2^jk}-1]\\&\quad+\sum_{j=-\infty}^{-1}\hat{f}(\xi-2\pi 2^j)\chi_{2^{-j}K_2}(\xi)[e^{-i2\pi 2^jk}-1]=0\end{aligned}$$

for all $k \in \mathbb{Z}$. Observe that $K_1 \subset (-\infty, 0)$ and $K_2 \subset (0, +\infty)$. The above equation is thus equivalent to

$$\sum_{j=1}^{+\infty} \hat{f}(\xi + 2\pi 2^{-j})\chi_{2^j K_1}(\xi)[e^{i2\pi 2^{-j}k} - 1]$$
$$= \sum_{j=1}^{+\infty} \hat{f}(\xi - 2\pi 2^{-j})\chi_{2^j K_2}(\xi)[e^{-i2\pi 2^{-j}k} - 1] = 0. \tag{3.4}$$

Since K_1 and K_2 are subsets contained in $[-2\pi, -\delta_0] \cup [\delta_0, 2\pi]$, there exists a positive integer C such that

$$\sum_{j=1}^{\infty} \chi_{2^j K_1}(\xi) + \sum_{j=1}^{\infty} \chi_{2^j K_2}(\xi) \le C, \qquad \forall \xi \in \mathbb{R}.$$

Hence, for each fixed ξ, the summation in (3.4) is a finite sum with at most C nonzero terms. By Lemma 3.2, we conclude that

$$\hat{f}(\xi + 2\pi 2^{-j})\chi_{2^j K_1}(\xi) = \hat{f}(\xi - 2\pi 2^{-j})\chi_{2^j K_2}(\xi) = 0, \qquad \forall j \in \mathbb{N}.$$

Equivalently,

$$\hat{f}(\xi) = 0 \qquad \forall \xi \in \bigcup_{j=1}^{\infty} [(2^j K_1 - 2\pi 2^{-j}) \cup (2^j K_2 + 2\pi 2^{-j})]. \tag{3.5}$$

We now demonstrate that $\operatorname{supp} \hat{f} \subseteq (-\delta_0, \delta_0)$. From (3.5), it suffices to show that

$$[\delta_0, +\infty) \subseteq \bigcup_{j=1}^{\infty} (2^j [2\pi 2^{-2J}, \delta_0] + 2\pi 2^{-j}) \subseteq \bigcup_{j=1}^{\infty} (2^j K_2 + 2\pi 2^{-j}).$$

Let $I_j := 2^j [2\pi 2^{-2J}, \delta_0] + 2\pi 2^{-j} = [2\pi 2^{j-2J} + 2\pi 2^{-j}, 2^j \delta_0 + 2\pi 2^{-j}]$. It is easy to verify that

$$2\pi 2^{j-2J} + 2\pi 2^{-j} < 2\pi 2^{j+1-2J} + 2\pi 2^{-1-j} < 2^j \delta_0 + 2\pi 2^{-j}, \qquad \forall j \ge J.$$

Therefore,

$$[\delta_0, +\infty) \subseteq [2\pi 2^{1-J}, +\infty) \subseteq [2\pi 2^{J-2J} + 2\pi 2^{-J}, +\infty) \subseteq \bigcup_{j=J}^{\infty} I_j \subseteq \bigcup_{j=1}^{\infty} I_j.$$

Therefore, $\hat{f}(\xi) = 0$ for all $|\xi| \ge \delta_0$.

On the other hand, it follows from (3.3) and (3.4) that

$$\hat{\psi}(\xi) = \hat{f}(\xi) \sum_{j \in \mathbb{Z}} \chi_{2^j K}(\xi) + \sum_{j=0}^{+\infty} \hat{f}(\xi + 2\pi 2^j)\chi_{2^{-j} K_1}(\xi)$$
$$+ \sum_{j=0}^{+\infty} \hat{f}(\xi - 2\pi 2^j)\chi_{2^{-j} K_2}(\xi). \tag{3.6}$$

However, for any $\xi \in (\delta_1, 2\delta_2)$ which is a subset of $(\delta_0, +\infty)$, we have $\hat{f}(\xi) = \hat{f}(\xi + 2\pi 2^j) = \hat{f}(\xi - 2\pi 2^j) = 0$ for all $j \geq 0$. This fact implies that for any $\xi \in (\delta_1, 2\delta_2)$, the right-hand side of (3.6) is 0 while the left-hand side of (3.6) is $\hat{\psi}(\xi) = 1$. Thus, we obtain a contradiction and therefore, $P(\{\psi\}) \neq 1$. ■

