Biorthogonal Bases of Compactly Supported Wavelets

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Abstract

Orthonormal bases of compactly supported wavelet bases correspond to subband coding schemes with exact reconstruction in which the analysis and synthesis filters coincide. We show here that under fairly general conditions, exact reconstruction schemes with synthesis filters different from the analysis filters give rise to two dual Riesz bases of compactly supported wavelets. We give necessary and sufficient conditions for biorthogonality of the corresponding scaling functions, and we present a sufficient condition for the decay of their Fourier transforms. We study the regularity of these biorthogonal bases. We provide several families of examples, all symmetric (corresponding to "linear phase" filters). In particular we can construct symmetric biorthogonal wavelet bases with arbitrarily high preassigned regularity; we also show how to construct symmetric biorthogonal wavelet bases "close" to a (nonsymmetric) orthonormal basis.

1. Introduction

Wavelets are functions generated from one basic function by dilations and translations. They are used as analyzing tools, by both pure mathematicians (in harmonic analysis, for the study of Calderón-Zygmund operators) and electrical engineers (in signal analysis). A particularly interesting development is the recent discovery of orthonormal bases of wavelets. For particular functions $\psi \in L^2(\mathbb{R})$, the family

$$\psi_{jk}(x) = 2^{-j/2}\,\psi(2^{-j}x - k), \quad j,k \in \mathbb{Z}\,, \tag{1.1}$$

constitutes an orthonormal basis for $L^2(\mathbb{R})$. The oldest example of such a basis is the Haar basis; smoother choices for ψ were constructed by Stromberg in [32], Meyer in [26], Lemarié in [24], Battle in [3], and Daubechies in [10]. There exist generalizations of (1.1) with dilation factors α different from 2 (α rational; for $\alpha = p/q > 1$ one needs $p - q$ different functions ψ, see [2]). Higher-dimensional extensions also exist (see, e.g., [2], [27]), in general the dilation factor can then be replaced by a matrix with integer entries and with eigenvalues strictly larger than 1 in absolute value; see [27].

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1.A. Multiresolution Analysis

In all the interesting examples, the orthonormal wavelet bases can be associated with a *multiresolution analysis* framework. The concept of multiresolution analysis was introduced by S. Mallat in [25]. For the purposes of this paper, the following brief summary will suffice; for proofs, more details, and examples the reader should consult [25], [27], or [28].

A multiresolution analysis consists of a ladder of spaces,

$$\cdots \subset V_2 \subset V_1 \subset V_0 \subset V_{-1} \subset V_{-2} \subset \dots ,$$

with $\overline{\cup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R})$, $\cap_{j\in\mathbb{Z}} V_j = \{0\}$, which satisfy the following two conditions:

(C1) $f \in V_j \Leftrightarrow f(2^j x) \in V_0$

(C2) there exists $\phi \in V_0$ such that the $\phi_{0n}(x) = \phi(x-n)$ constitute an orthonormal basis of V_0.

The spaces V_j can be considered as different approximation spaces: for a given f, the successive projections $\mathrm{Proj}_{V_j} f$ describe approximations of f with resolution 2^j.

If we define $\phi_{jk}(x) = 2^{-j/2}\,\phi(2^{-j}x - k)$, then it follows that, for every j, the $(\phi_{jk})_{k\in\mathbb{Z}}$ constitute an orthonormal basis for V_j,

$$\mathrm{Proj}_{V_j}(f) = \sum_{k\in\mathbb{Z}} \langle f, \phi_{jk}\rangle \phi_{jk} \; .$$

Note that, since $\phi \in V_0 \subset V_{-1} = \overline{\mathrm{Span}\{\phi_{-1n};\ n \in \mathbb{Z}\}}$, the function ϕ necessarily satisfies an equation of the type

$$\phi(x) = \sum_{n\in\mathbb{Z}} c_n\, \phi(2x-n) \; . \tag{1.2}$$

The c_n in (1.2) cannot be any arbitrary sequence. Orthogonality of the ϕ_{0k} immediately implies

$$\sum_n c_n c_{n+2k} = 2\delta_{k0} \; .$$

The orthonormal wavelet basis associated to this multiresolution analysis is then defined by

$$\psi(x) = \sum_{n\in\mathbb{Z}} (-1)^n\, c_{-n+1}\, \phi(2x-n) \; , \tag{1.3}$$

where the c_n are given by (1.2). (Note that we have assumed that the c_n are real. The whole analysis carries through, modulo some complex conjugations, for complex c_n. In particular, the c_{-n+1} in (1.3) should be replaced by $\overline{c_{-n+1}}$

if the c_n are complex. For the sake of convenience, we shall stick to real c_n, which corresponds to real functions ϕ and ψ, as in most interesting examples.) It is proved in [25], [27] that the $\psi_{jk}(x) = 2^{-j/2}\,\psi(2^{-j}x - k)$, $j, k \in \mathbb{Z}$, then constitute an orthonormal basis for $L^2(\mathbb{R})$. Moreover, for every fixed j, the $\{\langle f, \psi_{jk}\rangle; k \in \mathbb{Z}\}$ express the difference between the approximations of f with resolutions 2^j and 2^{j-1},

$$\text{Proj}_{V_{j-1}} f = \text{Proj}_{V_j} f + \sum_{k\in\mathbb{Z}} \langle f, \psi_{jk}\rangle \psi_{jk} \,. \tag{1.4}$$

1.B. Subband Coding Scheme Corresponding to a Multiresolution Analysis

The multiresolution ladder leads to a hierarchical scheme for the computation of the *wavelet coefficients* $\langle f, \psi_{jk}\rangle$. From (1.2), (1.3) one finds

$$\begin{aligned} \langle f, \psi_{jk}\rangle &= \sum_n g_{n-2k}\, \langle f, \phi_{j-1\,n}\rangle \,, \\ \langle f, \phi_{jk}\rangle &= \sum_n h_{n-2k}\, \langle f, \phi_{j-1\,n}\rangle \,, \end{aligned} \tag{1.5}$$

where $h_\ell = c_\ell/\sqrt{2}$, $g_\ell = (-1)^\ell\, c_{-\ell+1}/\sqrt{2}$. Note that both formulas in (1.5) have the structure of a convolution, followed by a "down-sampling" (only one out of every two entries of the convolution is retained). The expressions (1.5) show how to compute a coarser approximation from a finer one, as well as the difference in information between the two successive approximations. Recovering the finer approximation from the coarser one together with the difference information is just as easy. From (1.2), (1.3), and (1.4) we obtain

$$\begin{aligned} \langle f, \phi_{j-1\,m}\rangle &= \sum_k \left[\langle f, \phi_{jk}\rangle\, \langle \phi_{jk}, \phi_{j-1\,m}\rangle + \langle f, \psi_{jk}\rangle\, \langle \psi_{jk}, \phi_{j-1\,m}\rangle\right] \\ &= \sum_k \left[h_{m-2k}\langle f, \phi_{jk}\rangle + g_{m-2k}\langle f, \psi_{jk}\rangle\right] \,. \end{aligned} \tag{1.7}$$

The right-hand side of (1.7) can be read as a succession of three steps:

- "upsample" the $\langle f, \phi_{jk}\rangle$, i.e., consider them as the even entries of a sequence whose odd entries are zero
- convolve this upsampled sequence with the filter coefficients h_n
- do the same with $\langle f, \psi_{jk}\rangle$, with convolution with g_n, and add the two results.

The whole decomposition + reconstruction scheme (1.5) + (1.7) is therefore, in electrical engineering terms, a *subband coding scheme with exact reconstruction*, represented by Figure 1.1.

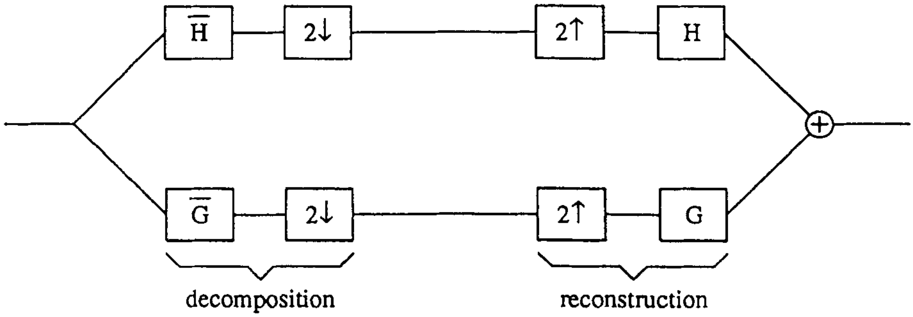

Figure 1.1. Diagram representing (1.5) and (1.7). The symbols H, $\overline{H}$ stand for convolution with h_n, h_{-n} respectively; the symbols 2 ↓ and 2 ↑ stand for the downsampling and upsampling described in the text.

For many orthonormal wavelet bases, the functions ϕ and ψ are supported on the whole line, and infinitely many c_n are different from zero. If ϕ and ψ have compact support (as in [10]), then all but finitely many c_n vanish, and the "filters" h and g have a finite number of "taps" (i.e., nonzero entries h_n, g_n). For every orthonormal basis of compactly supported wavelets there exists therefore an associated pair of finite filters for subband coding with exact reconstruction. The converse is not generally true.

One easily checks that exact reconstruction by the scheme represented in Figure 1.1 is only possible if

$$\sum_k [h_{m-2k} h_{n-2k} + g_{m-2k} g_{n-2k}] = \delta_{mn} \ .$$

For $g_n = (-1)^n \, h_{-n+1}$ this reduces to

$$\sum_\ell h_\ell h_{\ell+2m} = \delta_{m0} \ . \tag{1.8}$$

1.C. Orthonormal Wavelet Bases from Subband Coding Schemes

A function ψ can be a candidate for a "mother wavelet" (i.e., generating an orthonormal basis of wavelets) only if

$$\int d\xi |\xi|^{-1} \, |\hat{\psi}(\xi)|^2 < \infty \ , \tag{1.9}$$

where $\hat{\psi}$ denotes the Fourier transform of ψ,

$$\hat{\psi}(\xi) = (2\pi)^{-1/2} \int dx \ \exp\{-i\xi x\} \ \psi(x) \ .$$

Note that this condition is also necessary if the ψ_{jk} are merely a Riesz (rather than orthonormal) basis. (For a proof, see, e.g., Section 2.2.2.B in [11].) For $\psi \in L^1(\mathbf{R})$, $\hat{\psi}$ is continuous and (1.9) implies

$$\int dx\, \psi(x) = (2\pi)^{1/2}\, \hat{\psi}(0) = 0 \,.$$

Since the function ϕ in a multiresolution analysis has to satisfy $|\int dx\, \phi(x)| = 1$ (see [25]; note that if $\int dx\, \phi(x) = 0$, then $\overline{\cup_j V_j} \neq L^2(\mathbf{R})$), it follows from (1.3) that necessarily

$$\sum_n c_n (-1)^n = 0 \,,$$

hence (see (1.6))

$$\sum_n g_n = 0 \,. \tag{1.10}$$

There exist many pairs of exact reconstruction subband coding filters for which $\sum_n g_n$ is close to but not quite zero (see, e.g., [34]). Such pairs cannot possibly correspond to an orthonormal wavelet basis. If $\sum_n g_n = 0$, then all but a few pathological pairs do lead to an orthonormal basis. The following argument of W. Lawton (see [22]) shows why. Let us start from a subband coding scheme with exact reconstruction and with finite filters h_n, g_n, with $g_n = (-1)^n h_{-n+1}$ satisfying (1.10), and try to construct the corresponding ϕ, ψ. Once this is done, we can then ask whether the $\psi_j k$ do indeed constitute an orthonormal wavelet basis. By (1.2), (1.6) we have

$$\phi(x) = \sqrt{2} \sum_n h_n\, \phi(2x - n) \,.$$

By applying the Fourier transform, we find

$$\hat{\phi}(\xi) = m_0(\xi/2)\, \hat{\phi}(\xi/2) \,, \tag{1.11}$$

where m_0 is the periodic function

$$m_0(\xi) = 2^{-1/2} \sum_n h_n \exp\{-in\xi\} \,.$$

In terms of m_0, the condition (1.8) can be rewritten as

$$|m_0(\xi)|^2 + |m_0(\xi + \pi)|^2 = 1 \,, \tag{1.12}$$

while (1.10) becomes

$$m_0(\pi) = 0 .$$

If (1.10) holds we have therefore $|m_0(0)| = 1$, or $|\sum_n h_n| = \sqrt{2}$. By changing, if necessary, the sign of the filter coefficients h_n, we can therefore assume

$$m_0(0) = 1 . \tag{1.13}$$

It then follows from (1.11) that

$$\widehat{\phi}(\xi) = (2\pi)^{-1/2} \prod_{j=1}^{\infty} m_0(2^{-j}\xi) , \tag{1.14}$$

where we have used $\widehat{\phi}(0) = (2\pi)^{-1/2}$ (because $\int dx\, \phi(x) = 1$) and where the infinite product converges uniformly on compact sets because of (1.12). Using Fatou's lemma, one can show that this pointwise convergence, together with (1.12), implies that the right-hand side of (1.14) defines an element of $L^2(\mathbb{R})$ with norm bounded by 1 (see, e.g., [25]). By standard Paley-Wiener arguments (see Lemma 3.1 below) one sees that ϕ is compactly supported if only finitely many h_n are nonzero. We can also define

$$\psi(x) = \sqrt{2} \sum_n (-1)^n h_{-n+1}\, \phi(2x - n) , \tag{1.15}$$

which is again compactly supported. One can show (see [22]) that, for all $f \in L^2(\mathbb{R})$,

$$f = \sum_{j,k} \langle f, \psi_{jk} \rangle \psi_{jk} .$$

If $\|\psi\| = [\int dx |\psi(x)|^2]^{1/2} = 1$, then this implies that the ψ_{jk} are an orthonormal basis, and that ϕ characterizes the associated multiresolution analysis. To show $\|\psi\| = 1$, it is sufficient to prove $\|\phi\| = 1$, which turns out to be equivalent with

$$\int dx \phi(x)\, \phi(x - k) = \delta_{k0} . \tag{1.16}$$

(Even though (1.16) seems stronger than merely $\|\phi\| = 1$, it really is not, because of the special structure of ϕ.) The following argument by Lawton shows that (1.15) holds for almost all choices of the h_n. Define

$$\alpha_k = \int dx\, \phi(x)\, \phi(x - k) .$$

Then

$$\alpha_k = 2\sum_{m,n} h_m\, h_n \int dx\, \phi(2x-m)\phi(2x-2k-n)$$

$$= \sum_{m,n} h_m\, h_n\, \alpha_{2k-n+m} = \sum_{\ell} A_{k\ell}\alpha_\ell$$

where A is the matrix defined by

$$A_{k\ell} = \sum_m h_m\, h_{m+2k-\ell} \ .$$

In other words, α is an eigenvector of A with eigenvalue 1,

$$A\alpha = \alpha \ .$$

One can moreover show that $\sum_k \alpha_k = 1$. By (1.8) the vector β defined by $\beta_\ell = \delta_{\ell 0}$ is another such eigenvector, $A\beta = \beta$, and $\sum_k \beta_k = 1$. It follows that $\alpha \neq \beta$ is only possible if the eigenvalue 1 of A is degenerate. Among all the possible choices for the h_n (assuming we impose a fixed filter length) satisfying (1.8), this degeneracy is only possibly for a set of measure zero. Consequently for almost all choices of h_n,

$$\int dx\, \phi(x)\phi(x-k) = \delta_{k0} \ ,$$

and the ψ_{jk} constructed from ψ, ϕ as given by (1.14), (1.15) constitute an orthonormal basis. Note that the above argument gives only a sufficient condition ensuring that the ψ_{jk} are an orthonormal basis. It is conceivable that the eigenvalue 1 of A is degenerate, but that (1.16) would hold nevertheless. At the end of his paper (see [22]), Lawton raises the question whether this can be excluded, i.e., whether his condition might be necessary. There also exists another characterization, due to one of us (see [8]), slightly more technical to formulate than Lawton's condition, which does give a necessary and sufficient condition on the h_n ensuring orthonormality of the ψ_{jk}. In a very recent paper (see [23]), Lawton uses the theorem proved in [8] to show that his own sufficient condition is also necessary, thereby answering the question he raised in [22]. We shall come back below to this question; in particular we prove a generalization of [8] for the biorthogonal case which also provides an independent proof of the necessity of Lawton's condition. (See Section 4.A below.)

1.D. Subband Coding Schemes with Different Synthesis and Analysis Filters — Biorthogonal Wavelet Bases

The subband coding schemes we have discussed so far use the same filters h_n, g_n (or rather their mirror images h_{-n}, g_{-n}) for the reconstruction as for the decomposition. Such filter banks are well-known in the ASSP (Acoustics, Sound and Signal Processing) literature, where they were developed (see [30], [35]) a little earlier than multiresolution analysis was discovered in mathematics. In the same context, there have also been constructions of exact reconstruction filter banks in which the synthesis filters are different from the decomposition filters; see [35], [31]. Such filter banks have more flexibility, and are therefore easier to design. Moreover, they have the advantage that symmetric filters can be used, which is impossible in the case where synthesis and analysis filters are the same; see [30], [10]. It is natural to wonder what these generalizations on the filter bank side mean for wavelet bases and multiresolution analysis. This paper provides an answer to this question. We generalize orthonormal wavelet bases by constructing *biorthogonal wavelet bases*, i.e., two dual bases ψ_{mn}, $\tilde{\psi}_{mn}$, each given by the dilates and translates of one single function, ψ or $\tilde{\psi}$. One such pair of dual (non-orthonormal) bases was already constructed a few years ago by Ph. Tchamitchian (see [33]); we shall find back his construction as a special case in one of our examples. The multiresolution analysis for biorthogonal wavelet bases becomes a little more complicated than in the orthonormal case. Basically we will have *two* hierarchies of approximation spaces,

$$\cdots \subset V_2 \subset V_1 \subset V_0 \subset V_{-1} \subset V_{-2} \subset \cdots$$
$$\cdots \subset \tilde{V}_2 \subset \tilde{V}_1 \subset \tilde{V}_0 \subset \tilde{V}_{-1} \subset \tilde{V}_{-2} \subset \cdots .$$

Every space W_j will be a complement to V_j in V_{j-1}, but not the orthogonal complement as before. In the orthonormal case we had

$$\sum_k |\langle f, \phi_{j-1\,k}\rangle|^2 = \sum_k \left[|\langle f, \phi_{jk}\rangle|^2 + |\langle f, \psi_{jk}\rangle|^2\right] .$$

Because $W_j \not\perp V_j$ now, we have

$$A\sum_k \left[|\langle f,\phi_{jk}\rangle|^2 + |\langle f,\psi_{jk}\rangle|^2\right] \leqq \sum_k |\langle f,\phi_{j-1\,k}\rangle|^2$$
$$\leqq B\sum_k \left[|\langle f,\phi_{jk}\rangle|^2 + |\langle f,\psi_{jk}\rangle|^2\right] ,$$

with $A < 1 < B$. It is clear that bounds like these are not sufficient to establish that the ψ_{jk} constitute a Riesz basis: repeating them many times leads to a blowup of the constants. This is where the dual hierarchy steps in. We have complement spaces $\tilde{W}_j$ there as well, and it turns out that $\tilde{W}_j \perp V_j$, $W_j \perp \tilde{V}_j$.

The two multiresolution scales fit together like a giant zipper, and this allows us to control expressions like $\sum_{j,k} |\langle f, \psi_{jk}\rangle|^2$.

This paper is organized as follows. In Section 2 we discuss subband coding schemes with exact reconstruction; we derive necessary and sufficient conditions on the *four* filters (two for analysis, two for synthesis) to lead to exact reconstruction. In Section 3 we mimic the construction given in Section 1.C to construct ϕ, $\tilde{\phi}$, ψ and $\tilde{\psi}$. We prove that with some minimal conditions on these functions, the ψ_{jk}, $\tilde{\psi}_{jk}$ are indeed two dual Riesz bases. In Section 4 we relate the conditions on ϕ, $\tilde{\phi}$ necessary for Section 3 to conditions on the filters themselves. This involves a generalization of the arguments in [8] and [22]. In Section 5 we discuss the regularity of ψ, $\tilde{\psi}$, and finally, in Section 6 we construct several families of examples. The discussion of biorthogonal wavelet bases in this paper starts from the filter coefficients, from which everything else is constructed. In this it parallels the construction of orthonormal bases of wavelets as done in [10] rather than the construction from a multiresolution analysis framework as in [25]. One of us (J.-C. Feauveau in [20]) also developed an approach to biorthogonal bases which is closer in spirit to Mallat's original paper.

The biorthogonal bases constructed in this paper are a special case of wavelet "frames", as defined in [18], [11], or the "ϕ-transform", developed independently and around the same time in [21]. While we were completing this work, we became aware of similar results, obtained independently and simultaneously by other groups. In particular, M. Vetterli and C. Herley constructed linear phase filters with vanishing moments which are identical to our examples in Sections 6.A and 6.B. Their approach is complementary to ours, in that we are here concerned mainly with mathematical proofs that the wavelets do indeed constitute Riesz bases, etc., while they explore more the signal analysis applications of these filters; see [36]. In a less direct way our work is also related to a recent paper by De Vore, Jawerth, and Popov; see [19]. The examples in Section 6.A lead to expansions of the form $f = \sum_{j,k} \langle f, \tilde{\psi}_{jk}\rangle\, \psi_{jk}$ where ψ is a finite linear combination of B-splines,

$$\psi(x) = \sum_{\ell} g_\ell \phi(2x - \ell)\ ,$$

with ϕ a B-spline. They can therefore be rewritten as expansions in B-splines,

$$f = \sum_{j,k} \alpha_{jk}(f)\, \phi_{jk}\ ,$$

with $\phi_{jk}(x) = 2^{-j/2}\phi(2^{-j}x - k)$, $\alpha_{jk}(f) = \sum_m g_{k-2m}\langle f, \tilde{\psi}_{j+1\, m}\rangle$. In this sense, the two dual filters and the associated biorthogonal wavelet bases permit another way of attaining the spline decompositions featured in [19].

