



A Decay Theorem for Refinable Functions

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Abstract—We show that a refinable function with absolutely summable mask cannot have exponential decay in both time and frequency.

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1. INTRODUCTION

An important feature of orthonormal wavelets is that expansions with respect to these bases can be computed by fast algorithms via underlying subband filtering schemes. This is due to the “nested” structure of the associated multiresolution analysis [1,2]. In the past ten years, many wavelet bases with excellent localization and smoothness properties have been constructed. Examples are compactly supported orthonormal wavelets of smoothness C^k , for any fixed $k \geq 0$; the Meyer wavelet which is C^∞ and decays faster than any inverse polynomial; and the Battle-Lemarié wavelets which are C^k and have exponential decay. However, there is a limit to how smooth and how localized such orthonormal wavelets can be: It is now well known that orthonormal wavelets cannot simultaneously be C^∞ and have exponential decay at infinity. This fact is proved in [3] and in [4, Corollary 5.5.2]; these proofs are based on the fact that orthonormal wavelets which are in C^n must have $n + 1$ vanishing moments.

On the other hand, it is known that for *wavelet frames*, which allow redundancy, the smoothness is not inexorably linked to vanishing moments. For example, the Mexican hat wavelet $\psi(x) = (1 - x^2)e^{-x^2/2}$ is in C^∞ even though it has only two vanishing moments; it also has exponential decay at ∞ . On the other hand, the Mexican hat wavelet transform is not efficiently computable by the kind of fast algorithm that subband filtering schemes provide.

Therefore, the question remains of whether there can exist frames which are C^∞ and have exponential decay, and which are also associated with fast implementation schemes of generalized subband filtering type, such as the “algorithme à trous,” see [5,6]. More precisely, we consider wavelet frames generated by a function ψ which is a (finite) linear combination of translates of an L^1 -function ϕ satisfying a *refinement equation*:

$$\phi(x) = \sum_{k \in \mathbb{Z}} h_k \phi(2x - k), \quad (1.1)$$

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where $\{h_k\}$ is a fixed absolutely summable real scalar sequence called a *mask*. Since ψ inherits the properties of ϕ , the question above becomes: Do there exist refinable functions ϕ with the property that both $\phi(x)$ and its Fourier transform $\widehat{\phi}$,

$$\widehat{\phi}(\xi) = \int_{-\infty}^{+\infty} \phi(x) e^{-i\xi x} dx,$$

have exponential decay as $|x|$ and $|\xi|$ tend to ∞ , respectively? Refinable functions of this type would have interesting applications. Their exponential decay in x would provide excellent localization, with a fast decomposition algorithm, even though the mask is infinite: if the mask has exponential decay, then it can be truncated in practice with good control of the error. Because of their exponential decay in ξ , they would be analytic in a strip, which would allow wavelets derived from them to be used to characterize singularities of analytic functions away from the real axis, in much the same way as conventional wavelet bases can be used to characterize singularities on the real axis.

This note proves a “no-go” theorem stating that no such functions ϕ can exist, however desirable they might be for applications:

THEOREM. *If ϕ is an L^1 -function satisfying*

$$\phi(x) = \sum_{k \in \mathbb{Z}} h_k \phi(2x - k)$$

with $\sum_{k \in \mathbb{Z}} |h_k| < \infty$, such that $\phi(x)$ and $\widehat{\phi}(\xi)$ both have exponential decay at infinity, then $\phi \equiv 0$.

This result is stronger than the no-go theorem of Battle in [3], which says that there is no function ϕ such that ϕ and $\widehat{\phi}$ both have exponential decay at infinity, and such that the functions in $\{\phi(\cdot - k); k \in \mathbb{Z}\}$ are orthonormal. In fact, Battle’s result applies more generally to the case that $\{\phi(\cdot - k); k \in \mathbb{Z}\}$ is merely a Riesz basis for its closed linear span, for then the Gram-orthogonalization trick used in [7] can be used to reduce the problem to Battle’s result. However, our present result applies regardless of whether the $\phi(\cdot - k)$ are independent or not.

2. MAIN RESULT

We first prove that $\widehat{\phi}(\xi)$ and the mask coefficients h_k cannot both have exponential decay. Note that for practical implementations, exponential decay of the h_k is really what matters (rather than exponential decay of ϕ).

THEOREM. *If ϕ is a refinable L^1 -function satisfying (1.1) such that both $\widehat{\phi}$ and $\{h_k\}_{k \in \mathbb{Z}}$ have exponential decay, then $\phi \equiv 0$.*

PROOF.

1. By Fourier transforming (1.1) we obtain:

$$\widehat{\phi}(\xi) = m\left(\frac{\xi}{2}\right) \cdot \widehat{\phi}\left(\frac{\xi}{2}\right), \quad (2.1)$$

where

$$m(\xi) = \frac{1}{2} \sum_{k \in \mathbb{Z}} h_k e^{ik\xi}.$$

Thus, if we suppose $|h_n| \leq C \cdot e^{-\gamma|n|}$ for some positive C, γ , then $m(\xi)$ has an analytic extension to the strip $|\operatorname{Im} \xi| < \gamma$ and (2.1) holds pointwise for ξ in this strip. Furthermore, it follows from (2.1) that

$$\widehat{\phi}(\xi) = \widehat{\phi}(0) \prod_{j=1}^{\infty} m(2^{-j}\xi),$$

where the infinite product converges absolutely and uniformly on compact sets [8]. If $\widehat{\phi}(0) = 0$, then $\phi \equiv 0$ follows immediately, and we are done. So we assume $\widehat{\phi}(0) \neq 0$.