The above proof tells us that for our construction $P(\{\psi\})$ must be an even integer. By choosing arbitrarily large J, a similar argument yields that the corresponding $P(\{\psi\})$ must be either 0 or very large. (In fact, $P(\{\psi\})$ will tend to ∞ as $J \to \infty$.) It is not clear to us whether for a wavelet frame generated by $\{\psi^1, \dots, \psi^r\}$ we must always have $P(\{\psi^1, \dots, \psi^r\}) = 0$ or $P(\{\psi^1, \dots, \psi^r\}) = 2^J$ for some J. Note that $P(\{\psi^1, \dots, \psi^r\}) = 2^J$, with $J \geq 0$, means that the canonical dual frame still has the same wavelet structure, since it is generated by translates and dilates of $2^J r$ functions.

In the following, we demonstrate that the example in Theorem 3.3 is associated with a multiresolution hierarchy. Define three 2π-periodic functions m_0, m_1, and $\tilde{m}_1$ as follows,

$$m_0(\xi) := \chi_{(-\pi/2, \pi/2)}(\xi), \qquad \xi \in (-\pi, \pi],$$

and

$$m_1(\xi) := \hat{\psi}(2\xi) = \chi_K(2\xi), \qquad \tilde{m}_1(\xi) := \hat{\tilde{\psi}}(2\xi), \ \xi \in (-\pi, \pi].$$

Let $\hat{\phi} = \chi_{(-\pi,\pi)}$. Since in Theorem 3.3, $K \subseteq (-2\pi, 2\pi)$ and $\hat{\psi} = \chi_K$, it is easy to see that

$$\hat{\phi}(\xi) = m_0(\xi/2)\hat{\phi}(\xi/2), \qquad \hat{\psi}(\xi) = m_1(\xi/2)\hat{\phi}(\xi/2), \qquad \text{and}$$
$$\hat{\tilde{\psi}}(\xi) = \tilde{m}_1(\xi/2)\hat{\phi}(\xi/2).$$

In other words, ϕ is a refinable function and ψ, $\tilde{\psi}$ are two wavelet functions derived from the refinable function ϕ. Observe that

$$\overline{m_1(\xi)}\tilde{m}_1(\xi) = \tilde{m}_1(\xi), \qquad \overline{m_0(\xi + \pi)} m_0(\xi) = 0, \qquad \text{and} \qquad \overline{m_1(\xi + \pi)}\tilde{m}_1(\xi) = 0.$$

It is easy to verify that the following identities

$$\overline{m_0(\xi)} m_0(\xi)\Theta(2\xi) + \overline{m_1(\xi)}\tilde{m}_1(\xi) = \Theta(\xi)$$

and

$$\overline{m_0(\xi + \pi)} m_0(\xi)\Theta(2\xi) + \overline{m_1(\xi + \pi)}\tilde{m}_1(\xi) = 0$$

hold for a.e. $\xi \in \mathbb{R}$ with the 2π-periodic function Θ given by $\Theta(\xi) = \chi_{(-\delta_3, \delta_3)}(\xi)$, $\xi \in (-\pi, \pi]$. Therefore, by [4, Theorem 2.3], ψ and $\tilde{\psi}$ generate a pair of dual wavelet frames from the refinable function ϕ. So the example in Theorem 3.3 is indeed associated with a multiresolution hierarchy.

Now let us employ an argument in [6, Section 4] to modify the function $\hat{\psi}$ in Theorem 3.3 into a C^∞ function. It is well known (see [6, Lemma 4.2]) that there exists a C^∞ function θ supported on $[-1, +\infty)$ such that $\theta(x)^2 + \theta(-x)^2 = 1$ for all $x \in \mathbb{R}$. Given a closed interval $[a, b]$ and two positive numbers η_1, η_2 such that $\eta_1 + \eta_2 \leq b - a$, as in

[6, p. 407], we define

$$f_{([a,b];\eta_1,\eta_2)}(x) = \begin{cases} \theta\left(\dfrac{x-a}{\eta_1}\right), & \text{when } x < a+\eta_1, \\ 1, & \text{when } a+\eta_1 \le x \le b-\eta_2, \\ \theta\left(\dfrac{b-x}{\eta_2}\right), & \text{when } x > b-\eta_2. \end{cases}$$

Then $f_{([a,b];\eta_1,\eta_2)} \in C^\infty$ and is supported on $[a-\eta_1, b-\eta_2]$. In other words, $f_{([a,b];\eta_1,\eta_2)}$ is a C^∞ modification of the characteristic function of the interval $[a,b]$.