2. Necessary and Sufficient Conditions for Exact Reconstruction

We want to construct four sequences $h = (h_n)_{n\in\mathbf{Z}}$, $g = (g_n)_{n\in\mathbf{Z}}$, $\tilde{h} = (\tilde{h}_n)_{n\in\mathbf{Z}}$ and $\tilde{g} = (\tilde{g}_n)_{n\in\mathbf{Z}}$, two of which will be used for decomposition (h, g) and two for reconstruction $(\tilde{h}, \tilde{g})$. Starting from a data sequence $c^0 = (c^0_n)_{n\in\mathbf{Z}}$, we convolve with h, g and retain only one sample out of every two for the decomposition (see (1.5)),

$$\begin{aligned} c^1_n &= \sum_k h_{2n-k}\, c^0_k \\ d^1_n &= \sum_k g_{2n-k+1}\, c^0_k \,, \end{aligned} \tag{2.1}$$

where we have introduced a shift of 1 in the indices of g for later convenience. On each sequence c^1, d^1 we then perform a converse operation (we interleave zeros and convolve with the mirror images of respectively $\tilde{h}$, $\tilde{g}$) and we add the two results,

$$\tilde{c}^0_l = \sum_n \left[\tilde{h}_{2n-\ell}\, c^1_n + \tilde{g}_{2n-\ell+1}\, d^1_n\right] \,, \tag{2.2}$$

where the shift in the index of $\tilde{g}$ is again for convenience' sake. Requiring exact reconstruction means imposing $\tilde{c}^0 = c^0$, or

$$\sum_n \left[\tilde{h}_{2n-\ell}\, h_{2n-k} + \tilde{g}_{2n-\ell+1}\, g_{2n-k+1}\right] = \delta_{\ell k} \,. \tag{2.3}$$

This condition can be rewritten via the *z-notation.* In this notation we represent every sequence by a formal power series in z,

$$h(z) = \sum_n h_n\, z^n \,, \quad c^0(z) = \sum_n c^0_n\, z^n \,, \text{ etc. } \ldots$$

We can then rewrite (2.1) as

$$\begin{aligned} c^1(z^2) &= \frac{1}{2}\left[h(z)c^0(z) + h(-z)c^0(-z)\right] \\ z\, d^1(z^2) &= \frac{1}{2}\left[g(z)c^0(z) - g(-z)c^0(-z)\right] \;; \end{aligned}$$

(2.2) becomes

$$\tilde{c}^0(z) = \bar{\tilde{h}}(z)c^1(z^2) + \bar{\tilde{g}}(z)z\, d^1(z^2) \,.$$

Here we use the notation

$$\bar{a}(z) = \sum_n a_{-n}\, z^n = \sum_n a_n\, z^{-n} \;;$$

for $|z| = 1$ and $a_n \in \mathbb{R}$ we have $\overline{a(z)} = \bar{a}(z)$. It then follows that (2.3) is equivalent to

$$\frac{1}{2}\left[h(z)\bar{\tilde{h}}(z) + g(z)\bar{\tilde{g}}(z)\right] = 1 \tag{2.4}$$

$$\frac{1}{2}\left[h(-z)\bar{\tilde{h}}(z) - g(-z)\bar{\tilde{g}}(z)\right] = 0 \,. \tag{2.5}$$

In practice we shall want h, $\tilde{h}$, g, $\tilde{g}$ to be finite sequences; with $z = \exp\{i\xi\}$ their z-notations then correspond to trigonometric polynomials. In other words, h, $\tilde{h}$, g, $\tilde{g}$ can all be written as a product of a polynomial in z with an integer (possibly negative) power of z. Because of (2.4), $h(-z)$ and $g(-z)$ have no zeros in common. It follows from (2.5) that $\bar{\tilde{h}}(z) = 0$ whenever $g(-z) = 0$, and $\bar{\tilde{g}}(z) = 0$ whenever $h(-z) = 0$ (including multiplicity). Consequently

$$\bar{\tilde{h}}(z) = g(-z)\,p(z)$$
$$\bar{\tilde{g}}(z) = h(-z)p(z)\,,$$

where p is again an integer power of z multiplying a polynomial in z. Substituting this into (2.4) leads to

$$p(z)[h(z)\,g(-z) + h(-z)\,g(z)] = 2\,.$$

The only possible solutions to this are

$$\begin{aligned} p(z) &= \alpha z^k \\ h(z)g(-z) + h(-z)g(z) &= 2\alpha^{-1}z^{-k} \end{aligned} \tag{2.6a}$$

where $\alpha \in \mathbb{C} \setminus \{0\}$, $k \in \mathbb{Z}$. This amounts to

$$\bar{\tilde{h}}(z) = \alpha\, z^k g(-z)\,, \quad \bar{\tilde{g}}(z) = \alpha\, z^k h(-z)\,. \tag{2.6b}$$

Conditions (2.6a) and (2.6b) are necessary and sufficient to ensure exact reconstruction for the decomposition + synthesis scheme (2.1) + (2.2). For the sake of definiteness we choose $k = 0$ and $\alpha = -1$, i.e.

$$g_n = (-1)^{m+1}\,\tilde{h}_{-n}, \tilde{g}_n = (-1)^{n+1}h_{-n}\,. \tag{2.7}$$

In terms of h and $\tilde{h}$, (2.4) and (2.6a) becomes then

$$h(z)\bar{\tilde{h}}(z) + h(-z)\bar{\tilde{h}}(-z) = 2 \tag{2.8}$$

or

$$\sum_n h_m \tilde{h}_{n+2k} = \delta_{k0} \ . \tag{2.9}$$

Remarks.

1. If the same filters are used for synthesis as for decomposition, $\tilde{h} \equiv h$, $\tilde{g} \equiv g$, then (2.7), (2.9) are, as was to be expected, identical to the conditions in [10].
2. In many of our examples, h and $\tilde{h}$ will be symmetric, so that $\bar{\tilde{h}} = \tilde{h}$.
3. Assume that h and $\tilde{h}$ are of the form

$$h(z) = \sum_{n=N_1}^{N_2} h_n z^n = z^{N_1} p(z)$$

$$\tilde{h}(z) = \sum_{n=\tilde{N}_1}^{\tilde{N}_2} \tilde{h}_n z^n = z^{\tilde{N}_2} q(z^{-1}) \ ,$$

where p and q are two polynomials, and where we suppose $h_{N_1} \neq 0 \neq h_{N_2}$ and $\tilde{h}_{\tilde{N}_1} \neq 0 \neq \tilde{h}_{\tilde{N}_2}$. Then (2.8) can be rewritten as

$$z^{N_1-\tilde{N}_2} \left[p(z)\, q(z) + (-1)^{N_1-\tilde{N}_2} p(-z)\, q(-z)\right] = 2 \ . \tag{2.10}$$

It follows that $N_1 \leqq \tilde{N}_2$. If $N_1 = \tilde{N}_2$, then both p and q are constants, and the sequences $(h_n)_n$, $(\tilde{h}_n)_n$ each have only one nonvanishing entry. This solution is uninteresting for both signal analysis and wavelets. We therefore assume $N_1 < \tilde{N}_2$. A similar argument shows that $\tilde{N}_1 \leq N_2$ if more than one h_n or $\tilde{h}_n$ are different from zero. Suppose that $N_2 - \tilde{N}_1$ is even, $N_2 - \tilde{N}_1 = 2k > 0$, and let us compute the coefficient of z^{-2k} in (2.10); we find

$$2^h_{N_1} \tilde{h}_{\tilde{N}_2} = 0 \ ,$$

which is a contradiction with $h_{N_1} \neq 0 \neq \tilde{h}_{\tilde{N}_2}$. It follows that $N_2 - \tilde{N}_1$ is odd. Similarly $\tilde{N}_2 - N_1$ is odd. In the case where $\tilde{h}_n = h_n$, this reduces to the well-known fact that $N_2 - N_1$ has to be odd for nontrivial h_n.

3. Construction of the Two Multiresolution Hierarchies

We start by mimicking the construction of ϕ, ψ in the orthonormal case (see [25] or the summary in Section 1). Define

$$\begin{aligned} m_0(\xi) &= 2^{-1/2} \sum_n h_n \exp\{-in\xi\} \\ \tilde{m}_0(\xi) &= 2^{-1/2} \sum_n \tilde{h}_n \exp\{-in\xi\} \ . \end{aligned} \tag{3.1}$$

(For simplicity, we assume that only finitely many h_n, $\tilde{h}_n$ are nonzero. Several of our results can, however, be extended to infinite sequences which have sufficient decay for $|n| \to \infty$.) Then we define ϕ, $\tilde{\phi}$ by

$$\begin{aligned} \widehat{\phi}(\xi) &= (2\pi)^{-1/2} \prod_{j=1}^{\infty} m_0(2^{-j}\xi) \\ \widehat{\tilde{\phi}}(\xi) &= (2\pi)^{-1/2} \prod_{j=1}^{\infty} \tilde{m}_0(2^{-j}\xi) \,. \end{aligned} \tag{3.2}$$

These infinite products can only converge if

$$m_0(0) = 1 = \tilde{m}_0(0) \,, \tag{3.3}$$

i.e., if

$$\sum_n h_n = h(1) = \sqrt{2} = \tilde{h}(1) = \sum_n \tilde{h}_n \,. \tag{3.4}$$

If (3.3) is satisfied, then the infinite products in (3.2) converge uniformly and absolutely on compact sets, so that $\widehat{\phi}$ and $\widehat{\tilde{\phi}}$ are well-defined C^∞ functions. Clearly

$$\begin{aligned} \widehat{\phi}(\xi) &= m_0(\xi/2)\, \widehat{\phi}(\xi/2) \\ \widehat{\tilde{\phi}}(\xi) &= \tilde{m}_0(\xi/2)\, \widehat{\tilde{\phi}}(\xi/2) \,, \end{aligned} \tag{3.5}$$

or, equivalently,

$$\begin{aligned} \phi(x) &= \sqrt{2} \sum_n h_n\, \phi(2x-n) \\ \tilde{\phi}(x) &= \sqrt{2} \sum_n \tilde{h}_n\, \phi(2x-n) \,, \end{aligned} \tag{3.6}$$

at least in the sense of distributions. By the following lemma, borrowed from Deslauriers and Dubuc (see [15]), ϕ and $\tilde{\phi}$ have compact support.

LEMMA 3.1. *If* $\Gamma(\xi) = \sum_{n=N_1}^{N_2} \gamma_n \exp\{-in\xi\}$, *with* $\sum_{n=N_1}^{N_2} \gamma_n = 1$, *then* $\prod_{j=1}^{\infty} \Gamma(2^{-j}\xi)$ *is an entire function of exponential type. In particular, it is the Fourier transform of a distribution with support in* $[N_1, N_2]$.

Proof: By the Paley-Wiener theorem for distributions, it is sufficient to prove that $\prod_{j=1}^{\infty} \Gamma(2^{-j}\xi)$ is an entire function of exponential type with bounds

$$\left|\prod_{j=1}^{\infty} \Gamma(2^{-j}\xi)\right| \leqq C_1(1+|\xi|)^{M_1} \exp\{N_1|\text{Im }\xi|\} \quad \text{for} \quad \text{Im }\xi \geqq 0\,,$$

$$\left|\prod_{j=1}^{\infty} \Gamma(2^{-j}\xi)\right| \leqq C_2(1+|\xi|)^{M_2} \exp\{N_2|\text{Im }\xi|\} \quad \text{for} \quad \text{Im }\xi \leqq 0\,,$$

for some C_1, C_2, M_1, M_2. We shall only prove the first bound; the second is entirely analogous. Define

$$\Gamma_1(\xi) = \exp\{-iN_1\xi\}\,\Gamma(\xi) = \sum_{n=0}^{N_2-N_1} \gamma_{n+N_1} \exp\{-in\xi\}\,.$$

Then

$$\prod_{j=1}^{\infty} \Gamma(2^{-1}\xi) = \exp\{-i\,N_1\,\xi\} \prod_{j=1}^{\infty} \Gamma_1(2^{-j}\xi)\,,$$

so we only need to prove a polynomial bound for $\prod_{j=1}^{\infty} \Gamma_1(2^{-j}\xi)$ for Im $\xi \geqq 0$. For Im $\zeta \geqq 0$ we have

$$\begin{aligned} |\Gamma_1(\zeta) - 1| &\leqq \sum_{n=0}^{N_2-N_1} |\gamma_{n+N_1}|\,|\exp\{-in\zeta\} - 1| \\ &\leqq 2 \sum_{n=0}^{N_2-N_1} |\gamma_{n+N_1}| \min(1, n|\zeta|) \\ &\leqq C \min(1, |\zeta|)\,. \end{aligned}$$

Take ξ arbitrary, with Im $\xi \geqq 0$. If $|\xi| \leqq 1$, then

$$\begin{aligned} \left|\prod_{j=1}^{\infty} \Gamma_1(2^{-1}\xi)\right| &\leqq \prod_{j=1}^{\infty} [1 + C\,2^{-j}] \\ &\leqq \prod_{j=1}^{\infty} \exp\{2^{-j}C\} \leqq \exp\{C\}\,. \end{aligned} \tag{3.7}$$

If $|\xi| \geqq 1$, then there exists $j_0 \geqq 0$ so that $2^{j_0} \leqq |\xi| < 2^{j_0+1}$, and

$$\left|\prod_{j=1}^{\infty} \Gamma_1(2^{-j}\xi)\right| \leqq \prod_{j=1}^{j_0+1} [1+C] \left|\prod_{j=1}^{\infty} \Gamma_1(2^{-j}2^{-j_0-1}\xi)\right|$$

$$\leqq (1+C)^{j_0+1} \exp\{C\} \tag{3.8}$$

$$\leqq \exp\{C\}(1+C) \exp\{\ln(1+C) \ln|\xi|/\ln 2\}$$

$$\leqq (1+C) \exp\{C\} \, |\xi|^{\ln(1+C)/\ln 2} .$$

Combining (3.7) for $|\xi| \leqq 1$ and (3.8) for $|\xi| \geqq 1$ establishes the desired polynomial bound. ∎

Continuing to mimic the construction in the orthonormal case, we also define

$$\begin{aligned} \psi(x) &= \sqrt{2} \sum_n g_{n+1}\, \phi(2x-n) = \sqrt{2} \sum_n (-1)^n \tilde{h}_{-n-1}\, \phi(2x-n) \\ \tilde{\psi}(x) &= \sqrt{2} \sum_n \tilde{g}_{n+1}\, \tilde{\phi}(2x-n) = \sqrt{2} \sum_n (-1)^n h_{-n-1}\, \tilde{\phi}(2x-n) \end{aligned} \tag{3.9}$$

or, equivalently,

$$\begin{aligned} \hat{\psi}(\xi) &= \exp\{i\xi/2\}\, \overline{\tilde{m}_0\left(\frac{\xi}{2}+\pi\right)}\, \hat{\phi}(\xi/2) \\ \hat{\tilde{\psi}}(\xi) &= \exp\{i\xi/2\}\, \overline{m_0\left(\frac{\xi}{2}+\pi\right)}\, \hat{\tilde{\phi}}(\xi/2) . \end{aligned} \tag{3.10}$$

Note that, because of (2.7), the relationship between $\hat{\psi}(\xi)$ and $\hat{\phi}(\xi/2)$ is given by $\tilde{m}_0(\xi/2+\pi)$, and not by m_0 itself. The functions ψ, $\tilde{\psi}$ can only be candidates for generating Riesz bases of wavelets if they satisfy (1.9), i.e., if

$$\tilde{m}_0(\pi) = 0 = m_0(\pi) , \tag{3.11}$$

or

$$\sum_n (-1)^n h_n = h(-1) = 0 = \tilde{h}(-1) = \sum_n (-1)^n \tilde{h}_n . \tag{3.12}$$

Note that by (2.8), (3.12) necessarily implies

$$h(1)\, \overline{\tilde{h}(1)} = 2 ,$$

which means that a suitable normalization of the h_n, $\tilde{h}_n$ automatically satisfies (3.4).

Having constructed our candidate functions ψ, $\tilde{\psi}$, we can go about the business of proving that they generate Riesz bases of wavelets. A first obstruction is that there is no a priori estimate ensuring that $\tilde{\phi}$ or $\hat{\tilde{\phi}}$ are square integrable or bounded. This is unlike the orthonormal case, where $|m_0(\xi)| \leqq 1$ because of (1.12), so that $|\hat{\phi}(\xi)| \leqq 1$ automatically followed, without extra assumptions on m_0. Equation (1.12) for m_0 was even sufficient, in the orthonormal case, to ensure $\phi \in L^2(\mathbf{R})$ (see Section 1.C). In the present case, we have to impose extra restrictions on m_0, $\tilde{m}_0$ in order to ensure that ϕ, $\tilde{\phi} \in L^2(\mathbf{R})$. For the time being, we shall not state any explicit conditions (we shall be more explicit in Section 4), and merely assume that m_0, $\tilde{m}_0$ are such that $\hat{\phi}$, $\hat{\tilde{\phi}}$ have sufficient decay to ensure square integrability. This turns out to be sufficient to prove a large chunk of what we want.

THEOREM 3.2. *Suppose that ϕ, $\tilde{\phi}$, as defined by* (3.2), *satisfy*

$$|\hat{\phi}(\xi)| \leqq C(1+|\xi|)^{-1/2-\varepsilon}$$
$$|\hat{\tilde{\phi}}(\xi)| \leqq C(1+|\xi|)^{-1/2-\varepsilon} . \tag{3.13}$$

Define

$$\psi_{jk}(x) = 2^{-j/2}\psi(2^{-j}x-k) ,$$
$$\tilde{\psi}_{jk}(x) = 2^{-j/2}\tilde{\psi}(2^{-j}x-k) ,$$

with ψ, $\tilde{\psi} \in L^2(\mathbf{R})$ defined as in (3.9). *Then, for all $f \in L^2(\mathbf{R})$,*

$$f = \sum_{j,k\in\mathbf{Z}} \langle f, \tilde{\psi}_{jk}\rangle \psi_{jk} = \sum_{j,k\in\mathbf{Z}} \langle f, \psi_{jk}\rangle \tilde{\psi}_{jk} \tag{3.14}$$

where the sequences converge strongly.

We shall prove this theorem by a succession of lemmas. As in Section 1, we use the notation

$$\phi_{jk}(x) = 2^{-j/2}\phi(2^{-j}x-k) ,$$

with $\tilde{\phi}_{jk}$ defined analogously.

LEMMA 3.3. *Under the assumptions of Theorem* 3.2, *we have, for all $f_1, f_2 \in L^2(\mathbf{R})$,*

$$\sum_{k\in\mathbf{Z}} \left\langle f_1, \phi_{-1k}\right\rangle \left\langle \tilde{\phi}_{-1k}, f_2\right\rangle = \sum_{\ell\in\mathbf{Z}} \left[\left\langle f_1, \phi_{0\ell}\right\rangle \left\langle \tilde{\phi}_{0\ell}, f_2\right\rangle + \left\langle f_1, \psi_{0\ell}\right\rangle \langle \tilde{\psi}_{0\ell}, f_2\rangle\right] \tag{3.15}$$

Proof:

1. The following argument shows that all the sums are well-defined. By the Poisson summation formula

$$\sum_{k\in\mathbf{Z}} |\langle f, \phi_{0k}\rangle|^2 = \sum_{l\in\mathbf{Z}} \int d\xi\, \overline{\hat f(\xi)}\, \hat f(\xi+2\pi\ell)\, \widehat{\phi}(\xi)\, \overline{\widehat{\phi}(\xi+2\pi\ell)}$$

$$\leqq \sum_{\ell\in\mathbf{Z}} \left[\int d\xi \left|\hat f(\xi)\right|^2 \left|\widehat{\phi}(\xi+2\pi\ell)\right|^2\right]^{1/2} \left[\int d\xi \left|\hat f(\xi+2\pi\ell)\right|^2 \left|\widehat{\phi}(\xi)\right|^2\right]^{1/2}$$

$$\leqq \sum_{\ell\in\mathbf{Z}} \int d\xi \left|\hat f(\xi)\right|^2 \left|\widehat{\phi}(\xi+2\pi\ell)\right|^2 \leqq C\|f\|^2 ,$$

since $\sum_\ell |\widehat{\phi}(\xi+2\pi\ell)|^2 \leqq C \sum_\ell (1+|\xi+2\pi\ell|)^{-1-2\varepsilon}$ is bounded uniformly in ξ. Similarly $\sum_k |\left\langle f, \widetilde{\phi}_{0k}\right\rangle|^2 \leqq \infty$. Convergence of the ψ-term follows immediately from the ϕ-estimates, because ψ and $\widetilde{\psi}$ are finite linear combinations of $\phi_{-1k}, \widetilde{\phi}_{-1k}$.

2. Using (3.6), (3.9) we obtain

$$\sum_{\ell\in\mathbf{Z}} \left[\left\langle f_1, \phi_{0\ell}\right\rangle \left\langle \widetilde{\phi}_{0\ell}, f_2\right\rangle + \left\langle f_1, \psi_{0\ell}\right\rangle \langle \widetilde{\psi}_{0\ell}, f_2\rangle\right]$$

$$= \sum_{\ell\in\mathbf{Z}} \sum_{n,m} \left[h_n \tilde h_m + g_{n+1}\tilde g_{m+1}\right] \langle f_1, \phi_{-1\,2\ell+n}\rangle \left\langle \widetilde{\phi}_{-1\,2\ell+m}, f_2\right\rangle .$$

Since only finitely many n, m contribute, we can change this to

$$\sum_{n,m} \left\langle f_1, \phi_{-1n}\right\rangle \left\langle \widetilde{\phi}_{-1m}, f_2\right\rangle \sum_{\ell} \left[h_{n-2\ell}\, \tilde h_{m-2\ell} + g_{n+1-2\ell}\, \tilde g_{m+1-2\ell}\right]$$

$$= \sum_n \left\langle f_1, \phi_{-1n}\right\rangle \left\langle \widetilde{\phi}_{-1n}, f_2\right\rangle \qquad \text{by (2.3).} \ \blacksquare$$

Telescoping (3.15) leads to

LEMMA 3.4. *Under the assumptions of Theorem* 3.2, *we have, for all* $f_1, f_2 \in L^2(\mathbb{R})$,

$$\sum_{j,k} \langle f_1, \psi_{jk}\rangle \langle \widetilde{\psi}_{jk}, f_2\rangle = \langle f_1, f_2\rangle \tag{3.16}$$

Proof:

1. We first show that the left-hand side of (3.16) makes sense by proving that $\sum_{j,k} |\langle f_1, \psi_{jk}\rangle|^2 < \infty$, $\sum_{j,k} |\langle \tilde{\psi}_{jk}, f_2\rangle|^2 < \infty$. We have

$$\sum_k \left|\langle f_1, \psi_{jk}\rangle\right|^2 = \sum_k \left|\int d\xi\, \hat{f}_1(\xi)\, \overline{\widehat{\psi}_{jk}(\xi)}\right|^2$$

$$= 2^j \sum_k \left|\int_0^{2\pi 2^{-j}} d\xi \exp\{-ik2^j\xi\} \sum_\ell \hat{f}_1(\xi + 2\pi\ell 2^{-j})\, \overline{\widehat{\psi}(2^j\xi + 2\pi\ell)}\right|^2$$

$$= 2\pi \int_0^{2\pi 2^{-j}} d\xi \left|\sum_\ell \hat{f}_1(\xi + 2\pi\ell 2^{-j})\, \overline{\widehat{\psi}(2^j\xi + 2\pi\ell)}\right|^2 .$$

By Cauchy-Schwartz on the summation over ℓ this leads to

$$\sum_k \left|\left\langle f_1, \psi_{jk}\right\rangle\right|^2 \leqq 2\pi \int_0^{2\pi 2^{-j}} d\xi \left(\sum_\ell \left|\hat{f}_1\,(\xi + 2\pi\ell 2^{-j})\right|^2 \left|\widehat{\psi}\,(2^j\xi + 2\pi\ell)\right|^{2\delta}\right) \left(\sum_{\ell'} \left|\widehat{\psi}\,(2^j\xi + 2\pi\ell')\right|^{2(1-\delta)}\right)$$

$$= 2\pi \int_{-\infty}^{\infty} d\xi \left|\hat{f}_1(\xi)\right|^2 \left|\widehat{\psi}\,(2^j\xi)\right|^{2\delta} \sum_m \left|\widehat{\psi}\,(2^j\xi + 2\pi m)\right|^{2(1-\delta)} ,$$

where $\delta \in (0, 1)$ will be fixed below.