2. For each positive integer $L \geq 2$, define $\xi_L = 2\pi/(2^L - 1)$. Then ξ_L is a fixed point for multiplication by 2^L modulo 2π :

$$2^L \cdot \xi_L \equiv \xi_L \pmod{2\pi}.$$

3. Then for any L , we must have either $\widehat{\phi}(\xi_L) = 0$ or $m(2^j \xi_L) = 0$, for some $j \in \{0, \dots, L-1\}$. For otherwise, for any $k \in \mathbb{N}$, we have

$$\begin{aligned} \left| \widehat{\phi}(2^{kL} \xi_L) \right| &= \left| \left[\prod_{j=0}^{L-1} m(2^j \cdot \xi_L) \right]^k \cdot \widehat{\phi}(\xi_L) \right| \\ &\geq \left[\min_{j=0, \dots, L-1} |m(2^j \xi_L)| \right]^{kL} C_L \\ &\geq 2^{-\beta_L kL} C'_L \\ &\geq C''_L (1 + |2^{kL} \xi_L|)^{-\beta_L}, \end{aligned}$$

for some finite constant β_L and some strictly positive constant C''_L , which contradicts the exponential decay of $\widehat{\phi}(\xi)$.

4. Since $\widehat{\phi}(0) \neq 0$, there exists a positive integer N , such that $\widehat{\phi}(\xi) \neq 0$, for $|\xi| \leq 2^{-N}\pi$. Thus, if $\widehat{\phi}(\xi_L) = 0$, then

$$\widehat{\phi}(\xi_L) = \prod_{j=1}^N m(2^{-j} \xi_L) \widehat{\phi}(2^{-N} \xi_L)$$

will imply $m(2^{-j} \xi_L) = 0$ for some j , $1 \leq j \leq N$.

5. Therefore, for any $L > 1$, we always have

$$m(2^{jL} \xi_L) = 0,$$

for some $j_L \in \{-N, \dots, 0, \dots, L-1\}$. On the other hand, it is easy to check that all the $2^{jL} \xi_L$ are different elements of $(0, \pi)$, so that $m(\xi)$ vanishes on a bounded infinite set $\{2^{jL} \xi_L\}$. Since $m(\xi)$ is analytic by Step 1, $m(\xi)$ is identically zero; this implies $\phi \equiv 0$, and contradicts the assumption $\widehat{\phi}(0) \neq 0$ at the end of Step 1. \blacksquare

The no-go theorem announced in the introduction will then follow from combining this theorem with the lemma below:

LEMMA. *Let ϕ be a refinable L^1 -function with absolutely summable mask $\{h_k\}_{k \in \mathbb{Z}}$. If ϕ has exponential decay, then the h_k have exponential decay as well.*

PROOF. If ϕ has exponential decay, then $\widehat{\phi}(\xi)$ is analytic on some strip $|\operatorname{Im} \xi| < \gamma$. It follows that $m(\xi) = (\widehat{\phi}(2\xi)/\widehat{\phi}(\xi))$ is a meromorphic function on the strip $|\operatorname{Im} \xi| < \gamma/2$. Because the restriction of $m(\xi)$ to real ξ is 2π -periodic, its meromorphic extension to the strip $|\operatorname{Im} \xi| < \gamma/2$ is 2π -periodic as well. On the other hand, the absolute summability of the h_k implies that $m(\xi)$ is bounded for real ξ , which means that every zero of $\widehat{\phi}(\xi)$ on the real axis is also a zero of $\widehat{\phi}(2\xi)$ with at least the same multiplicity. Define now γ' by

$$\gamma' = \inf \left\{ |\operatorname{Im} \xi| \leq \frac{\gamma}{2}; \widehat{\phi}(\xi) = 0, -\pi \leq \operatorname{Re} \xi \leq 3\pi, \operatorname{Im} \xi \neq 0 \right\}.$$

Then $\gamma' > 0$, and $m(\xi)$ is analytic on $(-\pi, 3\pi) \times (-\gamma', \gamma')$. By the 2π -periodicity of $m(\xi)$, it follows that $m(\xi)$ is analytic on the whole strip $|\operatorname{Im} \xi| < \gamma'$. Thus, the h_k , being Fourier

coefficients of $m(\xi)$, decay exponentially (as can be checked directly by a standard Paley-Wiener type argument). \blacksquare

REMARK. Our proof requires absolute summability of the h_k . We do not know whether there exist nonsummable masks $\{h_k\}_{k \in \mathbb{Z}}$ for which the corresponding function ϕ would have exponential decay in both x and ξ . However, the following example suggests that such functions might exist. With the help of Meyer's scaling function ϕ^M [2], we can construct a refinable function with excellent localization in time and frequency, corresponding to a mask that is not absolutely summable. Define the following mask: $h_0^C = h_2^C = 2$, with the other h_n^C given by $h_{2n}^C = 1$, $h_{2n+1}^C = -1$. The corresponding refinable function is $\phi^C = \chi_{[0,2]}$. (This function is also given by the mask $h_0 = h_2 = 1$, all the other $h_n = 0$; nonuniqueness of the mask is a price we have to pay if the $\phi(x - n)$ are not independent.) Now convolve this ϕ^C with ϕ^M ; the function thus obtained is again a refinable function satisfying

$$\phi^C * \phi^M(x) = \sum_k (h^C * h^M)_k (\phi^C * \phi^M)(2x - k).$$

We can check that $\phi^C * \phi^M$ is compactly supported in frequency and decays faster than any inverse polynomial in time, while $\sum_k |(h^C * h^M)_k| = +\infty$. However, even if similar examples existed with exponential decay for both ϕ and $\hat{\phi}$, they would not be useful in practice, because they lack fast computational algorithms.

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