Now the same argument as in Theorem 3.3 yields the following result.

COROLLARY 3.4. *Let* $0 < \delta_0 < \delta_1 = \delta_2 < \pi/3$ *and* $J \ge 2 + \log_2(2\pi/\delta_0)$. *Let* $0 < \eta < 4\min(2\pi 2^{-1-2J}, \delta_1 - \delta_0)$. *Define*

$$g(x) = f_{([\delta_1, 2\delta_1];\eta,2\eta)}(x) + f_{([2\pi 2^{-2J},\delta_0];\eta,\eta)}(x) + f_{([2\pi 2^{-2J},\delta_0];\eta,\eta)}(2\pi - x).$$

Let $\hat{\psi}(\xi) = g(\xi) + g(-\xi)$ *and* $\hat{\tilde{\psi}}(\xi) = f_{([\delta_1,2\delta_1];\eta,2\eta)}(x) + f_{([\delta_1,2\delta_1];\eta,2\eta)}(-x)$. *Then* ψ *and* $\tilde{\psi}$ *generate a pair of dual wavelet frames in* $L_2(\mathbb{R})$ *and* $P(\{\psi\}) \ne 1$.

4. WAVELET FRAMES WITH COMPACT SUPPORT

In this section, we derive similar results for wavelet frames with compact support. To do this, we need a theorem which is a direct consequence of a result from Daubechies and Han [4].

THEOREM 4.1 [4, Corollary 3.3]. *Let* ϕ *be a refinable function such that* $\hat{\phi}(2\xi) = a(\xi)\hat{\phi}(\xi)$ *for a* 2π*-periodic trigonometric polynomial* a *satisfying* $a(\pi) = 0$. *For any positive integer* ℓ, *define*

$$\psi(x) = \sum_{k=0}^{\ell} (-1)^k \frac{\ell!}{k!(\ell-k)!} \phi(x-k).$$

Then ψ *generates a wavelet frame in* $L_2(\mathbb{R})$. *Moreover, there exist two compactly supported functions* $\tilde{\psi}^1$ *and* $\tilde{\psi}^2$ *with arbitrary smoothness such that* $\{\psi_{1,0}, \psi_{1,1}\}$ *and* $\{\tilde{\psi}^1, \tilde{\psi}^2\}$ *generate a pair of dual wavelet frames in* $L_2(\mathbb{R})$.

EXAMPLE 4.2. Let $\phi = \chi_{[0,1]}$ and $\ell = 1$ in Theorem 4.1. Then $\psi = \chi_{[0,1]} - \chi_{[1,2]}$ generates a wavelet frame in $L_2(\mathbb{R})$. This wavelet frame can also be generated by $\{\eta^1, \eta^2\}$, where $\eta^2 = \eta^1(\cdot + 1/2)$ and $\eta^1 = \chi_{[0,1/2]} - \chi_{[1/2,1]}$ is the Haar wavelet function. Using the shift-invariant orthonormal basis $\{\phi(\cdot - k)\}_{k\in\mathbb{Z}} \cup \{\eta^1_{j,k}\colon\ j \in \mathbb{N}\cup\{0\},\ k \in \mathbb{Z}\}$ in $L_2(\mathbb{R})$, we can verify by a direct computation that $S^{-1}[\eta^2_{0,1}] \ne S^{-1}[\eta^2_{0,0}](\cdot - 1)$. Thus $P(\{\eta^1, \eta^2\}) \ne 1$ and therefore, its canonical dual frame cannot be generated by two functions, though by Theorem 4.1 there exist arbitrarily smooth compactly supported functions $\tilde{\psi}^1$ and $\tilde{\psi}^2$ such that $\{\eta^1, \eta^2\}$ and $\{\tilde{\psi}^1, \tilde{\psi}^2\}$ generate a pair of dual wavelet frames in $L_2(\mathbb{R})$. The computation also suggests that the canonical dual frame does not have compact support.

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