From (3.10) and (3.13) we have

$$|\widehat{\psi}(\xi)| \leqq C(1 + |\xi|)^{-1/2-\varepsilon} . \tag{3.17}$$

It follows that $\sum_m |\widehat{\psi}(\xi + 2\pi m)|^{2(1-\delta)}$ is bounded uniformly in ξ if $2(1-\delta) \times(1/2 + \varepsilon) > 1$, or $\delta < 2\varepsilon(1 + 2\varepsilon)^{-1}$. For $\delta < 2\varepsilon(1 + 2\varepsilon)^{-1}$ we find therefore

$$\sum_{j,k} \left|\left\langle f_1, \psi_{jk}\right\rangle\right|^2 \leqq C \int d\xi \left|\hat{f}_1(\xi)\right|^2 \sum_j \left|\widehat{\psi}\,(2^j\xi)\right|^{2\delta}$$

$$\leqq C \left[\sup_{1\leqq|\xi|\leqq 2} \sum_j \left|\widehat{\psi}\,(2^j\xi)\right|^{2\delta}\right] \|f_1\|^2 .$$

Since $|\widehat{\psi}|$ is bounded and $\widehat{\psi}(0) = 0$, we have $|\widehat{\psi}(\xi)| \leqq C|\xi|$ (remember that $\widehat{\psi}$ extends to an entire function), hence, for $1 \leqq |\xi| \leqq 2$,

$$\sum_{j=-\infty}^{0} \left|\widehat{\psi}(2^j\xi)\right|^{2\delta} \leqq C^{2\delta} \sum_{j=0}^{\infty} 2^{-2j\delta} |\xi|^{2\delta} \leqq (2C)^{2\delta} \left(1 - 2^{-2\delta}\right)^{-1} .$$

On the other hand, by (3.17)

$$\sum_{j=1}^{\infty} |\widehat{\psi}(2^j\xi)|^{2\delta} \leqq C^{2\delta} \sum_{j=1}^{\infty} (1+2^j|\xi|)^{-\delta}$$
$$\leqq C^{2\delta} \sum_{j=1}^{\infty} 2^{-j\delta} \leqq C^{2\delta} \left(1-2^{-\delta}\right)^{-1} .$$

This completes the proof that

$$\sum_{j,k} |\langle f_1, \psi_{jk}\rangle|^2 \leqq C\|f_1\|^2 \tag{3.18}$$

for some $C \geqq 0$. Similarly

$$\sum_{j,k} \left|\langle f_2, \widetilde{\psi}_{jk}\rangle\right|^2 \leqq C\,\|f_2\|^2 . \tag{3.19}$$

2. A simple dilation argument shows that (3.15) still holds true if we replace the indexes -1 by $j-1$, 0 by j,

$$\begin{aligned}&\sum_{k\in\mathbb{Z}} \left\langle f_1, \phi_{j-1\,k}\right\rangle \left\langle \widetilde{\phi}_{j-1\,k}, f_2\right\rangle \\ &= \sum_{\ell\in\mathbb{Z}} \left[\left\langle f_1, \phi_{j\ell}\right\rangle \left\langle \widetilde{\phi}_{j\ell}, f_2\right\rangle + \left\langle f_1, \psi_{j\ell}\right\rangle \langle \widetilde{\psi}_{j\ell}, f_2\rangle\right] .\end{aligned} \tag{3.20}$$

Summing (3.20) for all the j-values between $-J$ and J, we obtain

$$\begin{aligned}&\sum_{k\in\mathbb{Z}} \left\langle f_1, \phi_{-J-1\,k}\right\rangle \left\langle \widetilde{\phi}_{-J-1\,k}, f_2\right\rangle \\ &= \sum_{j=-J}^{J} \sum_{\ell\in\mathbb{Z}} \langle f_1, \psi_{j\ell}\rangle \langle \widetilde{\psi}_{j\ell}, f_2\rangle + \sum_{\ell\in\mathbb{Z}} \left\langle f_1, \phi_{J\ell}\right\rangle \left\langle \widetilde{\phi}_{J\ell}, f_2\right\rangle .\end{aligned} \tag{3.21}$$

By the bounds proved in point 1, we know that the first term in the right-hand side of (3.21) converges to the left-hand side of (3.16) as $J \to \infty$.

3. The second term in the right-hand side of (3.21) is bounded by

$$\left[\sum_{\ell} \left|\left\langle f_1, \phi_{J\ell}\right\rangle\right|^2\right]^{1/2} \left[\sum_{\ell} \left|\left\langle f_2, \widetilde{\phi}_{J\ell}\right\rangle\right|^2\right]^{1/2} .$$

The same estimates as in point 1 show that, for $\delta < 2\varepsilon(1+2\varepsilon)^{-1}$,

$$\sum_\ell \left|\left\langle f_1, \phi_{J\ell}\right\rangle\right|^2 \leqq C \int d\xi \left|\hat{f}_1(\xi)\right|^2 \left|\widehat{\phi}(2^J\xi)\right|^{2\delta} .$$

By (3.13) this becomes

$$\sum_\ell \left|\left\langle f_1, \phi_{J\ell}\right\rangle\right|^2 \leqq C \int d\xi \left|\hat{f}_1(\xi)\right|^2 (1+2^J|\xi|)^{-\delta} ,$$

which tends to 0 for $J \to \infty$, by dominated convergence.

4. Using again the same manipulations as in point 1, we rewrite the left-hand side of (3.21) as

$$\begin{aligned}
&\sum_k \left\langle f_1, \phi_{-J-1\,k}\right\rangle \left\langle \tilde{\phi}_{-J-1\,k}, f_2\right\rangle \\
&= 2\pi \sum_m \int_{-\infty}^{\infty} d\xi\, \hat{f}_1(\xi)\, \overline{\hat{f}_2(\xi+2\pi m 2^{J+1})}\, \overline{\widehat{\phi}(2^{-J-1}\xi)}\, \widehat{\tilde{\phi}}(2^{-J-1}\xi+2\pi m) \\
(3.22)\quad &= 2\pi \int_{-\infty}^{\infty} d\xi\, \hat{f}_1(\xi)\, \overline{\hat{f}_2(\xi)}\, \overline{\widehat{\phi}(2^{-J-1}\xi)}\, \widehat{\tilde{\phi}}(2^{-J-1}\xi) \\
&+ 2\pi \sum_{m\neq 0} \int_{-\infty}^{\infty} d\xi\, \hat{f}_1(\xi)\, \overline{\hat{f}_2(\xi+2\pi m 2^{J+1})}\, \overline{\widehat{\phi}(2^{-J-1}\xi)}\, \widehat{\tilde{\phi}}(2^{-J-1}\xi+2\pi m)
\end{aligned}$$

Since $\widehat{\phi}(\xi)$, $\widehat{\tilde{\phi}}(\xi)$ are bounded and continuous, and $\widehat{\phi}(0) = (2\pi)^{-1/2} = \widehat{\tilde{\phi}}(0)$, the first term converges to $\langle f_1, f_2\rangle$ for $J \to \infty$, by dominated convergence. It remains therefore to show that the second term converges to zero for $J \to \infty$ to complete the proof.

5. Using again (3.13), estimates similar to those in points 1 and 3 show that

$$\begin{aligned}
&\sum_{m\neq 0} \int_{-\infty}^{\infty} d\xi \left|\hat{f}_1(\xi)\right| \left|\hat{f}_2\left(\xi+2\pi m 2^{J+1}\right)\right| \left|\widehat{\phi}\left(2^{-J-1}\xi\right)\right| \left|\widehat{\tilde{\phi}}\left(2^{-J-1}\xi+2\pi m\right)\right| \\
&\qquad \leqq C \|f_1\| \|f_2\| ,
\end{aligned}$$

so that it suffices to prove convergence to zero for $J \to \infty$ for $\hat{f}_1$, $\hat{f}_2$ in the dense set of compactly supported L^2-functions. Assume support $\hat{f}_1$, support $\hat{f}_2 \subset \{\xi, |\xi| \leqq R\}$, and take $J \geqq \ln R/\ln 2$. Then $|2\pi m 2^{J+1}| \geqq 4\pi R > 2R$

for all $m \in \mathbb{Z}$, $m \neq 0$, so that $|\xi| \leqq R$ and $|\xi + 2\pi m 2^{J+1}| \leqq R$ are incompatible, and the second term in (3.22) vanishes identically. ■

The next lemma shows how the bounds (3.18), (3.19) suffice to turn this weak convergence into strong convergence.

LEMMA 3.5. *Under the assumptions of Theorem* 3.2, *we have, for all* $f \in L^2(\mathbf{R})$,

$$\lim_{J,K\to\infty} \sum_{\substack{|j|\leqq J\\ |k|\leqq K}} \langle f, \psi_{jk}\rangle\, \tilde{\psi}_{jk} = \lim_{J,K\to\infty} \sum_{\substack{|j|\leqq J\\ |k|\leqq K}} \langle f, \tilde{\psi}_{jk}\rangle\, \psi_{jk} = f\,,$$

where the limits are in the strong L^2*-topology.*

Proof:

$$\left\| f - \sum_{\substack{|j|\leqq J\\ |k|\leqq K}} \langle f, \psi_{jk}\rangle\, \tilde{\psi}_{jk} \right\| = \sup_{\|f_2\|=1} \left| \langle f, f_2\rangle - \sum_{\substack{|j|\leqq J\\ |k|\leqq K}} \langle f, \psi_{jk}\rangle \langle \tilde{\psi}_{jk}, f_2\rangle \right|$$

$$= \sup_{\|f_2\|=1} \left| \sum_{\substack{|j|> J\\ \text{or } |k|> K}} \langle f, \psi_{jk}\rangle \langle \tilde{\psi}_{jk}, f_2\rangle \right|$$

$$\leqq \sup_{\|f_2\|=1} \sum_{|j|+|k|>\min(J,K)} |\langle f, \psi_{jk}\rangle|\, |\langle \tilde{\psi}_{jk}, f_2\rangle|$$

$$\leqq \sup_{\|f_2\|=1} \left(\sum_{j,k} |\langle \tilde{\psi}_{jk}, f_2\rangle|^2 \right)^{1/2} \left(\sum_{|j|+|k|>\min(J,K)} |\langle f, \psi_{jk}\rangle|^2 \right)^{1/2}$$

$$\leqq \sup_{\|f_2\|=1} C\|f_2\| \left(\sum_{|j|+|k|>\min(J,K)} |\langle f, \psi_{jk}\rangle|^2 \right)^{1/2},$$

by (3.19). Consequently

$$\left\| f - \sum_{\substack{|j|\leqq J\\ |k|\leqq K}} \langle f, \psi_{jk}\rangle\, \tilde{\psi}_{jk} \right\| \leqq C \left(\sum_{|j|+|k|>\min(J,K)} |\langle f, \psi_{jk}\rangle|^2 \right)^{1/2} \underset{J,K\to\infty}{\longrightarrow} 0\,.$$

The other limit is entirely similar. ■

We have now proved all the assertions in Theorem 3.2. Note that (3.16) is *not* sufficient to prove that the ψ_{jk} and $\tilde{\psi}_{jk}$ constitute two dual Riesz bases. A Riesz basis can be defined in several ways. Two useful characterizations are the following:

1. $(u_n)_{n\in\mathbb{N}}$ is a Riesz basis in a Hilbert space $\mathcal{H}$ if and only if
   - The closure of the finite linear span of the u_n is $\mathcal{H}$, and
   - $\exists A > 0,\ B < \infty$ so that

$$A\sum_n |c_n|^2 \leqq \left\|\sum_n c_n u_n\right\|^2 \leqq B\sum_n |c_n|^2 \tag{3.23}$$

   for all $c = (c_n)_{n\in\mathbb{N}} \in \ell^2(\mathbb{N})$.

2. $(u_n)_{n\in\mathbb{N}}$ is a Riesz basis if and only if
   - The u_n are independent, i.e., no u_{n_0} lies within the closure of the finite linear span of the other u_n, and
   - $\exists A > 0,\ B < \infty$ so that

$$A\|f\|^2 \leqq \sum_n |\langle f, u_n\rangle|^2 \leqq B\|f\|^2 \tag{3.24}$$

   for all $f \in \mathcal{H}$.

It is easy to show that these two characterizations are equivalent (see, e.g., [37]); the first definition seems to be used more frequently. Note that (3.23) automatically implies linear independency of the u_n; (3.24), on the contrary, implies that the u_n span $\mathcal{H}$. A collection of u_n for which (3.24) holds, regardless of whether they are independent or not, is called a *frame* ([16], [37]). Because of (3.14) and (3.18), (3.19), the ψ_{jk}, $\tilde{\psi}_{jk}$ constitute a frame: the upper bound is immediate from (3.18), (3.19) and the lower bound follows from the following argument:

$$\begin{aligned}
\|f\| &= \sup_{\|g\|=1} |\langle f, g\rangle| \\
&\leqq \sup_{\|g\|=1} \sum_{j,k} |\langle f, \psi_{jk}\rangle|\,|\langle \tilde{\psi}_{jk}, g\rangle| && \text{(by (3.14)} \\
&\leqq \left(\sum_{j,k} |\langle f, \psi_{jk}\rangle|^2\right)^{1/2} \sup_{\|g\|=1}\left(\sum_{j,k} |\langle \tilde{\psi}_{jk}, g\rangle|^2\right)^{1/2} \\
&\leqq C\left(\sum_{j,k} |\langle f, \psi_{jk}\rangle|^2\right)^{1/2} && \text{(by (3.19))}\ .
\end{aligned}$$

In order to show that the ψ_{jk}, $\tilde{\psi}_{jk}$ constitute dual Riesz bases, we therefore only need to establish linear independence.

LEMMA 3.6. *Let ϕ, $\tilde{\phi}$, ψ, $\tilde{\psi}$ be as in Theorem 3.2. Then the ψ_{jk}, respectively $\tilde{\psi}_{jk}$, are linearly independent if and only if*

$$\langle \psi_{jk}, \tilde{\psi}_{j'k'} \rangle = \delta_{jj'}\delta_{kk'} \tag{3.25}$$

Proof:

1. If (3.25) is satisfied, then any f in the closed linear span of the ψ_{jk} with $(j,k) \neq (j_0, k_0)$ satisfies $\langle f, \tilde{\psi}_{j_0k_0} \rangle = 0$. It follows that $\psi_{j_0k_0}$ is not in this closed linear span.

2. By (3.14),

$$\psi_{j_0k_0} = \sum_{j,k} \langle \psi_{j_0k_0}, \tilde{\psi}_{jk} \rangle \, \psi_{jk} \, ,$$

hence

$$[1 - \langle \psi_{j_0k_0}, \tilde{\psi}_{j_0k_0} \rangle] \, \psi_{j_0k_0} = \sum_{(j,k)\neq(j_0,k_0)} \langle \psi_{j_0k_0}, \tilde{\psi}_{jk} \rangle \, \psi_{jk} \, .$$

If the ψ_{jk} are linearly independent, then this implies

$$\langle \psi_{j_0k_0}, \tilde{\psi}_{jk} \rangle = \delta_{jj_0}\delta_{kk_0} \, . \, \blacksquare$$

Because of the special structure of the ψ_{jk}, $\tilde{\psi}_{jk}$, (3.25) reduces to a condition on the ϕ_{0k}, $\tilde{\phi}_{0k}$ (i.e., one fixed dilation level).

LEMMA 3.7. *Let ϕ, $\tilde{\phi}$, ψ, $\tilde{\psi}$ be as in Theorem 3.2. A necessary and sufficient condition for (3.25) to hold is*

$$\left\langle \phi_{0k}, \tilde{\phi}_{0\ell} \right\rangle = \delta_{k\ell} \tag{3.26}$$

Proof:

1. We first prove (3.26)⇒(3.25).
By (3.9),

$$\begin{aligned} \langle \psi_{0k}, \tilde{\psi}_{0\ell} \rangle &= \sum_{n,m} g_{n+1} \, \tilde{g}_{m+1} \left\langle \phi_{-1\,2k-n}, \tilde{\phi}_{-1\,2\ell-m} \right\rangle \\ &= \sum_{n} g_{n+1} \, \tilde{g}_{2(l-k)+n+1} \qquad \text{(3.27))} \\ &= \sum_{m} \tilde{h}_m \, h_{m+2(k-1)} = \delta_{k\ell} \qquad \text{(by (2.7), (2.9))} \, . \end{aligned}$$

Similarly

$$\left\langle \psi_{0k}, \tilde{\phi}_{0\ell} \right\rangle = \sum_n g_{n+1}\, \tilde{h}_{2(l-k)+n}$$
$$= \sum_n (-1)^{n+1}\, \tilde{h}_{-n-1}\, \tilde{h}_{2(l-k)+n} .$$

Upon substituting $n' = 1 - n - 2(\ell - k)$, this last expression becomes its own negative, so that

$$\left\langle \psi_{0k}, \tilde{\phi}_{0\ell} \right\rangle = 0 . \tag{3.28}$$

Similarly

$$\langle \phi_{0k}, \tilde{\psi}_{0\ell} \rangle = 0 . \tag{3.29}$$

By a simple dilation, (3.27), (3.28), and (3.29) imply, for arbitrary j,

$$\langle \psi_{jk}, \tilde{\psi}_{j\ell} \rangle = \delta_{k\ell} , \quad \left\langle \psi_{jk}, \tilde{\phi}_{j\ell} \right\rangle = 0 = \langle \phi_{jk}, \tilde{\psi}_{j\ell} \rangle .$$

Since, for $j < j'$, $\tilde{\psi}_{j'k'}$ can be written as a linear combination of the $\tilde{\phi}_{j\ell}$, it follows that

$$\langle \psi_{jk}, \tilde{\psi}_{j'k'} \rangle = 0 \quad \text{if} \quad j < j' .$$

A similar argument proves (3.25) for $j > j'$.

2. Next we prove (3.25)⇒(3.26). We have

$$\begin{aligned} \delta_{\ell 0} &= \langle \psi_{1\ell}, \tilde{\psi}_{10} \rangle \\ &= \sum_{n,m} g_{n+1}\, \tilde{g}_{m-1} \left\langle \phi_{0\,2\ell-n}, \tilde{\phi}_{0\,-m} \right\rangle \\ &= \sum_{n,m} (-1)^{n+m}\, h_m\, \tilde{h}_n\, \alpha_{2\ell+n-m} \end{aligned} \tag{3.30}$$

where $\alpha_k = \int dx\, \phi(x-k)\, \tilde{\phi}(x)$.

If we define $\alpha(z) = \sum_n \alpha_n z^n$, then (3.30) is exactly the coefficient of $z^{2\ell}$ in $\bar{h}(-z)\tilde{h}(-z)\alpha(z)$ (using the notation of Section 2). Since this coefficient is equal to $\delta_{\ell 0}$, we have

$$\bar{h}(-z)\tilde{h}(-z)\alpha(z) + \bar{h}(z)\tilde{h}(z)\alpha(-z) = 2 .$$

Combined with (2.8), this becomes

$$h(z)\,\bar{\tilde{h}}(z)\beta(-z) + h(-z)\,\bar{\tilde{h}}(-z)\beta(z) = 0 , \tag{3.31}$$

where $\beta(z) = \bar{\alpha}(z) - 1$.

We know that only finitely many h_n, $\tilde{h}_n$ are different from zero. Let us assume (as in Remark 3 in Section 2)

$$\begin{aligned} h_n &= 0 \quad \text{for} \quad n < N_1,\, n > N_2 \\ \tilde{h}_n &= 0 \quad \text{for} \quad n < \tilde{N}_1,\, n > \tilde{N}_2 \,. \end{aligned}$$

Then $h(z)$ has $N_2 - N_1$ zeros in the complex plane, and $\tilde{h}(z)$ has $\tilde{N}_2 - \tilde{N}_1$ zeros (counting multiplicity). On the other hand, support $\phi \subset [N_1, N_2]$ and support $\tilde{\phi} \subset [\tilde{N}_1, \tilde{N}_2]$ by Lemma 3.1. Consequently α_k can only be nonvanishing if $\tilde{N}_1 - N_2 < k < \tilde{N}_2 - N_1$. It follows that $\alpha(z)$ can be written as the product of $z^{\tilde{N}_1 - N_2 + 1}$ with a polynomial of degree $(\tilde{N}_2 - \tilde{N}_1) + (N_2 - N_1) - 2$. By Remark 3 at the end of Section 2, $\tilde{N}_1 - N_2 < 0 < \tilde{N}_2 - N_1$, so that $\beta(z)$ is of the same form as $\bar{\alpha}(z)$. Because of (2.8), $h(z)\bar{\tilde{h}}(z)$ and $h(-z)\bar{\tilde{h}}(-z)$ have no common zeros. From (3.31) we see therefore that $\beta(z)$ is zero whenever $h(z)\bar{\tilde{h}}(z)$ vanishes, so that β has at least $N_2 - N_1 + \tilde{N}_2 - \tilde{N}_1$ zeros (counting multiplicity). Since $z^{-N_1+\tilde{N}_2-1}\,\beta(z)$ is a polynomial of degree strictly less than $N_2 - N_1 + \tilde{N}_2 - \tilde{N}_1$, it follows that $\beta(z) \equiv 0$, hence $\alpha(z) \equiv 1$ or $\alpha_k = \delta_{k0}$. This proves (3.26). ■

Putting everything together, we have therefore the following theorem.

THEOREM 3.8. *Let $(h_n)_n$, $(\tilde{h}_n)_n$ be finite real sequences satisfying*

$$\sum_n h_n \tilde{h}_n + 2k = \delta_{k0} \,.$$

Define

$$\begin{aligned} m_0(\xi) &= 2^{-1/2} \sum_n h_n \exp\{-in\xi\} \\ \tilde{m}_0(\xi) &= 2^{-1/2} \sum_n \tilde{h}_n \exp\{-in\xi\} \\ \hat{\phi}(\xi) &= (2\pi)^{-1/2} \prod_{j=1}^{\infty} m_0(2^{-j}\xi) \\ \hat{\tilde{\phi}}(\xi) &= (2\pi)^{-1/2} \prod_{j=1}^{\infty} \tilde{m}_0(2^{-j}\xi) \,. \end{aligned}$$

Suppose that, for some C, $\varepsilon > 0$,

$$\begin{aligned} |\hat{\phi}(\xi)| &\leqq C\,(1 + |\xi|)^{-1/2-\varepsilon} \\ |\hat{\tilde{\phi}}(\xi)| &\leqq C\,(1 + |\xi|)^{-1/2-\varepsilon} \,. \end{aligned} \tag{3.32}$$

Define

$$\psi(x) = \sqrt{2} \sum_n (-1)^n \tilde{h}_{-n+1} \phi(2x+n)$$

$$\tilde{\psi}(x) = \sqrt{2} \sum_n (-1)^n h_{-n+1} \tilde{\phi}(2x+n) .$$

Then the $\psi_{jk}(x) = 2^{-j/2} \psi(2^{-j}x - k)$, $j, k \in \mathbb{Z}$, *constitute a frame in* $L^2(\mathbb{R})$. *Their dual frame is given by the* $\tilde{\psi}_{jk}(x) = 2^{-j/2} \tilde{\psi}(2^{-j}x - k)$, $j, k \in \mathbb{Z}$; *for any* $f \in L^2(\mathbb{R})$,

$$f = \sum_{j,k \in \mathbb{Z}} \langle f, \tilde{\psi}_{jk} \rangle \psi_{jk} = \sum_{j,k \in \mathbb{Z}} \langle f, \psi_{jk} \rangle \tilde{\psi}_{jk} ,$$

where the series converge strongly.

Moreover, the ψ_{jk}, $\tilde{\psi}_{jk}$ *constitute two dual Riesz bases, with*

$$\langle \psi_{jk}, \tilde{\psi}_{j'k'} \rangle = \delta_{jj'} \delta_{kk'}$$

if and only if

$$\int dx\, \phi(x) \tilde{\phi}(x-k) = \delta_{k0} . \tag{3.33}$$

In the next section we shall see several strategies to ensure that the sequences $(h_n)_n$, $(\tilde{h}_n)_n$ lead to functions ϕ, $\tilde{\phi}$ satisfying (3.32) and (3.33).

4. A Closer Look at the Conditions

We have two conditions: (3.32) demands decay of $\hat{\phi}$ and $\hat{\tilde{\phi}}$, whereas (3.33) is a biorthogonality condition. Decay of $\hat{\phi}$ and $\hat{\tilde{\phi}}$ will correspond to divisibility of m_0, $\tilde{m}_0$ by $(1+e^{i\xi})$, while the biorthogonality follows from the structure of the set of zeros of m_0, $\tilde{m}_0$. We first concentrate on biorthogonality.

4.A. Biorthogonality

In the orthonormal case there exist two strategies to ensure orthonormality of the ϕ_{0k} (see Section 1.C): a sufficient condition due to Lawton (see [22]) and a different, necessary, and sufficient condition due to one of us (see [8]). We shall generalize both here to the nonorthonormal case, and discuss their relationship. We start with a generalization of [8].

DEFINITION 4.1. A compact set K is said to be congruent to $[-\pi, \pi]$ modulo 2π if

(1) for every $x \in [-\pi, \pi]$, there exists $k \in \mathbb{Z}$ so that $x + 2\pi k \in K$ and
(2) the total Lebesgue measure of K is 2π.

Such a set K will consist of a union of disjoint closed intervals, which, by means of translations by multiples of 2π, can be puzzled together to constitute exactly $[-\pi, \pi]$, with no overlaps (except for the endpoints of the translated intervals). Such compact sets were first introduced by one of us (see [8]) in the study of the orthonormal case. They can be used to formulate a necessary and sufficient condition on m_0 guaranteeing that the associated ϕ_{0k} are orthonormal. (Recall that for some rare choices of m_0 satisfying (1.12) the associated ϕ_{0k} may fail to be orthonormal — see [22] or Section 1.C.) More precisely, the following theorem was proved in [8]:

THEOREM 4.2. *Let m_0, $\widehat{\phi}$ be as in* (1.12), (1.14) *respectively. Then the functions $\phi_{0k}(x) = \phi(x-k)$ form an orthonormal set of L^2-functions if and only if there exists a compact set K, congruent to $[-\pi, \pi]$ modulo 2π, so that*

(1) K contains a neighborhood of 0,

(2) $\inf\{|m_0(2^{-k}\xi)|;\, k \geqq 1,\, \xi \in K\} > 0$.

Note that the orthonormality of the ϕ_{0k} is necessary and sufficient for the associated ψ_{jk} to constitute an orthonormal basis (see Section 1.C or Lemma 3.7).

In our present, nonorthonormal case, matters are more complicated, and to generalize Theorem 4.2 to the biorthogonal setting we shall have to introduce an additional condition. This condition turns out to be related to Lawton's condition (see [22]), generalized to a biorthogonal framework. We define two operators P_0 and $\tilde{P}_0$, acting on 2π-periodic functions f by

$$\begin{aligned}(P_0 f)(\xi) &= \left|m_0\left(\frac{\xi}{2}\right)\right|^2 f\left(\frac{\xi}{2}\right) + \left|m_0\left(\frac{\xi}{2}+\pi\right)\right|^2 f\left(\frac{\xi}{2}+\pi\right)\\ (\tilde{P}_0 f)(\xi) &= \left|\tilde{m}_0\left(\frac{\xi}{2}\right)\right|^2 f\left(\frac{\xi}{2}\right) + \left|\tilde{m}_0\left(\frac{\xi}{2}+\pi\right)\right|^2 f\left(\frac{\xi}{2}+\pi\right) .\end{aligned} \tag{4.1}$$

Expanding m_0, f into their trigonometric series shows that

$$(P_0 f)(\xi) = \sum_k (P_0 f)_k \exp\{-2\pi i k \xi\}$$

with

$$\begin{aligned}(P_0 f)_k &= \sum_\ell p_{k\ell}\, f_\ell\\ p_{k\ell} &= \sum_m h_m\, h_{m+\ell-2k} .\end{aligned} \tag{4.2}$$

Similar formulas hold for $\tilde{P}_0$. Note the similarity between the entries p_{kl} of the infinite matrix corresponding to P_0 and the entries of Lawton's matrix

$A_{k\ell}$ in Section 1.C (see also Remark 1 below). We are now all set to state the generalization of Theorem 4.2 to the biorthogonal setting. Without any assumptions of decay on $\widehat{\phi}$ or $\widehat{\tilde{\phi}}$ we have the following theorem:

THEOREM 4.3. *Let h_n, $\tilde{h}_n$ be finite real sequences satisfying* (2.9) *and* (3.4). *Define m_0, $\tilde{m}_0$, ϕ, $\tilde{\phi}$ as in* (3.1), (3.2). *Then the following three statements are equivalent:*

C1. *$\phi, \tilde{\phi} \in L^2(\mathbb{R})$ and $\int dx\, \phi(x-k)\tilde{\phi}(x-\ell) = \delta_{k\ell}$.*

C2. *There exist strictly positive trigonometric polynomials f_0, $\tilde{f}_0$ and a compact set K congruent to $[-\pi, \pi]$ modulo 2π so that*

- *$P_0 f_0 = f_0$ and $\tilde{P}_0 \tilde{f}_0 = \tilde{f}_0$*
- *$0 \in$ interior of K*
- *for all $\xi \in K$, all $k \in \mathbb{N} \setminus \{0\}$, and some strictly positive C (independent of ξ and k)*

$$|m_0(2^{-k}\xi)|\,, \quad |\tilde{m}_0(2^{-k}\xi)| \geqq C \tag{4.3}$$

C3. *There exist strictly positive trigonometric polynomials f_0, $\tilde{f}_0$ so that $P_0 f_0 = f_0$, $P_0 \tilde{f}_0 = \tilde{f}_0$ and these are the only trigonometric polynomials (up to normalization) invariant under P_0, $\tilde{P}_0$ respectively.*

Remarks.

1. Suppose the trigonometric polynomial m_0 is of the type

$$m_0(\xi) = \frac{1}{\sqrt{2}} \sum_{n=0}^{N} h_n \exp\{-in\xi\}\,,$$

so that $|m_(\xi)|^2$ and $|m_0(\xi+\pi)|^2$ are both trigonometric polynomials with frequencies ranging from $-N$ to N. It is then easy to check that any trigonometric polynomial f which is also an eigenfunction of P_0, $P_0 f = \lambda f$, is necessarily also of the type $\sum_{n=-N}^{N} f_n \exp\{-in\xi\}$, i.e., it contains no "frequencies" larger than N. This means that the discussion of invariant trigonometric polynomials is restricted to the finite-dimensional space of trigonometric polynomials f with $f_n = 0$ for $|n| > N$. This space is invariant under P_0; the restriction of P_0 to this invariant space is a $(2N+1) \times (2N+1)$ matrix with matrix elements given by (4.2) (which is essentially Lawton's matrix).

2. In the orthonormal case, m_0 satisfies (1.12), so that the constant functions are automatically invariant for $P_0 = \tilde{P}_0$ in this case. Condition C3 in Theorem 4.3 is then only a rephrasing of Lawton's condition (the eigenvalue 1 of the matrix A in Section 1.C should be nondegenerate). It is therefore a

consequence of Theorem 4.3 that, when specialized to the orthonormal case, both Lawton's and Cohen's conditions are necessary and sufficient. This answers the question raised by Lawton at the end of his paper (see [22]). After this work was completed, we learned that W. Lawton independently also proved his own conjecture, using the results in [8] (see [23]).

The proof of Theorem 4.3 consists for a large part of the study of the operators P_0, $\tilde{P}_0$. We shall borrow several lemmas of J. P. Conze and A. Raugi, who in their recent paper (see [9]) proved many interesting results for the operators

$$P_u f(x) = u\left(\frac{x}{2}\right) f\left(\frac{x}{2}\right) + u\left(x + \frac{1}{2}\right) f\left(x + \frac{1}{2}\right),$$

defined on continuous functions f on $[0, 1]$. In their study they assume that u is non-negative, continuous, and $u(x)+u(x+1/2) = 1$, for $0 \leqq x \leqq 1/2$. This last condition is not satisfied in our present case (with $u(x) = |m_0(2\pi x)|^2$ or $u(x) = |\tilde{m}_0(2\pi x)|^2$), but nevertheless several of their lemmas turn out to be useful.

The following lemmas are needed to prove Theorem 4.3. Lemmas 4.4 to 4.6 are borrowed from [9]; Lemma 4.7 from [8]. We include their proofs for the sake of completeness.

LEMMA 4.4. *For any 2π-periodic function f and any $n \in \mathbb{N}$,*

$$\int_{-\pi}^{\pi} d\xi\, (P_0^n f)(\xi) = \int_{-2^n\pi}^{2^n\pi} d\xi \left[\prod_{k=1}^{n} \left|m_0(2^{-k}\xi)\right|^2\right] f(2^{-n}\xi)\,. \tag{4.4}$$

Proof: By induction. For $n = 0$ the statement is trivial. If the statement is true for $n = j$, then

$$\begin{aligned}\int_{-\pi}^{\pi} d\xi\, (P_0^{j+1} f)(\xi) &= \int_{-2^j\pi}^{2^j\pi} d\xi \left[\prod_{k=1}^{j} \left|m_0(2^{-k}\xi)\right|^2\right] (P_0 f)(2^{-j}\xi) \\ &= \int_{-2^j\pi}^{2^j\pi} d\xi \left[\prod_{k=1}^{j} \left|m_0(2^{-k}\xi)\right|^2\right] \left[\left|m_0(2^{-j-1}\xi)\right|^2 f(2^{-j-1}\xi)\right. \\ &\qquad \left. + \left|m_0(2^{-j-1}\xi + \pi)\right|^2 f(2^{-j-1}\xi + \pi)\right]\end{aligned}$$

$$= \int_{-2^j\pi}^{2^j(\pi+2\pi)} d\xi \left[\prod_{k=1}^{j+1} \left|m_0(2^{-k}\xi)\right|^2\right] f(2^{-j-1}\xi)$$

$$= \int_{-2^{j+1}\pi}^{2^{j+1}\pi} d\xi \left[\prod_{k=1}^{j+1} \left|m_0(2^{-k}\xi)\right|^2\right] , f(2^{-j-1}\xi) ,$$

where we have repeatedly used the periodicity of both m_0 and f. This proves (4.4) for $n = j + 1$, and the lemma is proved. ■

The next lemma shows how trigonometric polynomials invariant under $P_0 \tilde{P}_0$ can be used to prove ϕ, $\tilde{\phi} \in L^2(\mathbb{R})$.

LEMMA 4.5. *Suppose that there exists a strictly positive trigonometric polynomial f_0 invariant under P_0. Then $\phi \in L^2(\mathbb{R})$.*

Proof:

1. Since f_0 is strictly positive, periodic with period 2π, and continuous, there exists $C > 0$ so that $f_0(\xi) \geqq C$ for all $\xi \in \mathbb{R}$.

2. Define

$$F_n(\xi) = \left[\prod_{j=1}^{n} m_0(2^{-j}\xi)\right] \chi(2^{-n}\xi) ,$$

where $\chi(\xi) = 1$ if $|\xi| \leqq \pi$, 0 otherwise. Then $F_n \to \hat{\phi}$ pointwise. By Fatou's lemma, $\hat{\phi} \in L^2(\mathbb{R})$ if $\int d\xi\, |F_n(\xi)|^2$ is uniformly bounded.

3. By Lemma 4.4,

$$\int d\xi\, |F_n(\xi)|^2 f_0(2^{-n}\xi) = \int_{-\pi}^{\pi} d\xi\, (P_0^n f_0)(\xi)$$

$$= \int_{-\pi}^{\pi} d\xi\, f_0(\xi) .$$

Hence

$$\int d\xi\, |F_n(\xi)|^2 \leqq \int d\xi\, |F_n(\xi)|^2 [C^{-1} f_0(2^{-n}\xi)]$$

$$\leqq C^{-1} \int_{-\pi}^{\pi} d\xi\, f_0(\xi) ,$$

which finishes the proof. ■

The next lemma explores the structure of the set of zeros of a non-negative function invariant under P_0. First we define a "shift" operation τ on $[0, 2\pi[$. For $x \in [0, 2\pi[$, we define $\tau x \in [0, 2\pi[$ by $\tau x = 2x$ modulo 2π (i.e., $\tau x = 2x$ for $x < \pi$, $\tau x = 2x - 2\pi$ for $x \geqq \pi$). Then the following lemma holds.

LEMMA 4.6. *Let f be a non-negative trigonometric polynomial invariant under P_0. Then the set of zeros of f can be written as a disjoint union of cyclic sets for τ. Moreover, if $f(\xi) = 0$, then $m_0(\xi + \pi) = 0$.*

Proof:

1. If f has no zeros, or if the only zero of f is $\xi = 0$, then we have nothing to prove (since $m_0(\pi) = 0$; see (3.11)). Without loss of generality we can therefore assume $f(\xi) = 0$, for $0 \neq \xi \in [0, 2\pi[$. Then

$$0 = f(\xi) = (P_0 f)(\xi) = \left| m_0\left(\frac{\xi}{2}\right)\right|^2 f\left(\frac{\xi}{2}\right) + \left| m_0\left(\frac{\xi}{2} + \pi\right)\right|^2 f\left(\frac{\xi}{2} + \pi\right) .$$

Since (use (3.1) and (2.8))

$$m_0(\zeta)\,\overline{\tilde{m}_0(\zeta)} + m_0(\zeta + \pi)\,\overline{\tilde{m}_0(\zeta + \pi)} = 1 ,$$

$|m_0(\frac{1}{2}\xi)|^2$ and $|m_0(\frac{1}{2}\xi + \pi)|^2$ cannot vanish simultaneously. Since $f \geqq 0$, this implies either $f(\frac{1}{2}\xi) = 0$ or $f(\frac{1}{2}\xi + \pi) = 0$.

2. It follows that if we pick one zero $0 \neq \xi_1 \in [0, 2\pi[$ of f, we can associate to it a chain of zeros in $[0, 2\pi[$, $\xi_2, \ldots, \xi_k, \ldots$, with the property that ξ_{j+1} equals either $\frac{1}{2}\xi_j$ or $\frac{1}{2}\xi_j + \pi$, or, equivalently, $\xi_j = \tau\,\xi_{j+1}$. As a trigonometric polynomial f has only finitely many zeros, so that this chain cannot go on ad infinitum. Note that the chain has at least two elements, since $\xi_2 = \xi_1$ would imply $\xi_1 = 0$. Let r be the first index for which recurrence occurs, i.e., $\xi_r = \xi_k$ for some $k < r$. Then necessarily $k = 1$, because $k > 1$ would lead to $\xi_1 = \tau^{k-1}\,\xi_k = \tau^{k-1}\,\xi_r = \xi_{r-k+1}$ with $1 < r - k + 1 < r$, so that r would not be the first index for recurrence. It follows that we have a cycle of zeros, $\xi_1, \ldots, \xi_{r-1}$, with $\tau\xi_{j+1} = \xi_j$ for $j = 1, \ldots, r-2$, and $\tau\xi_1 = \xi_{r-1}$. Note that $\tau^{r-1}\xi_j = \xi_j$ for every zero in this cycle.

3. If this cycle of zeros does not exhaust the set of zeros different from 0, then we can find $0 \neq \zeta_1 \neq \xi_j$, $j = 1, \ldots, r-1$, for which $f(\zeta_1) = 0$. This can again be taken as a seed for a chain of zeros, $\zeta_1, \zeta_2, \ldots, \zeta_l, \ldots$. Every element of this new chain is necessarily different from all the ξ_j, since $\zeta_\ell = \xi_j$ would imply $\zeta_1 = \tau^{\ell-1}\zeta_\ell = \tau^{\ell-1}\xi_j$, i.e., ζ_1 would equal some ξ_k. By the same argument as above, ζ_1 generates therefore a cycle of zeros for f, invariant under τ, and disjoint from the first cycle. We can keep on constructing such cycles until exhaustion of the finite set of zeros of f. This proves the first part of the lemma.

4. To prove the second part, we first note that if $f(\xi) = 0$, then necessarily $f(\xi + \pi) \neq 0$. Indeed, since $\tau\xi = \tau(\xi + \pi)$, both ξ and $\xi + \pi$ would belong to the same cycle of zeros if $f(\xi) = 0 = f(\xi + \pi)$. If this cycle has length n, then it would follow that $\xi = \tau^n\xi = \tau^{n-1}\tau\xi = \tau^{n-1}\tau(\xi + \pi) = \xi + \pi$, which is impossible.

5. Take now any ξ so that $f(\xi) = 0$. Then $\tau\xi$ is also a zero for f, and

$$0 = f(\tau\xi) = (P_0 f)(\tau\xi) = |m_0(\xi)|^2 f(\xi) + |m_0(\xi + \pi)|^2 f(\xi + \pi) .$$

Since $f(\xi) = 0$ and $f(\xi + \pi) \neq 0$, this implies $m_0(\xi + \pi) = 0$. ∎

Finally we prove a technical lemma, borrowed from [8], which we will use to construct the compact set K of C2 if the functions ϕ, $\tilde{\phi}$ satisfy a technical condition.

LEMMA 4.7. *Suppose that* $F(\xi) = \prod_{j=1}^{\infty} |m_F(2^{-j}\xi)|$, *where* m_F *is a trigonometric polynomial satisfying* $m_F(0) = 1$. *Assume that*

$$\sum_{\ell \in \mathbf{Z}} F(\xi + 2\pi\ell) \geqq C \quad a.e. \ , \tag{4.5}$$

for some $C > 0$. *Then there exists a compact set* K, *congruent to* $[-\pi, \pi]$ *modulo* 2π, *containing a neighborhood of 0, so that*

$$\inf_{k \geqq 1, \xi \in K} |m_F(2^{-k}\xi)| > 0 . \tag{4.6}$$

Remark. We shall apply this lemma several times, for different choices of F : $F(\xi) = 2\pi\, |\hat{\phi}(\xi)|\, |\hat{\tilde{\phi}}(\xi)|$, $F(\xi) = 2\pi\, |\hat{\phi}(\xi)|^2$, and $F(\xi) = 2\pi\, |\hat{\tilde{\phi}}(\xi)|^2$.

Proof:

1. Note that Lemma 3.1 applies to $\prod_{j=1}^{\infty} m_F(2^{-j}\xi)$; in particular F is continuous. We first want to show that there exists $\ell_0 \in \mathbf{N}$ so that

$$\sum_{|\ell| \leqq \ell_0} F(\xi + 2\pi\ell) \geqq C/2 \tag{4.7}$$

for all ξ in $[-\pi, \pi]$. By (4.5) there exists, for almost all ξ in $[-\pi, \pi]$, ℓ_ξ so that

$$\sum_{|\ell| \leqq \ell_\xi} F(\xi + 2\pi\ell) \geqq \frac{3}{4} C .$$

Since F is continuous, the finite sum $\sum_{|\ell|\leqq \ell_\xi} F(\cdot + 2\pi\ell)$ is continuous as well. Therefore there exists, for every ξ in $[-\pi, \pi]$, a neighborhood $\{\zeta;\, |\zeta - \xi| \leqq R_\xi\}$ so that, for all ζ in this neighborhood,

$$\sum_{|\ell|\geqq \ell_\xi} F(\zeta + 2\pi\ell) \geqq C/2 .$$

Since $[-\pi, \pi]$ is compact, there exists a finite subset of the collection of intervals $\{\zeta;\, |\zeta - \xi| \leqq R_\xi\}$ which still covers $[-\pi, \pi]$. Take ℓ_0 to be the maximum of the ℓ_{ξ_j} associated to this finite covering; (4.7) holds for that ℓ_0.

2. We can now use (4.7) to construct a compact set K, congruent to $[-\pi, \pi]$ modulo 2π, on which F is bounded below away from zero. From (4.7) we know that for any $\xi \in [-\pi, \pi]$, there exists ℓ between ℓ_0 and $-\ell_0$ so that $F(\xi + 2\pi\ell) \geqq C/[2(2\ell_0 + 1)]$. It follows that if we define sets S_ℓ, $-\ell_0 \leqq \ell \leqq \ell_0$, by

$$S_0 = \{\xi \in [-\pi, \pi];\quad F(\xi) \geqq C/[2(2\ell_0 + 1)]\}$$

and, for $\ell \neq 0$,

$$S_\ell = \left\{\xi \in [-\pi, \pi] \setminus \left(\bigcup_{k=-\ell_0}^{\ell-1} S_k \bigcup S_0\right);\quad F(\xi + 2\pi\ell) \geqq C/[2(2\ell_0 + 1)]\right\} ,$$

then the S_ℓ, $-\ell_0 \leqq \ell \leqq \ell_0$ form a partition of $[-\pi, \pi]$. Since $F(0) = 0$, and since F is continuous, S_0 contains a neighborhood of 0. Define now

$$K = \bigcup_{\ell=-\ell_0}^{\ell_0} \overline{(S_\ell + 2\pi\ell)} .$$

The set K is clearly compact and congruent to $[-\pi, \pi]$ modulo 2π. By construction, $F \geqq C/[2(2\ell_0 + 1)]$ on K. Moreover K contains a neighborhood of 0.

3. Next we show that K satisfies (4.6). We need only check (4.6) for a finite number of k. Indeed, m_F is continuous, and $m_F(0) = 1$. It follows that there exists r so that $|m_F(\zeta)| \geqq 1/2$ for $|\zeta| \leqq r$. Consequently $|m_F(2^{-k}\xi)| \geqq 1/2$ for $\xi \in K$ if $2^{-k}|\xi| \leqq 2^{-k}(2\ell_0 + 1)\pi \leqq r$ or $k \geqq k_0 = \lceil \log_2 (2\ell_0 + 1)\pi/r \rceil$. Let us now treat $1 \leqq k \leqq k_0$. For $\xi \in K$, we have that

$$F(\xi) = \left[\prod_{k=1}^{k_0} \left|m_F(2^{-k}\xi)\right|\right] F(2^{-k}\xi)$$

is bounded below away from zero. Consequently the first factor has no zeros on the compact set K. As a finite product of continuous functions it is itself continuous, whence

$$\prod_{k=1}^{k_0} \left| m_F(2^{-k}\xi) \right| \geqq C_1 > 0 \quad \text{for} \quad \xi \in K .$$

Since m_F is uniformly bounded by, say, C_2, we find therefore, for any k, $1 \leqq k \leqq k_0$,

$$|m_F(2^{-k}\xi)| \geqq C_1\, C_2^{-k_0+1} > 0 ,$$

which proves (4.6). ∎

We are now ready to attack the proof of Theorem 4.3.

Proof of Theorem 4.3:

1. We start by proving $C1 \Rightarrow C2$. We therefore assume C_1 holds, and we construct f_0 and $\tilde{f}_0$.

Since ϕ, hence $\widehat{\phi} \in L^2(\mathbb{R})$, the function

$$f_0(\xi) = \sum_{\ell \in \mathbb{Z}} \left| \widehat{\phi}(\xi + 2\pi\ell) \right|^2$$

is in $L^1([-\pi, \pi])$. One has

$$\int_{-\pi}^{\pi} d\xi\, f_0(\xi) \exp\{-in\xi\} = \int_{-\infty}^{\infty} d\xi \left| \widehat{\phi}(\xi) \right|^2 \exp\{-in\xi\} = \int_{-\infty}^{\infty} dx\, \phi(x)\phi(x-n) .$$

Since ϕ has compact support, this vanishes for large $|n|$, so that f_0 is a trigonometric polynomial. We define $\tilde{f}_0$ entirely analogously.

2. Since $\widehat{\phi}$, $\widehat{\tilde{\phi}}$ are in $L^2(\mathbb{R})$, the sum

$$\sum_{\ell \in \mathbb{Z}} \widehat{\phi}(\xi + 2\pi\ell)\, \overline{\widehat{\tilde{\phi}}(\xi + 2\pi\ell)}$$

converges absolutely for almost all ξ. It defines again an L^1-function on $[-\pi, \pi]$, with Fourier coefficients

$$\int_{-pi}^{\pi} d\xi\, \exp\{-in\xi\} \sum_{\ell \in \mathbb{Z}} \widehat{\phi}(\xi + 2\pi\ell)\, \overline{\widehat{\tilde{\phi}}(\xi + 2\pi\ell)}$$

$$= \int_{-\infty}^{\infty} d\xi\, \exp\{-in\xi\}\, \widehat{\phi}(\xi)\, \overline{\widehat{\tilde{\phi}}(\xi)} = \int_{-\infty}^{\infty} dx\, \phi(x)\tilde{\phi}(x-n) = \delta_{n0} ,$$

so that

$$\sum_{\ell \in \mathbb{Z}} \widehat{\phi}(\xi + 2\pi\ell)\, \overline{\widehat{\tilde{\phi}}(\xi + 2\pi\ell)} = 1 \quad \text{a.e.} \tag{4.8}$$

This implies

$$\sum_{\ell \in \mathbb{Z}} \left|\widehat{\phi}(\xi + 2\pi\ell)\right| \left|\widehat{\tilde{\phi}}(\xi + 2\pi\ell)\right| \geqq 1 \quad \text{a.e.}\ , \tag{4.9}$$

whence, by Cauchy-Schwarz,

$$\left[f_0(\xi)\right]^{1/2} \left[\tilde{f}_0(\xi)\right]^{1/2} \geqq 1\ . \tag{4.10}$$

Since both f_0, $\tilde{f}_0$ are bounded (they are trigonometric polynomials), (4.10) implies that they are bounded below away from zero.

3. We have moreover

$$\begin{aligned}(P_0 f_0)(\xi) &= \left|m_0\left(\frac{\xi}{2}\right)\right|^2 \sum_{\ell \in \mathbb{Z}} \left|\widehat{\phi}\left(\frac{\xi}{2} + 2\ell\pi\right)\right|^2 \\ &\quad + \left|m_0\left(\frac{\xi}{2} + \pi\right)\right|^2 \sum_{\ell \in \mathbb{Z}} \left|\widehat{\phi}\left(\frac{\xi}{2} + (2\ell + 1)\pi\right)\right|^2 \\ &= \sum_{k \in \mathbb{Z}} \left|m_0\left(\frac{\xi}{2} + k\pi\right)\right|^2 \left|\widehat{\phi}\left(\frac{\xi}{2} + k\pi\right)\right|^2 \\ &= \sum_{k \in \mathbb{Z}} \left|\widehat{\phi}(\xi + 2k\pi)\right|^2 = f_0(\xi)\ ,\end{aligned}$$

so that f_0 is invariant under P_0. Similarly $\tilde{f}_0$ is invariant under $\tilde{P}_0$.

4. Points 1 to 3 prove the first part of C2. By (4.9), the function $F(\xi) = 2\pi\, |\widehat{\phi}(\xi)|\, |\widehat{\tilde{\phi}}(\xi)|$ satisfies all the conditions of Lemma 4.7. There exists therefore a compact set K congruent to $[-\pi, \pi]$ modulo 2π, containing a neighborhood of the origin, so that $|m_0(2^{-k}\xi)|\, |\tilde{m}_0(2^{-k}\xi)| \geqq C > 0$ for all $k \geqq 1$, all $\xi \in K$. Since both m_0 and $\tilde{m}_0$ are bounded by, say, C', it follows that

$$\left|m_0(2^{-k}\xi)\right|,\ \left|\tilde{m}_0(2^{-k}\xi)\right| \geqq C/C' \quad \text{for all } \xi \in K,\ k \geqq 1\ .$$

This proves the second part of C2, and ends the proof $C1 \Rightarrow C2$.

5. We now tackle $C2 \Rightarrow C3$.

Assume C2 holds. We only need to prove that f_0, $\tilde{f}_0$ are the unique trigonometric polynomials (up to normalization) invariant for P_0, $\tilde{P}_0$ respectively. We shall do this by showing that the existence of another invariant trigonometric polynomial for either P_0 or $\tilde{P}_0$ contradicts the existence of the compact set K in C2. Suppose $f_0^\sharp$ is an invariant trigonometric polynomial for P_0, with $f_0^\sharp \neq \gamma f_0$. Define $f_t = f_0 + t f_0^\sharp$. For some value t_1 of t, this function takes strictly negative as well as strictly positive values. Consider then $f^t = f_0 + t f_{t_1}$. Since $f^{t=0} = f_0 \geqq C > 0$, since $f_{t_1}(\xi) < 0$ for some ξ, and since $f^t(\xi)$ is continuous in t as well as in ξ, there exist $t_- < 0 < t_+$ so that f^{t_-}, f^{t_+} have zeros, but are non-negative. Since $f_0(0) \neq 0$, at least one of the two functions f^{t_-}, f^{t_+} does not vanish in $\xi = 0$; we denote this function by f. By construction, f is a non-negative trigonometric polynomial, invariant for P_0, which has at least one zero, and which satisfies $f(0) \neq 0$. By Lemma 4.6, the existence of f implies the existence of a cyclic set $\xi_1, \ldots, \xi_n$ for τ, with $\xi_j = \tau \xi_{j+1}$, $j = 1, \ldots, n-1$, $\xi_1 = \tau \xi_n$, so that $m_0(\xi_j + \pi) = 0$ for all j. Since $f(0) \neq 0$, we have $\xi_j \neq 0$.

6. We now show how these zeros $\xi_j + \pi$ for m_0 are incompatible with the existence of K.

Since $\tau\xi_{j+1} = \xi_j$, $\tau\xi_n = \xi_1$, and in particular $\xi_j = \tau^n \xi_j$, we have $\xi_j = 2\pi x_j$, where the $x_j \in [0, 1[$ have the following representations in binary:

$$\begin{array}{ll} x_1 = & .d_1 d_2 \ldots \quad d_n d_1 \ldots \quad d_n d_1 \ldots \quad d_n \ldots \\ x_2 = & .d_2 \ldots \quad d_n d_1 \ldots \quad d_n d_1 \ldots \quad d_n \ldots \\ \vdots & \\ x_n = & .d_n d_1 \ldots \quad d_n d_1 \ldots \quad d_n \ldots . \end{array} \qquad (d_j = 0 \text{ or } 1)$$

Since $\xi_1 \neq 0$, not all the d_j are zero. Let us, for this point only, define $\bar{d} = 1 - d$ for $d = 0$ or 1. Then $\xi_j + \pi = 2\pi y_j$ modulo 2π, with y_j given by

$$\begin{array}{ll} y_1 = & .\bar{d}_1 d_2 d_3 \ldots \quad d_n d_1 \ldots \quad d_n d_1 \ldots \quad d_n \ldots \\ y_2 = & .\bar{d}_2 d_3 \ldots \quad d_n d_1 \ldots \quad d_n d_1 \ldots \quad d_n \ldots \\ \vdots & \\ y_n = & .\bar{d}_n d_1 \ldots \quad d_n d_1 \ldots \quad d_n \ldots . \end{array}$$

We have $m_0(2\pi y_j) = 0$, $j = 1, \ldots, n$. Suppose a compact set K existed with all the properties listed in $C2$. Then there would be an integer ℓ, with a binary expansion with at most a certain preassigned number L of digits (L depends only on the size of K), so that $2\pi y = 2\pi(2y_1 + \ell)$ has the property that $m_0(2\pi 2^{-k} y) \neq 0$ for all $k \geqq 0$. We have

$$y = e_L \ldots e_2 e_1 .d_2 d_3 \ldots d_n d_1 \ldots d_n d_1 \ldots d_n \ldots$$

with $e_j = 1$ or 0 for $j = 1, \dots, L$. We can also rewrite this by inserting n extra zeros at the front end, i.e.,

$$y = e_{L+n} \dots e_{L+1} e_L \dots e_2 e_1 .d_2 d_3 \dots d_n d_1 \dots d_n d_1 \dots d_n$$

where $e_j = 1$ or 0 for $j = 1, \dots, L$ and $e_j = 0$ if $j > L$. The $2^{-k}y$ are obtained by shifting the decimal point to the left. Since m_0 is 2π-periodic, only the "tail", i.e., the part of the expansion of $2^{-k}y$ to the right of the decimal point, decides whether $m_0(2\pi 2^{-k}y)$ vanishes or not. If $e_1 = \bar{d}_1$, then $y/2$ would have the same decimal part as y_1, hence $m_0(2\pi y/2) = 0$ would follow. Since $m_0(2\pi y/2) \neq 0$, we have therefore $e_1 = d_1$. Similarly we conclude $e_2 = d_n$, $e_3 = d_{n-1}$, etc. It follows that $e_{L+1}, \dots, e_{L+n}$ are also successively equal to $d_k, d_{k-1}, \dots, d_1, d_n, \dots, d_k + 1$ for some $k \in \{1, 2, \dots, n\}$. Since the d_j are not all equal to 0, whereas $e_{L+1} = \dots = e_{L+n} = 0$, this is a contradiction. This finishes the proof of $C2 \Rightarrow C3$.

7. We now attack $C3 \Rightarrow C1$.

Assume $C3$ holds. By Lemma 4.5, the existence of strictly positive invariant trigonometric polynomials f_0, $\tilde{f}_0$ for P_0, $\tilde{P}_0$ respectively implies ϕ, $\tilde{\phi} \in L^2(\mathbb{R})$. We only need to prove that $\int dx\, \phi(x-k)\, \tilde{\phi}(x-l) = \delta_{k\ell}$. We shall do this by proving that ϕ, $\tilde{\phi}$ are the L^2-limits of functions that have this biorthogonality property.

Since ϕ, $\tilde{\phi} \in L^2(\mathbb{R})$, we can repeat the argument in point 1 of this proof, showing that $\sum_{\ell\in\mathbb{Z}} |\hat{\phi}(\xi+2\pi\ell)|^2$, $\sum_{\ell\in\mathbb{Z}} |\hat{\tilde{\phi}}(\xi+2\pi\ell)|^2$ are trigonometric polynomials, invariant for P_0, $\tilde{P}_0$ respectively. Since P_0, $\tilde{P}_0$ each have only invariant trigonometric polynomial, which is moreover strictly positive, it follows that

$$\sum_{\ell} \left|\hat{\phi}(\xi + 2\pi\ell)\right|^2 \geqq C\,, \quad \sum_{\ell} \left|\hat{\tilde{\phi}}(\xi + 2\pi\ell)\right|^2 \geqq C$$

for some $C > 0$. We can therefore apply Lemma 4.7 to $F(\xi) = 2\pi\, |\hat{\phi}(\xi)|^2$ or $F(\xi) = 2\pi\, |\hat{\tilde{\phi}}(\xi)|^2$. It follows that there exist compact sets K and $\tilde{K}$, both congruent to $[-\pi, \pi]$ modulo 2π, and both containing a neighborhood of the origin, so that

$$|m_0(2^{-k}\xi)| \geqq C \quad \text{for all} \quad \xi \in K,\ k \geqq 1$$

$$|\tilde{m}_0(2^{-k}\zeta)| \geqq C \quad \text{for all} \quad \zeta \in \tilde{K},\ k \geqq 1$$

for some $C > 0$. Note that K and $\tilde{K}$ need not be the same set here (unlike $C2$).

8. Define now $F_n(\xi) = (2\pi)^{-1/2} \left[\prod_{j=1}^n m_0(2^{-j}\xi)\right] \chi_K(2^{-n}\xi)$,

$$\tilde{F}_n(\xi) = (2\pi)^{-1/2} \left[\prod_{j=1}^{n} \tilde{m}_0(2^{-j}\xi)\right] \chi_{\tilde{K}}(2^{-n}\xi) ,$$

where $\chi_K(\zeta) = 1$ if $\zeta \in K, \chi_K(\zeta) = 0$ otherwise ($\chi_{\tilde{K}}$ is defined analogously). In this step of the proof we show that $\|F_n - \widehat{\phi}\|_{L^2}, \|\tilde{F}_n - \widehat{\tilde{\phi}}\|_{L^2} \to 0$ as $n \to \infty$.

For $\xi \in K$ we have $|m_0(2^{-k}\xi)| \geqq C > 0$. On the other hand, we also have, for any ξ, $|m_0(\xi)| \geqq 1 - |m_0(\xi) - m_0(0)| \geqq 1 - C'|\xi|$. Since K is compact and therefore bounded, we can find k_0 so that $2^{-k}C'|\xi| < 1/2$ if $\xi \in K$ and $k \geqq k_0$. Using $1 - x \geqq e^{-2x}$ for $0 \leqq x \leqq 1/2$, we find

$$\begin{aligned}\left|\widehat{\phi}(\xi)\right| &= (2\pi)^{-1/2} \prod_{k=1}^{k_0} \left|m_0(2^{-k}\xi)\right| \prod_{k=k_0+1}^{\infty} \left|m_0(2^{-k}\xi)\right| \\ &\geqq (2\pi)^{-1/2} C^{k_0} \prod_{k=k_0+1}^{\infty} \exp\left\{-2C' 2^{-k}|\xi|\right\} \\ &\geqq (2\pi)^{-1/2} C^{k_0} \exp\left\{-C' 2^{-k_0+1} \max_{\zeta \in K} |\zeta|\right\} = C'' > 0 ,\end{aligned}$$

or $|\widehat{\phi}(\xi)/C''| \geqq 1$ for $\xi \in K$, which we can also rewrite as $|\widehat{\phi}(\xi)| \geqq C'' \chi_K(\xi)$. Consequently

$$\begin{aligned}|F_n(\xi)| &= (2\pi)^{-1/2} \prod_{j=1}^{n} |m_0(2^{-j}\xi)| \chi_K(2^{-n}\xi) \\ &\leqq (C'')^{-1} \prod_{j=1}^{n} |m_0(2^{-j}\xi)| |\widehat{\phi}(2^{-n}\xi)| \\ &\leqq |\widehat{\phi}(\xi)|/C'' .\end{aligned}$$

Since $F_n \to \widehat{\phi}$ pointwise as $n \to \infty$, and since $\widehat{\phi} \in L^2$, it follows by dominated convergence that $\|F_n - \widehat{\phi}\|_{L^2} \to 0$ for $n \to \infty$. Similarly $\lim_{n\to\infty} \|\tilde{F}_n - \widehat{\tilde{\phi}}\|_{L^2} = 0$.

9. The compact set K contains a neighborhood $\{\xi;\ |\xi| \leqq \alpha\}$ of 0. Define $A_n(\xi) = (2\pi)^{-1/2} \left[\prod_{j=1}^n m_0(2^{-j}\xi)\right] \chi_{[-\alpha,\alpha]}(2^{-n}\xi)$, where $\chi_{[-\alpha,\alpha]}(\zeta) = 1$ if $|\zeta| \leqq \alpha$, 0 otherwise. Clearly $|A_n(\xi)| \leqq |F_n(\xi)| \leqq |\widehat{\phi}(\xi)|/C''$, while also $A_n \to \widehat{\phi}$

pointwise as $n \to \infty$. Consequently we have again $\|A_n - \widehat{\phi}\|_{L^2} \to 0$ as $n \to \infty$. This implies that $B_n = F_n - A_n$ tends to 0, in L^2-sense, as $n \to \infty$. We have

$$
\begin{aligned}
\|B_n\|_{L^2}^2 &= (2\pi)^{-1} \int\limits_{\substack{2^{-n}\xi \in K \\ |2^{-n}\xi| \geq \alpha}} d\xi \left| \prod_{j=1}^{n} m_0(2^{-j}\xi) \right|^2 \\
&= (2\pi)^{-1} \int\limits_{\alpha \leq |2^{-n}\xi| \leq \pi} d\xi \left| \prod_{j=1}^{n} m_0(2^{-j}\xi) \right|^2 ,
\end{aligned}
\tag{4.11}
$$

where we have used the 2π-periodicity of m_0 and the congruence of K to $[-\pi, \pi]$ modulo 2π, which since $[-\alpha, \alpha] \subset K$, implies the congruence of $K \setminus [-\alpha, \alpha]$ to $[-\pi, \pi] \setminus [-\alpha, \alpha]$. We introduce one more sequence of functions,

$$
\hat{u}_n(\xi) = (2\pi)^{-1/2} \left[\prod_{j=1}^{n} m_0(2^{-j}\xi) \right] \chi_{[-\pi, \pi]}(2^{-n}\xi) .
$$

Clearly $\hat{u}_n \in L^2(\mathbb{R})$, hence $u_n \in L^2(\mathbb{R})$. Moreover

$$
\hat{u}_n(\xi) = A_n(\xi) + (2\pi)^{-1/2} \left[\prod_{j=1}^{n} m_0(2^{-j}\xi) \right] \chi_{[-\pi, \pi] \setminus [-\alpha, \alpha]}(2^{-n}\xi) .
$$

By (4.11) the L^2-norm of the second term is exactly equal to $\|B_n\|_{L^2}$, which tends to 0 as $n \to \infty$. Since $\lim_{n\to\infty} \|A_n - \widehat{\phi}\|_{L^2} = 0$, it follows that $\lim_{n\to\infty} \|\hat{u}_n - \widehat{\phi}\|_{L^2} = 0$, or L^2-$\lim_{n\to\infty} u_n = \phi$. Similarly L^2-$\lim_{n\to\infty} \tilde{u}_n = \widetilde{\phi}$, where

$$
\hat{\tilde{u}}_n(\xi) = (2\pi)^{-1/2} \left[\prod_{j=1}^{n} \widetilde{m}_0(2^{-j}\xi) \right] \chi_{[-\pi, \pi]}(2^{-n}\xi) .
$$

Note that

$$
\hat{u}_{n+1}(\xi) = m_0(\xi/2)\, \hat{u}_n(\xi/2) ,
$$

or equivalently

$$
u_{n+1}(x) = \sqrt{2} \sum_k h_k\, u_n(2x - k) .
$$

Similarly

$$
\tilde{u}_{n+1}(x) = \sqrt{2} \sum_k \tilde{h}_k\, \tilde{u}_n(2x - k) .
$$

10. We have

$$\int dx\, u_0(x)\,\overline{\tilde{u}_0(x-k)} = \int d\xi\, \hat{u}_0(\xi)\,\overline{\hat{\tilde{u}}_0(\xi)}\, e^{ik\xi}$$
$$= \int_{-\pi}^{\pi} d\xi\, (2\pi)^{-1}\, e^{ik\xi} = \delta_{k0}\, .$$

On the other hand, if $\int dx\, u_n(x)\,\overline{\tilde{u}_n(x-k)} = \delta_{k0}$, then, by (2.9),

$$\int dx\, u_{n+1}(x)\,\overline{\tilde{u}_{n+1}(x-k)} = 2\sum_{r,s} h_r\,\tilde{h}_s \int dx\, u_n\,(2x-r)\,\overline{\tilde{u}_n(2x-2k-s)}$$
$$= \sum_r h_r\,\tilde{h}_{r-2k} = \delta_{k0}\, .$$

By induction this proves

$$\int dx\, u_n(x)\,\overline{\tilde{u}_n(x-k)} = \delta_{k0}$$

for all n. Since $L^2\text{-}\lim_{n\to\infty} u_n = \phi$, $L^2\text{-}\lim_{n\to\infty} \tilde{u}_n = \tilde{\phi}$, this implies ($\tilde{\phi}$ is real)

$$\int dx\, \phi(x)\,\tilde{\phi}(x-k) = \delta_{k0}\, ,$$

which proves $C1$.

This proves $C3 \Rightarrow C1$, thereby establishing the desired equivalence $C1 \Leftrightarrow 2C2 \Leftrightarrow C3$. ∎

Theorem 4.3 gives a satisfactory (since necessary and sufficient) as well as easy criterium for biorthogonality of the $\phi(x-k)$ and $\tilde{\phi}(x-\ell)$: it suffices to check that the matrices corresponding to P_0, $\tilde{P}_0$ have 1 as an eigenvalue, that this eigenvalue is nondegenerate in both cases, and that the trigonometric polynomials having the entries of the corresponding eigenvectors as their Fourier coefficients are strictly positive.

Remark. It seems quite striking that the equivalent conditions $C2$, $C3$ already imply ϕ, $\tilde{\phi} \in L^2(\mathbb{R})$, without any assumptions of decay for the infinite products $\prod_{j=1}^{\infty} m_0(2^{-j}\xi)$ or $\prod_{j=1}^{\infty} \tilde{m}_0(2^{-j}\xi)$. (In the orthonormal case, as noted in the Introduction, (1.12) is already sufficient to ensure $\phi \in L^2(\mathbb{R})$. The biorthogonal equivalent, $m_0(\xi)\,\overline{\tilde{m}_0(\xi)} + m_0(\xi+\pi)\,\overline{\tilde{m}_0(\xi+\pi)} = 1$, is no longer sufficient to ensure ϕ, $\tilde{\phi} \in L^2(\mathbb{R})$. An example is $m_0(\xi) = -1/2 + e^{-i\xi}/2 + e^{-2i\xi}$, $\tilde{m}_0(\xi) = 3e^{i\xi}/2 + 1/2 + e^{-i\xi}$. In this case P_0 and $\tilde{P}_0$ have each only one invariant trigonometric polynomial, given by $1-4\cos\xi$, $1-12\cos\xi$

respectively. As point 1 in the proof of Theorem 4.3 shows, $\phi \in L^2(\mathbf{R})$ would imply the existence of a non-negative invariant trigonometric polynomial. It follows that neither ϕ or $\tilde{\phi}$ are square integrable in this example. With the extra condition $C2$ or $C3$, square integrability is restored.) The fact that $\hat{\phi}$ and $\hat{\tilde{\phi}}$ can both be in $L^2(\mathbf{R})$ and yet not satisfy the decay condition (3.13) is due to the lacunarity of these functions: they often have narrow bumps, recurring infinitely often, but tending to be less frequent when ξ goes to ∞. These mar the decay of $\hat{\phi}$ and $\hat{\tilde{\phi}}$, but not their square integrability. In fact, we already know that the decay condition (3.13) is not necessary for the strong convergence of (3.14) or to have dual Riesz bases $\{\psi_{jk}\}$, $\{\tilde{\psi}_{jk}\}$: there exist orthonormal wavelet bases for which (3.13) is not satisfied.

Using a different approach, involving a further study of the operators P_0 and $\tilde{P}_0$ and their eigenvalues, two of us have derived recently (see [7]) (after completion of the present work — this "Remark" was added a year later) a set of necessary and sufficient conditions on m_0 and $\tilde{m}_0$ under which the same results as in Theorem 3.8 can be obtained, side stepping the decay condition (3.13). (This new technique in [7] was developed more specifically for higher dimensions, but applies also to one dimension.)

For many practical purposes, however, decay of $\hat{\phi}$ and $\hat{\tilde{\phi}}$ is desirable, even if not strictly necessary to make the theorem work.

4.B. Decay at Infinity

The following proposition gives a family of sufficient conditions ensuring that (3.13) holds. It was already stated (with proof) in [10]; see Lemmas 3.2 and 4.6 there. The argument is due to Ph. Tchamitchian; it is very short, so we repeat it here for the reader's convenience.

PROPOSITION 4.8. *Suppose m_0 can be factored as*

$$m_0(\xi) = \left(\frac{1+e^{-i\xi}}{2}\right)^L \mathcal{F}(\xi)\,, \tag{4.12}$$

where $\mathcal{F}$ is again a trigonometric polynomial. Suppose that, for some $k \geqq 0$,

$$B_k = \max_{\xi} |\mathcal{F}(\xi)\,\mathcal{F}(2\xi)\ldots\mathcal{F}(2^{k-1}\xi)|^{1/k} < 2^{L-1/2}\,. \tag{4.13}$$

Then $|\hat{\phi}(\xi)| \leqq C(1+|\xi|)^{-\frac{1}{2}-\varepsilon}$, with $\varepsilon = L - \frac{1}{2} - \frac{\log B_k}{\log 2} > 0$.

Note that since $m_0(\pi) = 0$ (see (3.11), we have always $L \geqq 1$.

Proof:

1. Since $|m_0(\xi)| \leqq 1 + C|\xi| \leqq \exp\{C|\xi|\}$, we have

$$\left|\widehat{\phi}(\xi)\right| \leqq \prod_{j=1}^{\infty} \exp\left\{2^{-j}\, C|\xi|\right\} \leqq \exp\left\{C|\xi|\right\} \ ,$$

which is uniformly bounded for $|\xi| \leqq 1$. We therefore need to concern ourselves only with $|\xi| \geqq 1$.

2. Since

$$\prod_{j=1}^{\infty} \left[\frac{1 + \exp\left\{-i2^{-j}\xi\right\}}{2}\right] = \prod_{j=1}^{\infty} \left[\exp\left\{-i2^{-j-1}\xi\right\} \cdot \cos\left(2^{-j-1}\xi\right)\right]$$

$$= \exp\{-i\xi/2\} \prod_{j=1}^{\infty} \frac{\sin 2^{-j}\xi}{2\sin 2^{-j-1}\xi} = \exp\{-i\xi/2\}\, \frac{\sin \xi/2}{\xi/2} \ ,$$

we have

$$\widehat{\phi}(\xi) = \prod_{j=1}^{\infty} m_0(2^{-j}\xi) = \exp\{-iL\xi/2\} \left(\frac{\sin \xi/2}{\xi/2}\right)^{L} \prod_{\ell=0}^{\infty} \mathcal{G}(2^{-k\ell}\xi) \ , \tag{4.14}$$

with $\mathcal{G}(\xi) = \mathcal{F}(\xi/2)\, \mathcal{F}(\xi/4) \cdots \mathcal{F}(2^{-k}\xi)$.

3. Since $|\xi| > 1$, there exists $\ell_0 \geqq 0$ so that $2^{k\ell_0} \leqq |\xi| < 2^{k(\ell_0+1)}$. By the same argument as in point 1,

$$\prod_{\ell=\ell_0+1}^{\infty} \left|\mathcal{G}\left(2^{-k\ell}\xi\right)\right| = \prod_{j=0}^{\infty} \left|\mathcal{G}\left(2^{-jk} 2^{-(\ell_0+1)k}\xi\right)\right|$$

is bounded independently of ξ, since $|2^{-(\ell_0+1)k}\xi| \leqq 1$. On the other hand

$$\prod_{\ell=0}^{\ell_0} \left|\mathcal{G}\left(2^{-k\ell}\xi\right)\right| \leqq B_k^{k(\ell_0+1)} \leqq B_k^{k+\log|\xi|/\log 2}$$

$$\leqq C(1 + |\xi|)^{\log B_k/\log 2} \ .$$

Putting all this together with (4.14), we find

$$\left|\widehat{\phi}(\xi)\right| \leqq C(1 + |\xi|)^{-L+\log B_k/\log 2} \ . \ \blacksquare$$

In [10] it was proved, for the orthonormal case, that $B_1 < 2^{L-1/2}$ implies not only decay of $\widehat{\phi}$, but also orthonormality of the $\phi(x-\ell)$. This proof used the auxiliary functions η_n, which are piecewise constant functions defined by

$$\begin{aligned} \eta_0(x) &= 1 \quad \text{for} \quad -1/2 \leqq x < 1/2 \\ &\quad 0 \quad \text{otherwise} \\ \eta_n(x) &= \sqrt{2} \sum_k h_k \, \eta_{n-1}(2x-k) \, . \end{aligned} \tag{4.15}$$

With this definition the η_n are piecewise constant on the intervals $[2^{-n}(\ell - 1/2), 2^{-n}(1+1/2)[$, $\ell \in \mathbb{Z}$. If $B_1 < 2^{L-1/2}$, then the η_n converge to ϕ in L^2-sense. This can then be used to prove orthonormality of the $\phi(x-\ell)$ by an argument similar to point 10 in the proof of Theorem 4.3: $\int dx\, \eta_n(x)\, \overline{\eta_n(x-\ell)} = \delta_{\ell 0}$ by induction (it is trivial for $n=0$, and the induction step follows from (1.12)), and L^2-convergence carries this over to ϕ. The following proposition uses a similar argument to prove that if both m_0 and $\widetilde{m}_0$ satisfy a condition of type (4.13), then the $\phi(x-\ell)$ and $\widetilde{\phi}(x-\ell')$ are biorthogonal.

Proposition 4.9. *Assume that both m_0 and $\widetilde{m}_0$ can be factored as in (4.12),*

$$m_0(\xi) = \left(\frac{1+e^{-i\xi}}{2}\right)^L \mathcal{F}(\xi) \, , \quad \widetilde{m}_0(\xi) = \left(\frac{1+e^{-i\xi}}{2}\right)^{\tilde{L}} \widetilde{\mathcal{F}}(\xi) \, ,$$

and suppose that, for some $k, \tilde{k} > 0$,

$$\begin{aligned} B_k &= \max_\xi \left| \mathcal{F}(\xi) \cdots \mathcal{F}\left(2^{k-1}\xi\right) \right|^{1/k} < 2^{L-1/2} \\ \widetilde{B}_{\tilde{k}} &= \max_\xi \left| \widetilde{\mathcal{F}}(\xi) \cdots \widetilde{\mathcal{F}}\left(2^{\tilde{k}-1}\xi\right) \right|^{1/\tilde{k}} < 2^{\tilde{L}-1/2} \, . \end{aligned} \tag{4.16}$$

Then $\phi, \widetilde{\phi} \in L^2(\mathbb{R})$ and $\int dx\, \phi(x)\, \widetilde{\phi}(x-n) = \delta_{n0}$.

Proof:

1. We introduce again the functions u_n, $\tilde{u}_n$ of point 9 in the proof of Theorem 4.3,

$$\begin{aligned} \hat{u}_n(\xi) &= (2\pi)^{-1/2} \left[\prod_{j=1}^n m_0(2^{-j}\xi)\right] \chi_{[-\pi,\pi]}(2^{-n}\xi) \\ \hat{\tilde{u}}_n(\xi) &= (2\pi)^{-1/2} \left[\prod_{j=1}^n \widetilde{m}_0(2^{-j}\xi)\right] \chi_{[-\pi,\pi]}(2^{-n}\xi) \, . \end{aligned}$$

It was already established in point 10 of the proof of Theorem 4.3 that

$$\int dx\, u_n(x)\, \overline{\tilde{u}_n(x-l)} = \delta_{\ell 0} \,.$$

To prove $\int dx\, \phi(x)\, \widetilde{\phi}(x-\ell) = d_{\ell 0}$, it suffices therefore to prove L^2-$\lim_{n\to\infty} \hat{u}_n = \widehat{\phi}$, L^2-$\lim_{n\to\infty} \hat{\tilde{u}}_n = \widehat{\tilde{\phi}}$.

2. Because of the factorization (4.12), we have

$$|\hat{u}_n(\xi)| = (2\pi)^{-1/2} \left| \frac{\sin(\xi/2)}{2^n \sin(2^{-n-1}\xi)} \right|^L \prod_{\ell=0}^{n} |\mathcal{F}(2^{-\ell}\xi)|\; \chi_{[-\pi,\pi]}(2^{-n}\xi) \,.$$

To bound this we use several ingredients. On the one hand, $|\sin\zeta| \geqq \frac{2}{\pi}|\zeta|$ for $|\zeta| \leqq \pi/2$, hence

$$\left|\sin\left(2^{-n-1}\xi\right)\right|^{-1} \chi_{[-\pi,\pi]}(2^{-n}\xi) \leqq \frac{2}{\pi} 2^{n+1} |\xi|^{-1} \,,$$

which implies

$$\left| \frac{\sin \xi/2}{2^n \sin(2^{-n-1}\xi)} \right| \chi_{[-\pi,\pi]}(2^{-n}\xi) \leqq \frac{2}{\pi} \left| \frac{\sin\xi/2}{\xi/2} \right| \leqq C(1+|\xi|)^{-1} \,.$$

On the other hand, writing $n = k\,n' + q$ with $0 \leqq q < k$,

$$\begin{aligned} \left| \prod_{\ell=0}^{n} \mathcal{F}\left(2^{-\ell}\xi\right) \right| &\leqq \left[\sup_{\zeta} |\mathcal{F}(\zeta)| \right]^{q} \prod_{\ell'=0}^{n'} \left| \mathcal{G}\left(2^{-k\ell'}\xi\right) \right| \\ &\leqq \left[\sup_{\zeta} |\mathcal{F}(\zeta)| \right]^{k-1} C(1+|\xi|)^{\log B_k/\log 2} \end{aligned}$$

by the same argument as in point 3 of the proof of Proposition 4.8. Putting it all together, we have

$$|\hat{u}_n(\xi)| \leqq C(1+|\xi|)^{-L+\log B_k \log 2} \,,$$

where C is independent of n. Since $\hat{u}_n$ converges pointwise that $\widehat{\phi}$, the Lebesgue dominated convergence theorem implies that $\hat{u}_n$ tends to $\widehat{\phi}$ in $L^2(\mathbb{R})$. The L^2-convergence of $\hat{\tilde{u}}_n$ is proved analogously. ∎

Remarks.

1. This proof is considerably simpler than the proofs in [10], mainly because the $\hat{u}_n$ are compactly supported; considerable effort in [10] was spent in dealing with the "tail" of the $\hat{\eta}_n$ (η_n as defined by (4.15) has compact support, so that $\hat{\eta}_n$ is supported on the whole real line).

2. Exactly the same arguments can be used to prove

$$(4.17) \qquad \begin{array}{llll} & L^1\text{-}\lim_{n\to\infty} \hat{u}_n = \widehat{\phi}, & L^1\text{-}\lim_{n\to\infty} \hat{\tilde{u}}_n = \widehat{\tilde{\phi}} & \\ \text{if} & B_k < 2^{L-1}, & \tilde{B}_{\tilde{k}} < 2^{\tilde{L}-1} & \text{for some } k, \tilde{k}\,. \end{array}$$

This leads to the uniform convergence of $u_n(x)$ to $\phi(x)$ and $\tilde{u}_n(x)$ to $\tilde{\phi}(x)$. In fact one even has uniform convergence in a Hölder space $C^\varepsilon(\mathbb{R})$ since

$$\int d\xi\, (1+|\xi|)^\varepsilon \left|\widehat{\phi}(\xi) - \hat{u}_n(\xi)\right| \underset{n\to\infty}{\longrightarrow} 0$$

and

$$\int d\xi\, (1+|\xi|)^\varepsilon \left|\widehat{\tilde{\phi}}(\xi) - \hat{\tilde{u}}_n(\xi)\right| \underset{n\to\infty}{\longrightarrow} 0\,,$$

for $0 \leqq \varepsilon < \min(L-1-\log_2 B_k,\ \tilde{L}-1-\log_2 \tilde{B}_{\tilde{k}})$. This can be used to prove pointwise convergence of the η_n, $\tilde{\eta}_n$: since $u_0(m) = \delta_{m0} = \eta_0(m)$, it easily follows from the recurrence relations for both u_n and η_n that $u_n(2^{-n}\ell) = \eta_n(2^{-n}\ell)$ for all $\ell \in \mathbb{Z}$. The argument in point 7 of the proof of Proposition 3.3. in [10] then proves pointwise convergence of $\eta_n(x)$ to $\phi(x)$. The η_n have one advantage that the u_n do not have: they are compactly supported; in fact support $\eta_n =$ support $\phi + [-2^{-n-1}, 2^{-n-1}]$. Moreover, for the η_n the recursion relation (4.15), relating $\eta_n(x)$ with the $\eta_{n-1}(2x-\ell)$, can be rewritten as a "local" recursion, where $\eta_n(x)$ is completely determined by $\eta_{n-1}(y)$ with $x + 2^{-n}N_1 \leqq y \leqq x + 2^{-n}N_2$ (if we assume that m_0 is of the form $m_0(\xi) = 2^{-1/2}\sum_{n=N_1}^{N_2} h_n\, e^{-in\xi}$). This translates into a graphical algorithm for the construction of the η_n, explained at length in [10], also called the "cascade algorithm" in [13]. It is akin to subdivision algorithms in CAD, which have the same "zoom-in" quality. The convergence of the η_n to ϕ is extremely (exponentially) fast; in fact, all the graphs of ϕ, $\tilde{\phi}$ given in the examples in this paper and in [10], [12] are graphs of some η_n, $\tilde{\eta}_n$ rather than ϕ, $\tilde{\phi}$, with $n = 6$, 7, or 8.

3. If B_k, $\tilde{B}_{\tilde{k}}$ satisfy even more stringent bounds,

$$B_k < 2^{L-1-m}, \quad \tilde{B}_{\tilde{k}} < 2^{\tilde{L}-1-\tilde{m}}, \quad \text{with } m, \tilde{m} \in \mathbb{N}$$

then one has $\phi \in C^m$, $\tilde{\phi} \in C^{\tilde{m}}$, implying $\psi \in C^m$, $\tilde{\psi} \in C^{\tilde{m}}$. In this case it is useful to define η_0^m to be the interpolating spline of order m, i.e., the spline function of order m with nodes at the integers if m is odd, at the integers $+ 1/2$ if m is even, and satisfying $\eta_0^m(\ell) = \delta_{\ell 0}$. (For $m = 1$, for instance, $\eta_0^1(x) = 1 - |x|$ if $|x| \leqq 1$, 0 otherwise.) Again we define η_n^m recursively by $\eta_n^m(x) = \sqrt{2} \sum_\ell h_\ell \, \eta_{n-1}^m (2x - l)$ ($\tilde{\eta}_n^m$ are defined analogously). Arguments similar to those above show then that, for any $m' \leqq m$, $(d^{m'}/(dx^{m'}) \eta_n^m \underset{n}{\longrightarrow} (d^{m'})/(dx^{m'}) \phi$ uniformly as $n \to \infty$.

4. It may seem artificial to impose a factorization of type (4.12) on m_0 and $\tilde{m}_0$. As noted above, $m_0(\pi) = 0 = \tilde{m}_0(\pi)$ implies that m_0 and $\tilde{m}_0$ are always divisible by $1 + e^{i\xi}$. Moreover, we shall see in the next section that more regularity (ψ, $\tilde{\psi} \in C^m$ with $m \geqq 1$) can only be attained if m_0, $\tilde{m}_0$ are both divisible by $(1 + e^{i\xi})^{m+1}$.

5. It is quite striking that the condition we imposed to ensure (3.13), namely (4.16), is also sufficient to prove (3.26). It looks like we might have dispensed altogether with all the technicalities in Section 4.A! Recall, however, that (3.13) is not strictly necessary to obtain dual Riesz bases (see [7]).

5. Regularity

Remark 3 at the end of the previous section showed that if m_0 can be factorized as in (4.12), with (4.13) replaced by the stronger condition $B_k \leq 2^{L-1-m}$, then $\phi \in C^m$. In [13] and [14] a more detailed study was made of general (not necessarily wavelet-related) functions satisfying a "two-scale difference equation" (i.e., an equation of the type $f(x) = \sum_n c_n f(2x - n)$). Again, special sum rules on the c_n, which are equivalent to the factorization (4.12), played an important role. For general solutions of two-scale difference equations, regularity is possible without these sum rules (see, e.g., [29]), although it has been proved (see [17]) that the cascade algorithm (with higher order splines replacing the piecewise constant η_n — see the end of Section 4.B or [10]) converges in C^m only if the associated trigonometric polynomial $\frac{1}{2} \sum_n c_n e^{in\xi}$ (generalizing m_0 to the non-wavelet case) can be factored as in (4.12), with $L \geqq m + 1$. In the case of orthonormal or biorthogonal wavelet bases, however, regularity of ψ, $\tilde{\psi}$ forces factorization of type (4.12) for m_0, $\tilde{m}_0$. Proofs of this fact for the orthonormal case can be found in [4] or in [27]. Both proofs work "in the Fourier domain" (i.e., they involve $\widehat{\psi}$, $\widehat{\phi}$ rather than ψ, ϕ directly). We present here an approach suitably generalized to accommodate the biorthogonal case, and our proof will not use the Fourier transform. It is similar to Battle's proof in that it does not even use multiresolution analysis or the fact that the ψ_{jk}, $\tilde{\psi}_{jk}$ constitute Riesz bases: biorthogonality is the only ingredient used.

PROPOSITION 5.1. *Suppose f, $\tilde{f}$ are two functions, not identically constant, such that*

$$\left\langle f_{jk}, \tilde{f}_{j'k'} \right\rangle = \delta_{jj'}\delta_{kk'}$$

with $f_{jk}(x) = 2^{-j/2} f(2^{-j}x - k)$, $\tilde{f}_{jk}(x) = 2^{-j/2} \tilde{f}(2^{-j}x - k)$. Suppose that $|\tilde{f}(x)| \leqq C(1 + |x|)^{-\alpha}$, with $\alpha < m + 1$, and suppose that $f \in C^m$, with $f^{(\ell)}$ bounded for $\ell \leqq m$. Then

$$\int dx\, x^\ell \tilde{f}(x) = 0 \quad \text{for} \quad \ell = 0, 1 \ldots m\,. \tag{5.1}$$

Proof:

1. The idea of the proof is very simple. Choose j, k, j', k' so that f_{jk} is rather spread out, and $\tilde{f}_{j'k'}$ very much concentrated. (For this expository point only, we assume that $\tilde{f}$ has compact support.) On the tiny support of $\tilde{f}_{j'k'}$ the slice of f_{jk} "seen" by $\tilde{f}_{j'k'}$ can be replaced by its Taylor series, with as many terms as are well-defined. Since, however, $\int dx\, \overline{f_{jk}(x)}\, \tilde{f}_{j'k'}(x) = 0$, this implies that the integral of the product of $\tilde{f}$ and a polynomial of order m is zero. We can then vary the locations of $\tilde{f}_{j'k'}$, as given by k'. For each location the argument can be repeated, leading to a whole family of different polynomials of order m which all give zero integral when multiplied with $\tilde{f}$. This leads to the desired moment condition. But let us be more precise.

2. We prove (5.1) by induction on ℓ. The following argument works for both the initial step and the inductive step. Assume $\int dx\, x^n \tilde{f}(x) = 0$ for $n \in \mathbb{N}$, $n < \ell$. (If $\ell = 0$, then this amounts to no assumption at all.) Since $f^{(\ell)}$ is continuous ($\ell \leqq m$), and since the dyadic rationals $2^{-j}k$ ($j, k \in \mathbb{Z}$) are dense in $\mathbb{R}$, there exist J, K so that $f^{(\ell)}(2^{-J}K) \neq 0$. (Otherwise $f^{(\ell)} \equiv 0$ would follow, implying $f \equiv$ constant if $\ell = 0$ or 1, which we know not to be the case, or, if $\ell \geqq 2$, $f \equiv$ polynomial of order $\ell - 1 \geqq 1$, which would imply that f is not bounded and is therefore also excluded.) Moreover, for any $\varepsilon > 0$ there exists $\delta > 0$ so that

$$\left| f(x) - \sum_{n=0}^{\ell} (n!)^{-1} f^{(n)}(2^{-J}K)\, (x - 2^{-J}K)^n \right| \leqq \varepsilon\, \left| x - 2^{-J}K \right|^\ell$$

if $|x - 2^{-J}K| \leqq \delta$. Take now $j > J$, $j > 0$. Then

$$
\begin{aligned}
0 &= \int dx\, f(x)\, \overline{\tilde{f}(2^j x - 2^{j-J}K)} \\
&= \sum_{n=0}^{\ell} (n!)^{-1} f^{(n)}(2^{-J}K) \int dx\, \left(x - 2^{-J}K\right)^n\, \overline{\tilde{f}(2^j x - 2^{j-J}K)} \\
&\quad + \int dx \left[f(x) - \sum_{n=0}^{\ell} (n!)^{-1} f^{(n)}(2^{-J}K) \left(x - 2^{-J}K\right)^n\right] \\
&\qquad \times \overline{\tilde{f}(2^j x - 2^{j-J}K)}\,.
\end{aligned} \tag{5.2}
$$

Since $\int dx\, x^n\, \tilde{f}(x) = 0$ for $n < \ell$, the first term is equal to

$$
(\ell!)^{-1} f^{(\ell)}\left(2^{-J}K\right) 2^{-(\ell+1)j} \int dx\, x^{\ell}\, \overline{f(x)}\,. \tag{5.3}
$$

Using the boundedness of the $f^{(n)}$, the second term can be bounded by

$$
\begin{aligned}
&\varepsilon \int_{|y|<\delta} dy\, |y|^{\ell} \left|\tilde{f}(2^j y)\right| + C' \int_{|y|>\delta} dy\, \left(1 + |y|^{\ell}\right) \left|\tilde{f}(2^j y)\right| \\
&\leqq 2\varepsilon C 2^{-j(\ell+1)} \int_0^{2^j\delta} dt\, t^{\ell} (1+t)^{-\alpha} + 2C'C \int_{\delta}^{\infty} dt\, (1+t)^{\ell} \left(1 + 2^j t\right)^{-\alpha} \\
&\leqq C_1\, \varepsilon\, 2^{-j(\ell+1)} + C_2\, 2^{-j\alpha}\, \delta^{-\alpha} (1+\delta)^{\ell+1}\,,
\end{aligned} \tag{5.4}
$$

where we replaced the upper integration bound by ∞ in the first term, and where we used in the second term that $(1 + 2^j t)^{-1} \leqq \frac{1+\delta}{1+2^j\delta} (1 + t)^{-1} \leqq 2^{-j} \frac{1+\delta}{\delta} (1 + t)^{-1}$ for $t \geqq \delta$. Note that C_1, C_2 only depend on C, α and ℓ; they are independent of ε, δ, and j. Combining (5.2), (5.3), and (5.4) leads to

$$
\left|\int dx\, x^{\ell} f(x)\right| \leqq (\ell!) \left[f^{(\ell)}\left(2^{-J}K\right)\right]^{-1} \left[\varepsilon\, C_1 + \delta^{-\alpha} (1+\delta)^{\ell+1}\, 2^{-j(\alpha-\ell-1)}\, C_2\right]\,.
$$

Here ε can be made arbitrarily small, and for the corresponding δ we can choose j sufficiently large to make the second term arbitrarily small as well. It follows that $\int dx\, x^{\ell} f(x) = 0$. ∎

When specialized to our compactly supported ψ, $\tilde{\psi}$ as given by Theorem 3.8, this leads immediately to

COROLLARY 5.2. *Let ψ, $\tilde{\psi}$ be as defined in Section 3. If $\langle \psi_{jk}, \tilde{\psi}_{j'k'} \rangle = \delta_{jj'}\delta_{kk'}$, in particular if the ψ_{jk}, $\tilde{\psi}_{j'k'}$ constitute dual Riesz bases, then*

$$\psi \in C^L \Rightarrow m_0(\xi) \quad \text{is divisible by} \quad (1 + e^{-i\xi})^{L+1}$$
$$\tilde{\psi} \in C^{\tilde{L}} \Rightarrow \tilde{m}_0(\xi) \quad \text{is divisible by} \quad (1 + e^{-i\xi})^{\tilde{L}+1} .$$

Proof:

1. Since ψ is compactly supported, $\psi \in C^L$ immediately implies that all the $\psi^{(\ell)}$, $\ell \leqq L$, are bounded. Since $\tilde{\psi}$ is also compactly supported, obviously $|\tilde{\psi}(x)| \leqq C(1 + |x|)^{-L-1-\varepsilon}$. Moreover $\psi \neq 0 \neq \tilde{\psi}$. All the conditions of Proposition 5.1 are thus satisfied, leading to

$$\int dx\, x^\ell\, \tilde{\psi}(x) = 0 \quad \ell = 0, 1, \ldots , L ,$$

or, equivalently,

$$\frac{d^\ell}{d\xi^\ell} \hat{\tilde{\psi}}\big|_{\xi=0} = 0 \qquad \ell = 0, 1, \ldots , L . \tag{5.5}$$

Since $\hat{\tilde{\psi}}(\xi) = e^{-i\xi/2}\, \overline{m_0(\frac{\xi}{2} + \pi)}\, \hat{\tilde{\phi}}(\xi/2)$, and $\hat{\tilde{\phi}}(0) = 1$, (5.5) implies

$$\frac{d^\ell}{d\zeta^\ell} m_0 \bigg|_{\zeta=\pi} = 0 \qquad \ell = 0, 1, \ldots , L .$$

But this is exactly the same as saying that the trigonometric polynomial $m_0(\xi)$ is divisible by $(1 + e^{-i\xi})^{L+1}$. ∎

Remarks.

1. If ψ, $\tilde{\psi}$ are merely continuous, then Corollary 5.2 does not lead to anything new, since m_0 and $\tilde{m}_0$ are always divisible by $(1+e^{i\xi})$, even if ψ, $\tilde{\psi}$ are not continuous. (See (3.11).)

2. If some minimum regularity is required for both ψ, $\tilde{\psi}$, then Corollary 5.2 implies a lower bound for their supportwidths. If

$$m_0(\xi) = \frac{1}{\sqrt{2}} \sum_{n=N_1}^{N_2} h_n\, e^{-n\xi} , \quad \tilde{m}_0(\xi) = \frac{1}{\sqrt{2}} \sum_{h=\tilde{N}_1}^{\tilde{N}_2} \tilde{h}_n\, e^{-in\xi} ,$$

with

$$h_{N_1} \neq 0 \neq h_{N_2} , \quad \tilde{h}_{\tilde{N}_1} \neq 0 \neq \tilde{h}_{\tilde{N}_2} ,$$

then

$$\text{support } \phi = [N_1, N_2] ,$$

$$\text{support } \tilde{\phi} = [\tilde{N}_1, \tilde{N}_2] ,$$

(see Lemma 3.1). Together with (3.9) this implies

$$\text{support } \psi = \left[\frac{N_1 - \tilde{N}_2 - 1}{2}, \frac{N_2 - \tilde{N}_1 - 1}{2}\right] ,$$

$$\text{support } \tilde{\psi} = \left[\frac{\tilde{N}_1 - N_2 - 1}{2}, \frac{\tilde{N}_2 - N_1 - 1}{2}\right] ,$$

hence $|\text{support } \psi| = |\text{support } \tilde{\psi}| = \frac{1}{2}(N_2 + \tilde{N}_2 - N_1 - \tilde{N}_1)$. On the other hand, $\psi \in C^k$, $\tilde{\psi} \in C^{\tilde{k}}$ implies that m_0(resp. $\tilde{m}_0$) is divisible by $(1+e^{-i\xi})^{k+1}$ (resp. $(1+e^{-i\xi})^{\tilde{k}+1}$). This implies, in particular, $N_2 - N_1 \geqq k+1$, $\tilde{N}_2 - \tilde{N}_1 \geqq \tilde{k}+1$. It follows that $\psi \in C^k$, $\tilde{\psi} \in C^{\tilde{k}}$ implies that $|\text{support } \psi| = |\text{support } \tilde{\psi}| \geqq \frac{k+\tilde{k}}{2} + 1$.

3. Regularity for ψ implies zero moments for $\tilde{\psi}$. No regularity for $\tilde{\psi}$ is required, however. In fact there exist examples (see Section 7) of biorthogonal wavelet bases in which one of the two wavelets, say ψ, is much more regular than the other, $\tilde{\psi}$. In the orthonormal case it is known that if $\psi \in C^r$ then it is possible to decide whether or not $f \in C^s$ $(0 \leqq s < r)$ by looking only at the wavelet coefficients $\langle f, \psi_{jk}\rangle$. (See [27]. More precisely: $f \in C^s(\mathbb{R})$ if and only if $|\langle f, \psi_{jk}\rangle| \leqq C\, 2^{-j(s+1/2)}$. Note that here s need not be integer: $f \in C^{n+\lambda}$, with $0 < \lambda < 1$ means that f is n times continuously differentiable, and that the n-th derivative of f is λ-Lipschitz, $|f^{(n)}(x) - f^{(n)}(y)| \leqq C|x-y|^\lambda$. For integer s, the characterization by means of wavelet coefficients is not really correct: a slightly larger class than the n-times continuously differentiable functions is obtained if $\lambda = 0$; see [27].) This is no longer true in the biorthogonal case: if $\psi \in C^r$, $\tilde{\psi} \in C^{\tilde{r}}$, with $r > \tilde{r}$, then the wavelet coefficients $\langle f, \psi_{jk}\rangle$ can certainly not be used to characterize C^s-spaces with $s \geq \tilde{r}$, even if $s < r$, since for the special choice $f = \tilde{\psi}$, the wavelet coefficients $\langle \tilde{\psi}, \psi_{jk}\rangle = \delta_{j0}\delta_{k0}$ would satisfy any candidate for such a criterium. Another way of seeing this is to consider the formula

$$f = \sum_{j,k} \langle f, \psi_{jk}\rangle\, \tilde{\psi}_{jk} .$$

It tells us that when we use the coefficients $\langle f, \psi_{jk}\rangle$, we are implicitly expanding f into the elementary building blocks $\tilde{\psi}_{jk}$. It is clear that any condition of a general nature on the $\langle f, \psi_{jk} \rangle$ cannot guarantee more regularity than the $\tilde{\psi}_{jk}$ themselves have. We can, however, also write

$$f = \sum_{j,k} \langle f, \tilde{\psi}_{jk} \rangle \psi_{jk} .$$

If ψ is much more regular than $\tilde{\psi}$, then this means that the wavelet coefficients $\langle f, \tilde{\psi}_{jk} \rangle$ with the less regular wavelets can be used to characterize $f \in C^s$ with $0 < s < r$, even if $s > \tilde{r}$: regular functions can be characterized by their inner products with much less regular wavelets.

6. Examples

Unlike the orthonormal case, it is possible, in the biorthogonal case, to choose m_0 so that it corresponds to a "linear phase" filter, or to a symmetric function ϕ. The filter associated with m_0 is linear phase if

$$m_0(\xi) = e^{i\lambda\xi} |m_0(\xi)| \tag{6.1}$$

for some $\lambda \in \mathbb{R}$. The 2π-periodicity of m_0 then forces $\lambda \in \mathbb{Z}$; by introducing a suitable integer translation in the indices of the h_n we can therefore assume $\lambda = 0$. For real h_n, (6.1) reduces to

$$m_0(-\xi) = m_0(\xi) , \tag{6.2}$$

or, equivalently, to

$$m_0(\xi) = \text{ polynomial in } \cos\xi . \tag{6.3}$$

It follows that $\hat{\phi}(\xi) = \hat{\phi}(-\xi)$, which implies $\phi(x) = \phi(-x)$ since ϕ is real. Note that this excludes the Haar case: the Haar scaling function ϕ is symmetric around $x = 1/2$ rather than around $x = 0$, i.e., $\phi_{\text{Haar}}(1-x) = \phi_{\text{Haar}}(x)$. Scaling functions ϕ with symmetry around $x = 1/2$ correspond to trigonometric polynomials m_0 satisfying

$$m_0(-\xi) = e^{i\xi} m_0(\xi) \tag{6.4}$$

rather than (6.2). It follows from (6.4) that $e^{i\xi/2} m_0(\xi)$ is invariant under the substitution $\xi \to -\xi$; since this function is also 4π-periodic, it is therefore a polynomial in $\cos\xi/2$. Since multiplication of this polynomial by $e^{-i\xi/2}$ reduces it to a trigonometric polynomial in ξ (rather than $\xi/2$), only odd powers of $\cos\xi/2$ are allowed. It follows that (6.4) is equivalent to

$$m_0(\xi) = e^{-i\xi/2} \cos\xi/2 \cdot \text{ polynomial in } \cos\xi . \tag{6.5}$$

In all our examples we shall concentrate on m_0 satisfying either (6.2) or (6.4).

Given m_0, we need to determine $\widetilde{m}_0$ so that (see Sections 2 and 3)

$$m_0(\xi)\,\overline{\widetilde{m}_0(\xi)} + m_0(\xi+\pi)\,\overline{\widetilde{m}_0(\xi+\pi)} = 1 \;. \tag{6.6}$$

The following proposition shows that we only need to concern ourselves with symmetric $\widetilde{m}_0$.

PROPOSITION 6.1. *Let m_0 be fixed. Suppose $\widetilde{m}_0$ is a solution to* (6.6). *If $m_0(-\xi) = m_0(\xi)$, then $\widetilde{m}_0^\sharp(\xi) = \frac{1}{2}\,[\widetilde{m}_0(\xi) + \widetilde{m}_0(-\xi)]$ is also a solution to* (6.6) *which moreover satisfies $\widetilde{m}_0^\sharp(-\xi) = \widetilde{m}_0^\sharp(\xi)$. If on the other hand $m_0(-\xi) = e^{i\xi}\,m_0(\xi)$, then $\widetilde{m}_0^\sharp(\xi) = \frac{1}{2}\,[\widetilde{m}_0(\xi) + e^{-i\xi}\,\widetilde{m}_0(-\xi)]$ is also a solution to* (6.6) *which moreover satisfies $\widetilde{m}_0^\sharp(-\xi) = e^{i\xi}\,\widetilde{m}_0^\sharp(\xi)$.*

Proof: Trivial (substitute $\widetilde{m}_0^\sharp$ into (6.6)). ∎

We shall therefore always assume that $\widetilde{m}_0$ has the same symmetry property (either (6.2) or (6.4)) as m_0.

On the other hand, our analysis in Sections 4 and 5 indicates that m_0, $\widetilde{m}_0$ should be divisible by $(1+e^{-i\xi})^L$, $(1+e^{-i\xi})^{\tilde{L}}$, respectively, with L, $\tilde{L}$ certainly $\geqq 1$, but even larger if we want ψ, $\widetilde{\psi}$ to be reasonably regular. It turns out that m_0 satisfying (6.2), respectively (6.4) can only be divisible by an even, respectively odd number of factors $\cos\xi/2$:

PROPOSITION 6.2. *Assume m_0 is a trigonometric polynomial with real coefficients. If m_0 satisfies* (6.2), *then it can be rewritten as*

$$m_0(\xi) = (\cos\xi/2)^{2\ell}\,p_0(\cos\xi) \tag{6.7}$$

where p_0 is a polynomial such that $p_0(-1) \neq 0$, and $\ell \in \mathbb{N}$.

If m_0 satisfies (6.4), *then it can be rewritten as*

$$m_0(\xi) = e^{-i\xi/2}\,(\cos\xi/2)^{2\ell+1}\,p_0(\cos\xi) \tag{6.8}$$

where p_0 is a polynomial such that $p_0(-1) \neq 0$, and $\ell \in \mathbb{N}$.

Proof:

1. If m_0 satisfies (6.2), then (see (6.3)) m_0 can be written as a polynomial in $\cos\xi$, $m_0(\xi) = p(\cos\xi)$. This polynomial can be written as

$$p(x) = (1+x)^\ell\,q(x)\;,$$

with $q(-1) \neq 0$, and $\ell \in \mathbb{N}$ (possibly $\ell = 0$ for general q; in our case $m_0(\pi) = 0$, or $p(-1) = 0$ hence $\ell \geqq 1$). Since $1+\cos\xi = 2\cos^2\xi/2$, (6.7) follows.

2. The same argument, applied to (6.5), leads to (6.8). ∎

Whether m_0 and $\tilde{m}_0$ are both of type (6.2) or of type (6.4), substituting their factorizations (6.7), (6.8) into (6.6) leads in both cases to the following equation:

$$(6.9)\quad \left(\cos\frac{\xi}{2}\right)^{2k} p_0(\cos\xi)\,\tilde{p}_0(\cos\xi) + \left(\sin\frac{\xi}{2}\right)^{2k} p_0(-\cos\xi)\,\tilde{p}_0(-\cos\xi) = 1\ ,$$

with $k = \ell + \tilde{\ell}$ in the first case, $k = \ell + \tilde{\ell} + 1$ in the second case. If we rewrite the product of p_0 and $\tilde{p}_0$ as a polynomial in $\frac{1-\cos\xi}{2} = \sin^2\xi/2$, then (6.9) reduces to

$$(6.10)\quad \begin{aligned} &(\cos\xi/2)^{2k}\,P(\sin^2\xi/2) + (\sin\xi/2)^{2k}\,P(\cos^2\xi/2) = 1 \\ &\text{or} \\ &(1-x)^k\,P(x) + x^k P(1-x) = 1\ , \end{aligned}$$

where $x = \sin^2\xi/2$. All our examples correspond therefore to: (1) a choice for ℓ, $\tilde{\ell}$ or equivalently for ℓ, k; (2) a choice of P solving (6.10) for that k; (3) a choice for the factorization of $P(\sin^2\xi/2)$ into $p_0(\cos\xi)\,\tilde{p}_0(\cos\xi)$. To solve (6.10) we use

THEOREM 6.3. *If p_1, p_2 are two polynomials of degree n_1, n_2 respectively, and if p_1, p_2 have no common zeros, then there exist unique polynomials q_1, q_2 of degree at most $n_2 - 1$, $n_1 - 1$ respectively, so that*

$$(6.11)\quad p_1(x)\,q_1(x) + p_2(x)\,q_2(x) = 1\ .$$

This theorem is known as Bezout's theorem. It can be proved by expanding the equation (6.11) into its Taylor series around every zero of p_1 or p_2. This leads to a constructive algorithm for q_1, q_2, for which we first have to find all the zeros of p_1, p_2. On the other hand (6.11) can also be viewed as a consequence of the fact that the polynomials constitute a Euclidean ring, and q_1, q_2 can then be constructed by means of Euclid's algorithm. This algorithm only requires the division of polynomials, i.e., solving linear systems of equations; as a result it is easier and more accurate than the zero-based algorithm if p_1, p_2 are large polynomials that cannot be factored straightforwardly. Moreover, the method immediately shows that if all the coefficients of p_1, p_2 are rational, then the same is true for q_1, q_2. In the examples presented later in this section, we use either Euclid's algorithm or the factorization algorithm, depending on which is easier.

We now apply Theorem 6.3 to the choice $p_1(u) = (1-u)^k$, $p_2(u) = u^k$. Since $p_1(1-u) = p_2(u)$ in this case, substitution of $1-u$ for u in (6.11) shows that $\tilde{q}_1(u) = q_2(1-u)$ and $\tilde{q}_2(u) = q_1(1-u)$ are polynomials satisfying the same equation and degree restriction. By the uniqueness of q_1, q_2 it follows that $q_2(u) = q_1(1-u)$, so that the choice $P(u) \equiv q_1(u)$ is indeed a solution of (6.11). In this case the equation is so simple that once the existence and uniqueness of a polynomial P of degree at most $k-1$, solving (6.10), is established, we do not even need Euclid's algorithm to determine P. Rewrite (6.10) as

$$P(u) = (1-u)^{-k} - u^k\,(1-u)^{-k}\,P(1-u)\,, \tag{6.12}$$

and expand the right-hand side in its Taylor series. Since we know that P is of degree $k-1$, we only need to compute the first k terms. Only the first part of the right-hand side of (6.12) contributes to this, leading to

$$P(u) = \sum_{n=0}^{k-1} \binom{k+n-1}{n} u^n\,. \tag{6.13}$$

This is therefore the unique lowest degree solution to (6.10). There also exist solutions of a higher degree. By the Taylor expansion argument above, their first k terms coincide with (6.13), so that a general solution can be written as

$$P(u) = \sum_{n=0}^{k-1} \binom{k+n-1}{n} u^n + u^k\,r(u)\,.$$

Substitution into (6.10) leads then to

$$r(u) + r(1-u) = 0$$

or $r(u) = R(\frac{1}{2} - u)$ where R is an odd polynomial.

Remark. Equation (6.10) was already encountered in [10], where it was solved by means of two combinatorial lemmas. The method used here is much simpler.

Putting everything together, we see that we have proved the following proposition:

PROPOSITION 6.4. *Let m_0 be a trigonometric polynomial (with real coefficients) satisfying either* (6.2) *or* (6.4), *i.e., m_0 can be written as either*

$$m_0(\xi) = (\cos\xi/2)^{2\ell}\,p_0(\cos\xi) \tag{a}$$

or

(b) $$m_0(\xi) = e^{-i\xi/2}\,(\cos\xi/2)^{2\ell+1}\,p_0(\cos\xi)\ ,$$

with $p_0(-1) \neq 0$, $\ell \in \mathbb{N}$.

If there exist any solutions $\tilde{m}_0$ *for* (6.6) *at all, then there exist solutions* $\tilde{m}_0$ *which have the same form as* m_0, *i.e.,*

$$\tilde{m}_0(\xi) = (\cos\xi/2)^{2\tilde{\ell}}\,\tilde{p}_0(\cos\xi) \quad \textit{in case (a)}$$

or

$$\tilde{m}_0(\xi) = e^{-i\xi/2}(\cos\xi/2)^{2\tilde{\ell}+1}\,\tilde{p}_0(\cos\xi) \quad \textit{in case (b)}\ ,$$

with $\tilde{p}_0(-1) \neq 0$, $\tilde{\ell} \in \mathbb{N}$.

Moreover, p_0 *and* $\tilde{p}_0$ *are constrained by*

(6.14)
$$p_0(\cos\xi)\,\tilde{p}_0(\cos\xi) = \sum_{n=0}^{k-1} \binom{k-1+n}{n} \left(\frac{\sin^2\xi}{2}\right)^n + \left(\frac{\sin^2\xi}{2}\right)^k R(\cos\xi)$$

where $k = \ell + \tilde{\ell}$ *in case* (a), $k = \ell + \tilde{\ell} + 1$ *in case* (b), *and where* R *is an odd polynomial.*

Note that since $m_0(\pi) = 0 = \tilde{m}_0(\pi)$, we shall need $\ell, \tilde{\ell} \geqq 1$ in case (a). Let us now look at some specific examples.

6.A. The Spline Case

In this case ${}_N\phi$ is a B-spline function of order N, translated so that its nodes are at the integers, regardless of whether N is even or odd. The first few cases are

piecewise constant:	${}_1\phi(x) =$	1	$0 \leqq x < 1$
		0	otherwise
piecewise linear:	${}_2\phi(x) =$	$1 + x$	$-1 \leqq x \leqq 0$
		$1 - x$	$0 \leqq x \leqq 1$
		0	otherwise
piecewise quadratic:	${}_3\phi(x) =$	$(x+1)^2/2$	$-1 \leqq x \leqq 0$
		$-(x-1/2)^2 + 3/4$	$0 \leqq x \leqq 1$
		$(x-2)^2/2$	$1 \leqq x \leqq 2$
		0	otherwise.

The Fourier transform of ${}_N\phi$ is given by

$$ {}_N\widehat{\phi}(\xi) = e^{-i\kappa\xi/2} \left(\frac{\sin\xi/2}{\xi/2}\right)^N = e^{i\xi\lfloor N/2\rfloor} \left(\frac{1-e^{-i\xi}}{i\xi}\right)^N , $$

where $\kappa = 0$ if N is even, $\kappa = 1$ if N is odd. An alternative characterization of ${}_N\phi$ is given by

$$ \frac{d^N}{dx^N}\, {}_N\phi(x) = \sum_{n=0}^{N} \binom{N}{n} (-1)^n \,\delta(x - n + \lfloor N/2\rfloor) $$

together with the restriction that $\int dx\, {}_N\phi(x) = 1$. (As usual, $\lfloor y\rfloor$ denotes the largest integer not exceeding y.) One easily checks that

$$ {}_{2L}\phi(-x) = {}_{2L}\phi(x) , \quad {}_{2L+1}\phi(1-x) = {}_{2L+1}\phi(x) . $$

The corresponding ${}_N m_0$ are given by a binomial formula:

$$ \begin{aligned} {}_N m_0(\xi) = \left(\frac{1+e^{-i\xi}}{2}\right) N e^{i\xi\lfloor N/2\rfloor} &= e^{-i\kappa\xi/2} \left(\frac{\cos\xi}{2}\right)^N \\ &= \sum_{n=-\lfloor N/2\rfloor}^{N-\lfloor N/2\rfloor} 2^{-N} \binom{N}{n+\lfloor N/2\rfloor} e^{-in\xi} ; \end{aligned} $$

one finds

$$ {}_{2L}m_0(-\xi) = {}_{2L}m_0(\xi) , \quad {}_{2L+1}m_0(-\xi) = e^{i\xi}\, {}_{2L+1}m_0(\xi) . $$

In terms of the parameters in Proposition 6.5, this choice for m_0 amounts to $\ell = L$, and $p_0 \equiv 1$. It follows that for these m_0, the possible solutions $\widetilde{m}_0$ to (6.6), with the same symmetry as ${}_N m_0$, are given by

$$ {}_{N,\tilde N}\widetilde{m}_0(\xi) = $$
$$ e^{-i\kappa\xi/2} (\cos\xi/2)^{\tilde N} \left[\sum_{n=0}^{k-1} \binom{k-1+n}{n} (\sin\xi/2)^{2n} + (\sin\xi/2)^{2k} R(\cos\xi)\right] $$

where $\widetilde{N} \geqq 1$, $N + \tilde N = 2k$ is even, R is an odd polynomial, and $\kappa = 1$ if N is odd, 0 if N is even, as above. We shall restrict ourselves here to the choice $R \equiv 0$. In this case, the "spline pairs" m_0, $\widetilde{m}_0$ constructed here have the remarkable property that all their coefficients are dyadic fractions (i.e., rational numbers whose denominator is a power of 2), whatever choice is made for N or $\tilde N$. (See also Table 6.1 below.) This makes these filters particularly easy to implement on a computer.

So far, the filters m_0, $\tilde{m}_0$ satisfy (6.6) as well as $m_0(0) = 1 = \tilde{m}_0(0)$ and $m_0(\pi) = 0 = \tilde{m}_0(\pi)$ (this is the reason why $\tilde{N}$ is restricted to $\tilde{N} \geqq 1$). This is, however, not sufficient to ensure that the corresponding ψ, $\tilde{\psi}$ define biorthogonal Riesz bases: we also need to ascertain that (3.13) and (3.26) are satisfied. The decay condition on $\hat{\phi}$ is no problem: even for $N = 1$, $|{}_N\hat{\phi}(\xi)| \leqq C|\xi|^{-1}$. What about decay of $\hat{\tilde{\phi}}$? We can use a result from [10] to see what happens for large $\tilde{N}, N$. It was proved in [10], pages 981–983, that, for $R \equiv 0$, and $N + \tilde{N}$ large

(6.15)

$$\log_2 \left[\sup_\xi \left|\tilde{\mathcal{F}}_{N,\tilde{N}}(\xi)\,\mathcal{F}_{N,\tilde{N}}(2\xi)\,\tilde{\mathcal{F}}_{N,\tilde{N}}(4\xi)\,\tilde{\mathcal{F}}_{N,\tilde{N}}(8\xi)\right|^{1/4}\right] \sim .8064\,(N + \tilde{N})\,.$$

Consequently $\hat{\tilde{\phi}}$ has sufficient decay at ∞ if $.8064\,(N + \tilde{N}) < \tilde{N} - 1/2$, or

$$\tilde{N} > (.8064\,N + .5)/.1936 \simeq 4.1653\,N + 2.5826\,. \tag{6.16}$$

If we require some regularity for $\tilde{\phi}$ as well, e.g., $\tilde{\phi} \in C^m$, then we impose $.8064(N + \tilde{N}) < \tilde{N} - 1 - m$. It follows that the spline examples provide us with an infinite family of pairs of biorthogonal bases with arbitrarily high regularity:

$${}_N\phi \in C^{N-2} \quad {}_{N,\tilde{N}}\tilde{\phi} \in C^m \quad \text{if} \quad \tilde{N} > 4.1653\,N + 5.1653(m+1) \tag{6.17}$$

and

$$\left|\text{support } {}_{N,\tilde{N}}\psi\right| = \left|\text{support } {}_{N,\tilde{N}}\tilde{\psi}\right| = N + \tilde{N} - 1\,. \tag{6.18}$$

Note that if we require the same regularity for ${}_{N,\tilde{N}}\tilde{\phi}$ as for ${}_N\phi$, then we need

$$\tilde{N} > 1.8064\,N/.1936 \simeq 9.3306\,N \tag{6.19}$$

resulting in $|\text{support } {}_{N,\tilde{N}}\psi| = |\text{support } {}_{N,\tilde{N}}\tilde{\psi}| > 18.6612\,N - 1$. The estimates (6.15)–(6.19) are all asymptotic for large N. For small values of N, the left-hand side of (6.15) can be estimated explicitly. Even sharper estimates for the regularity of ${}_{N,\tilde{N}}\tilde{\phi}$ can be obtained via the techniques of [14], when feasible.

In Table 6.1 we have listed the coefficients of ${}_Nm_0$ and ${}_{N,\tilde{N}}\tilde{m}_0$ for the first few values of $N, \tilde{N}$. Graphs of the corresponding ${}_N\phi$, ${}_{N,\tilde{N}}\tilde{\phi}$, ${}_{N,\tilde{N}}\psi$, and ${}_{N,\tilde{N}}\tilde{\psi}$ are given in Figures 6.1 to 6.3. For $N = 3$, $\tilde{N} = 1$, the distribution ${}_{N,\tilde{N}}\tilde{\phi}$ is not square integrable. (We prove this by checking the eigenvector for the

Table 6.1. We list ${}_N m_0$, ${}_{N,\tilde N}\tilde m_0$ for the first few values of $N, \tilde N$, with $z = e^{-i\xi}$. The corresponding filter coefficients ${}_N h_k$, ${}_{N,\tilde N}\tilde h_k$ are obtained by multiplying $\sqrt{2}$ with the coefficient of z^k in ${}_N m_0$, ${}_{N,\tilde N}\tilde m_0$, respectively. We also give an estimate for α such that $|{}_{N,\tilde N}\hat{\tilde\phi}(\xi)| \leqq C(1 + |\xi|)^{-\alpha}$ when we can prove $\alpha > .5$. Note that the coefficients of ${}_{N,\tilde N}\tilde m_0$ are always symmetric; for very long ${}_{N,\tilde N}\tilde m_0$ we only list about half the coefficients (the others can be deduced by symmetry).

N	${}_N m_0$	$\tilde N$	${}_{N,\tilde N}\tilde m_0$	Decay of ${}_{N,\tilde N}\tilde\phi$
1	$\frac{1}{2}(1+z)$	1	$\frac{1}{2}(1+z)$	Haar basis $\alpha = 1$
		3	$-\frac{z^{-2}}{16} + \frac{z^{-1}}{16} + \frac{1}{2} + \frac{z}{2} + \frac{z^2}{16} - \frac{z^3}{16}$	$\alpha > 1.6584$
		5	$\frac{3}{256}z^{-4} - \frac{3}{256}z^{-3} - \frac{11}{128}z^{-2} + \frac{11}{128}z^{-1} + \frac{1}{2} + \frac{z}{2} + \frac{11}{128}z^2 - \frac{11}{128}z^3 - \frac{3}{256}z^4 + \frac{3}{256}z^5$	$\alpha > 2.2777$
2	$\frac{1}{4}(z^{-1} + 2 + z)$	2	$-\frac{1}{8}z^{-2} + \frac{1}{4}z^{-1} + \frac{3}{4} + \frac{1}{4}z - \frac{1}{8}z^2$	$\alpha > 0.6584$
		4	$\frac{3}{128}z^{-4} - \frac{3}{64}z^{-3} - \frac{1}{8}z^{-2} + \frac{19}{64}z^{-1} + \frac{45}{64}z^{-1} + \frac{45}{64} + \frac{19}{64}z - \frac{1}{8}z^2 - \frac{3}{64}z^3 + \frac{3}{128}z^4$	$\alpha > 1.2777$
		6	$-\frac{5}{1024}z^{-6} + \frac{5}{512}z^{-5} + \frac{17}{512}z^{-4} - \frac{39}{512}z^{-3} - \frac{123}{1024}z^{-2} + \frac{81}{256}z^{-1} + \frac{175}{256} + \frac{81}{256}z - \frac{123}{1024}z^2 \cdots$	$\alpha > 1.7542$

Table 6.1 (continued).

N	${}_N m_0$	$\tilde{N}$	${}_{N,\tilde{N}}\tilde{m}_0$	Decay of ${}_{N,\tilde{N}}\tilde{\phi}$
		8	$2^{-15}(35z^{-8} - 70z^{-7} - 300z^{-6} + 670z^{-5} + 1228z^{-4} - 3126z^{-3} - 3796z^{-2} + 10718z^{-1} + 22050 + 10718z - 3796z^2 \cdots)$	$\alpha > 2.2550$
3	$\frac{1}{8}(z^{-1} + 3 + 3z + z^2)$	1	$-\frac{1}{4}z^{-1} + \frac{3}{4} + \frac{3}{4}z - \frac{1}{4}z^2$	Not in $L^2(\mathbb{R})$
		3	$\frac{3}{64}z^{-3} - \frac{9}{64}z^{-2} - \frac{7}{64}z^{-1} + \frac{45}{64} + \frac{45}{64}z - \frac{7}{64}z^2 - \frac{9}{64}z^3 + \frac{3}{64}z^4$	See footnote*
		5	$-\frac{5}{512}z^{-5} + \frac{15}{512}z^{-4} + \frac{19}{512}z^{-3} - \frac{97}{512}z^{-2} - \frac{13}{256}z^{-1} + \frac{175}{256} + \frac{175}{256}z - \frac{13}{256}z^2 \cdots$	$\alpha > .7542$
		7	$2^{-14}(35z^{-7} - 105z^{-6} - 195z^{-5} + 865z^{-4} + 336z^{-3} - 3489z^{-2} - 307z^{-1} + 11025 + 11025z \cdots)$	$\alpha > 1.2550$
		9	$2^{-17}(-63z^{-9} + 189z^{-8} + 469z^{-7} - 1911z^{-6} - 1308z^{-5} + 9188z^{-4} + 1140z^{-3} - 29676z^{-2} + 190z^{-1} + 87318 + 87318z \cdots)$	$\alpha > 1.7384$

* ${}_{3,3}\tilde{\phi}$ does not satisfy (3.13), but ${}_{3,3}\psi$ and ${}_{3,3}\tilde{\psi}$ nevertheless generate Riesz bases; one can prove $\int d\xi(1 + |\xi|)^{\lambda}|\hat{\tilde{\phi}}(\xi)|^2 < \infty$ for $\lambda < .35026$.

(nondegenerate) eigenvalue 1 of the operator ${}_{N,\tilde{N}}\tilde{P}_0$ corresponding to ${}_{N,\tilde{N}}\tilde{m}_0$, as defined by (4.1). If $\tilde{\phi} \in L^2(\mathbb{R})$, then the entries of this eigenvector are the coefficients of a non-negative trigonometric polynomial; see Section 4. For $N = 3$, $\tilde{N} = 1$, one finds that this trigonometric polynomial takes strictly negative values.) For $N = 3$, $\tilde{N} = 3$, we can prove that ${}_{N,\tilde{N}}\tilde{\phi} \in L^2(\mathbb{R})$; in fact the techniques of the Appendix in [10] can be used to show that $\int d\xi\,(1 + |\xi|)^\lambda\,|{}_{3,3}\hat{\tilde{\phi}}(\xi)|^2 < \infty$ for $\lambda < .35026$. Nevertheless, ${}_{3,3}\tilde{\phi}$ does not satisfy (3.13); using $\log_2(\sum_{n=0}^{2}\binom{n+2}{2}(3/4)^n) = 2.727920\ldots$, one finds that $|{}_{3,3}\hat{\tilde{\phi}}(2^k \frac{2\pi}{3})| \geqq C\,2^{k(-3+2.727920\ldots)} \geqq C\,[2^k \frac{2\pi}{3}]^{.272079\ldots}$. With the techniques developed in [7], however, one can prove that ${}_{3,3}\psi$ and ${}_{3,3}\tilde{\psi}$ nevertheless generate dual Riesz bases. Anyway, it can be readily seen from Figure 6.3 that ${}_{3,3}\tilde{\phi}$ is much more photogenic than regular. For all the other pairs listed in Table 6.1, ${}_{N,\tilde{N}}\tilde{\phi}$ satisfies (3.13); we list an estimate for its decay rate. Note that ${}_{N,\tilde{N}}\tilde{\phi}$ is symmetric around 0 for N even, around 1/2 for N odd. The symmetry axis for both ${}_{N,\tilde{N}}\psi$ and ${}_{N,\tilde{N}}\tilde{\psi}$ lies at $x = 1/2$ in all cases; for N even, they are symmetric, for N odd, antisymmetric.

A striking feature in Figures 6.2 and 6.3 is that from some point on, increasing $\tilde{N}$ (for fixed N) does not alter the shape of ${}_{N,\tilde{N}}\psi$ very much; one sees the "wrinkles" in the corresponding ${}_{N,\tilde{N}}\tilde{\phi}$ and ${}_{N,\tilde{N}}\tilde{\psi}$ get ironed out as $\tilde{N}$ increases.

The functions ${}_{1,3}\psi$ and ${}_{1,3}\tilde{\psi}$ were first constructed by Ph. Tchamitchian (see [33]) as an example of two dual wavelet bases with very different regularity properties. In our present construction they constitute the first non-orthonormal example. ($N = 1 = \tilde{N}$ gives the Haar basis.)

The different regularity properties of ${}_{N,\tilde{N}}\psi$ and ${}_{N,\tilde{N}}\tilde{\psi}$ may be useful in some applications. If all that is wanted is a fast algorithm of decomposing f into reasonably smooth wavelet building blocks, then decomposition by means of ${}_{N,\tilde{N}}\tilde{m}_0$ and reconstruction via ${}_N m_0$ may be a perfectly good answer, even if ${}_{N,\tilde{N}}\tilde{\phi}$ is not very regular. In fact, experiments with images have shown that such a scheme leads to a much better compression rate than a scheme that would use the same filters, but reverse the role of decomposition and reconstruction filters; see [5]. In image analysis, one prefers to use filters of comparable length, which is not the case here: the filter pairs of the spline examples typically have very dissimilar lengths. For small values of $N, \tilde{N}$ this is apparent from Table 6.1. For large N, the asymptotic estimate shows that $\sharp\{\text{coefficients in } {}_{N,\tilde{N}}\tilde{m}_0\} = N + 2\tilde{N} - 1 \gtrsim 9.3306\,N$. This is without any regularity for ${}_{N,\tilde{N}}\tilde{\phi}$; if we require that ${}_{N,\tilde{N}}\tilde{\phi}$ and ${}_N\phi$ are of comparable regularity, then we find $\sharp\{\text{coefficients in } {}_{N,\tilde{N}}\tilde{m}_0\} \gtrsim 18.6612\,N$, which is much larger than $\sharp\{\text{coefficients in } {}_N m_0\} = N + 1$. The next subsection gives a

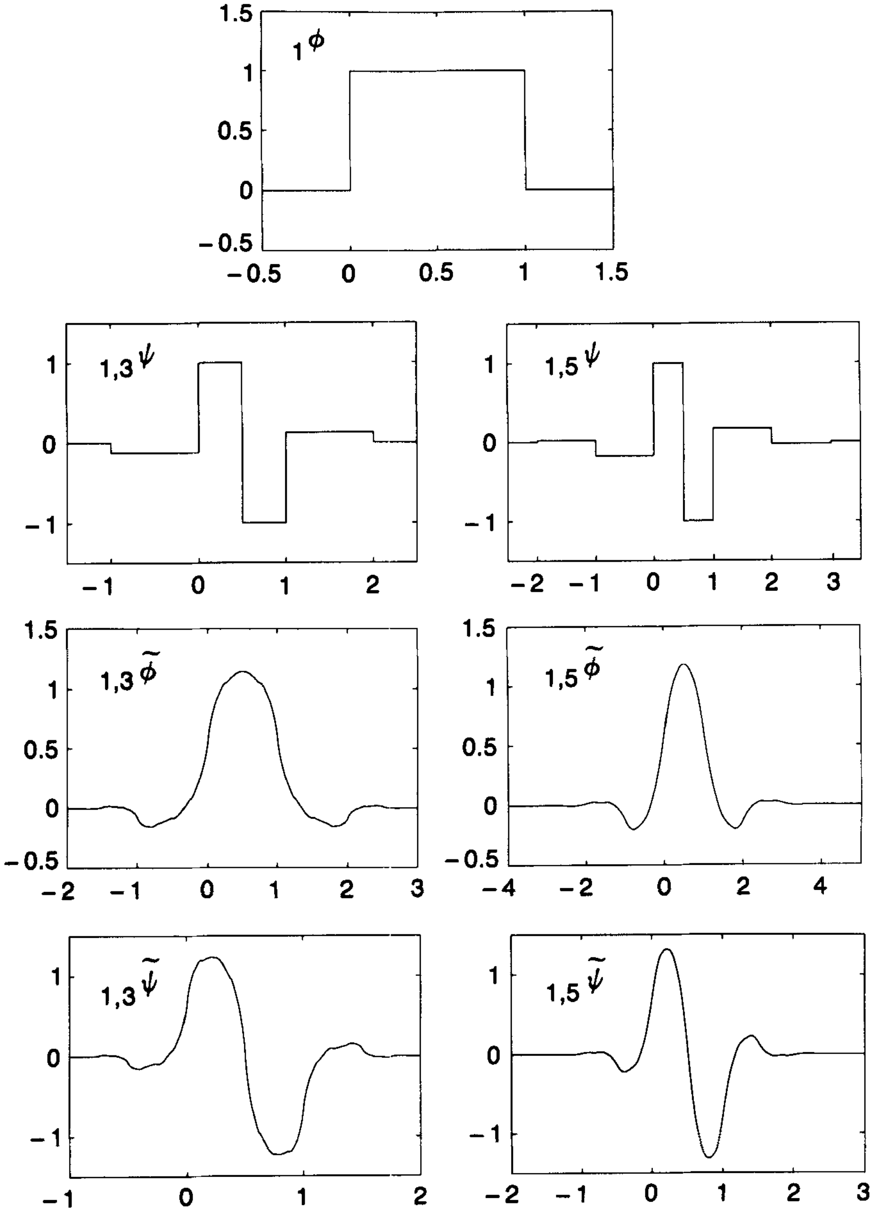

Figure 6.1. The functions $_N\phi$, $_{N,\tilde N}\psi$, $_{N,\tilde N}\tilde\phi$ and $_{N,\tilde N}\tilde\psi$ for $N = 1$, $\tilde N = 3, 5$. For $\tilde N = 1$ (not plotted) one finds the Haar basis, i.e., $_{1,1}\tilde\phi \equiv {}_1\phi$, $_{1,1}\tilde\psi \equiv {}_{1,1}\psi$, and $_{1,1}\psi(x) = 1$ for $0 \leqq x < 1/2$, -1 for $1/2 \leqq x < 1$, 0 otherwise. We have support $_{1,\tilde N}\tilde\phi = [-\tilde N + 1, \tilde N]$, support $_{1,\tilde N}\psi$ = support $_{1,\tilde N}\tilde\psi = \left[-\frac{\tilde N-1}{2}, \frac{\tilde N+1}{2}\right]$.

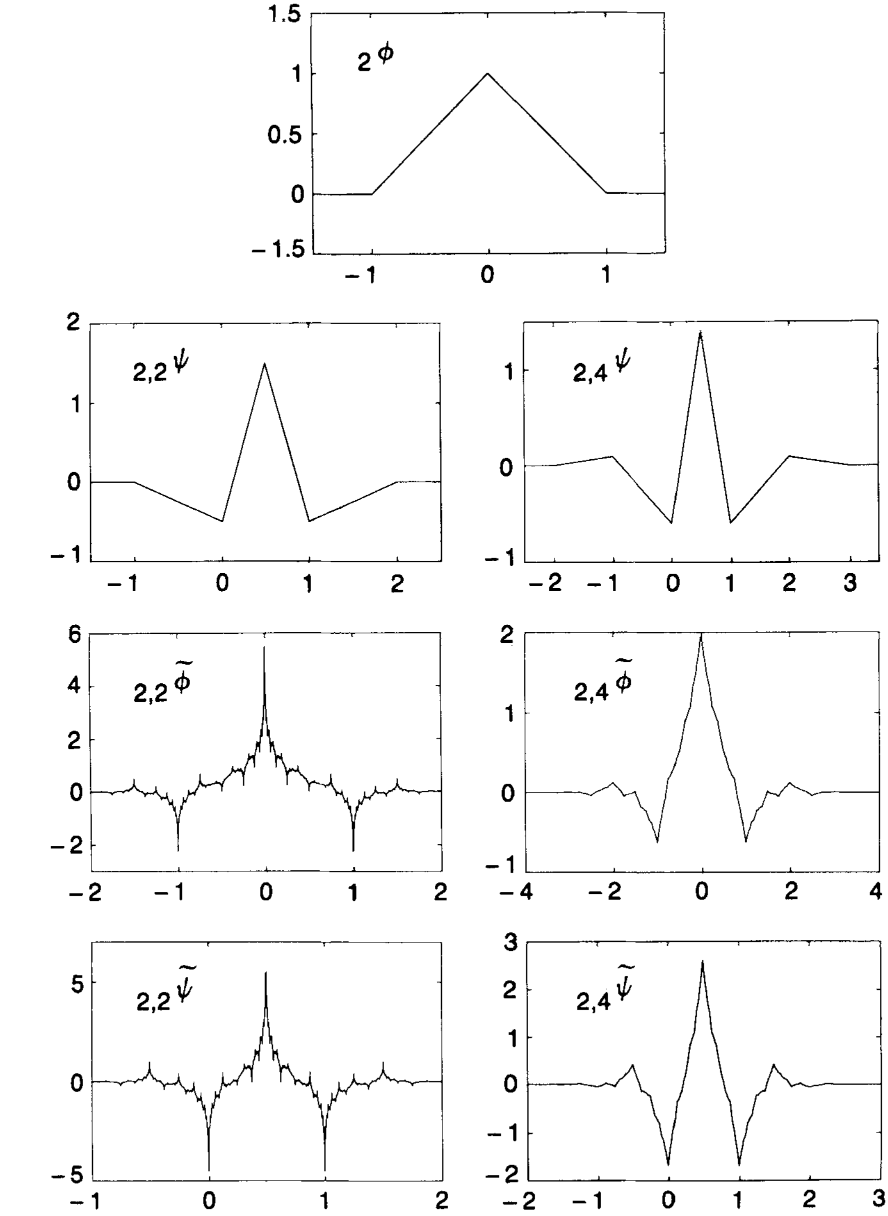

Figure 6.2. The functions $_N\phi$, $_{N,\tilde N}\psi$, $-{}_{N,\tilde N}\tilde\phi$ and $_{N,\tilde N}\tilde\psi$ for $N = 2$, $\tilde N = 2, 4, 6$, and 8. Notice how little $_{2,8}\psi$ differs from $_{2,6}\psi$. Support $_{2,\tilde N}\tilde\phi = [-\tilde N, \tilde N]$, support $_{2,\tilde N}\psi$ = support $_{2,\tilde N}\tilde\psi = [-\tilde N/2, \tilde N/2 + 1]$.

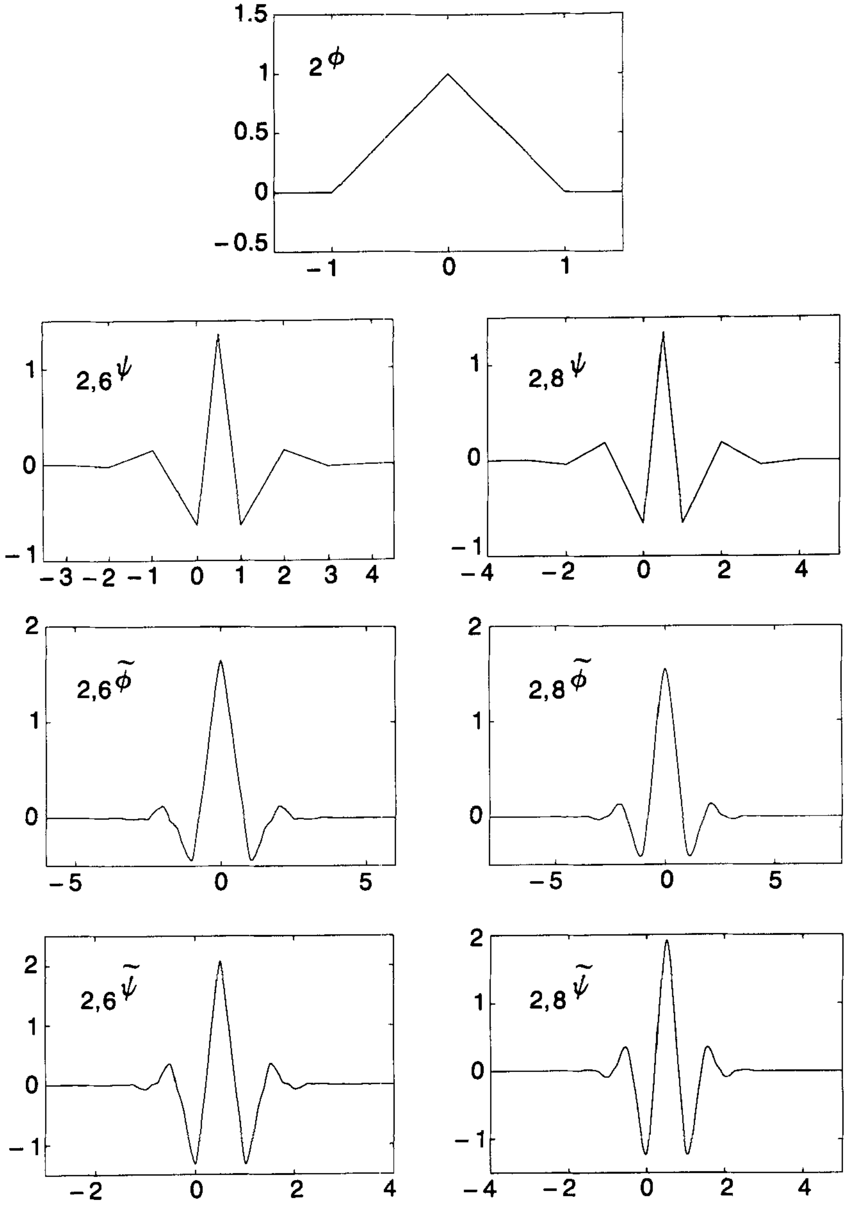

Figure 6.2 (continued).

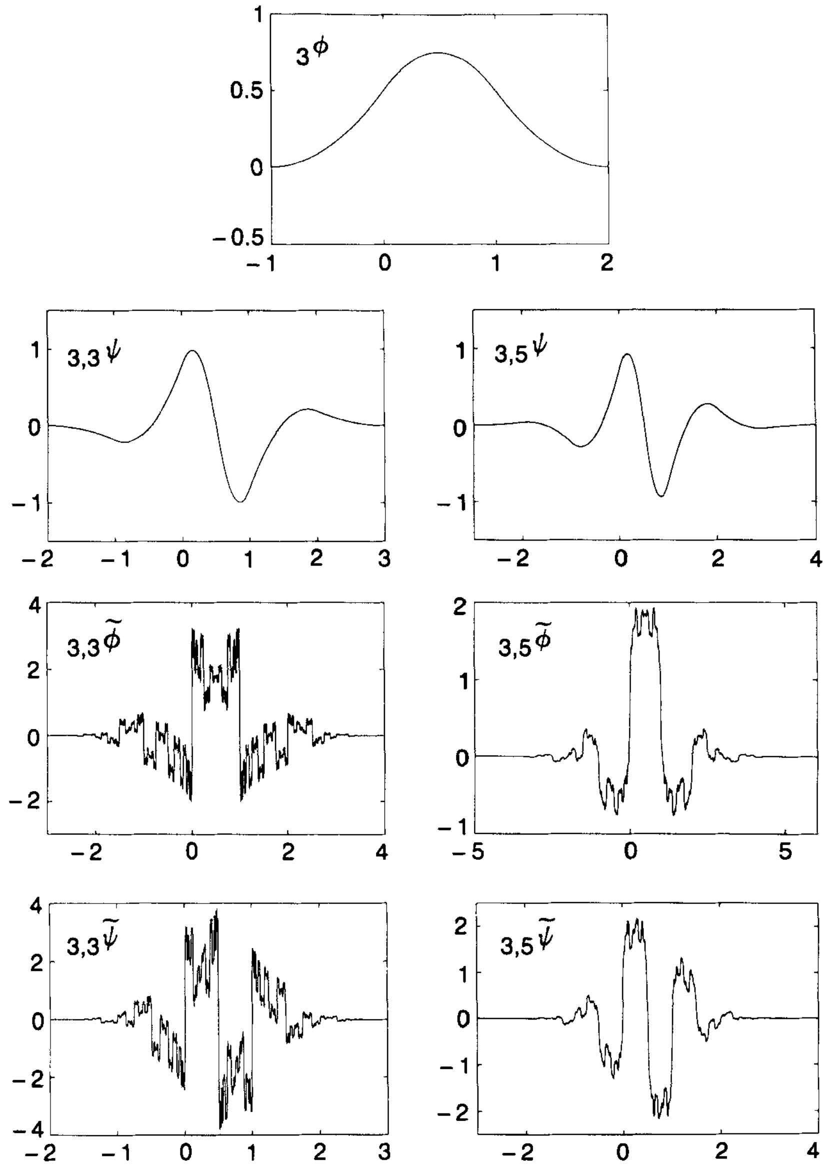

Figure 6.3. The functions ${}_N\phi$, ${}_{N,\tilde N}\psi$, ${}_{N,\tilde N}\tilde\phi$ and ${}_{N,\tilde N}\tilde\psi$ for $N = 3$, $\tilde N = 3, 5, 7$, and 9. For $\tilde N = 1$, ${}_{3,1}\tilde\phi$ (not plotted) is not square integrable. For $\tilde N = 3$, ${}_{3,3}\hat{\tilde\phi} \in L^2(\mathbb{R})$, but $\sup_\xi (1 + |\xi|)^{1/2}\,|{}_{3,3}\hat{\tilde\phi}(\xi)| = \infty$. Support ${}_{3,\tilde N}\tilde\phi = [-\tilde N, \tilde N + 1]$, support ${}_{3,\tilde N}\psi$ = support ${}_{3,\tilde N}\tilde\psi = \left[-\frac{\tilde N+1}{2}, \frac{\tilde N+3}{2}\right]$.

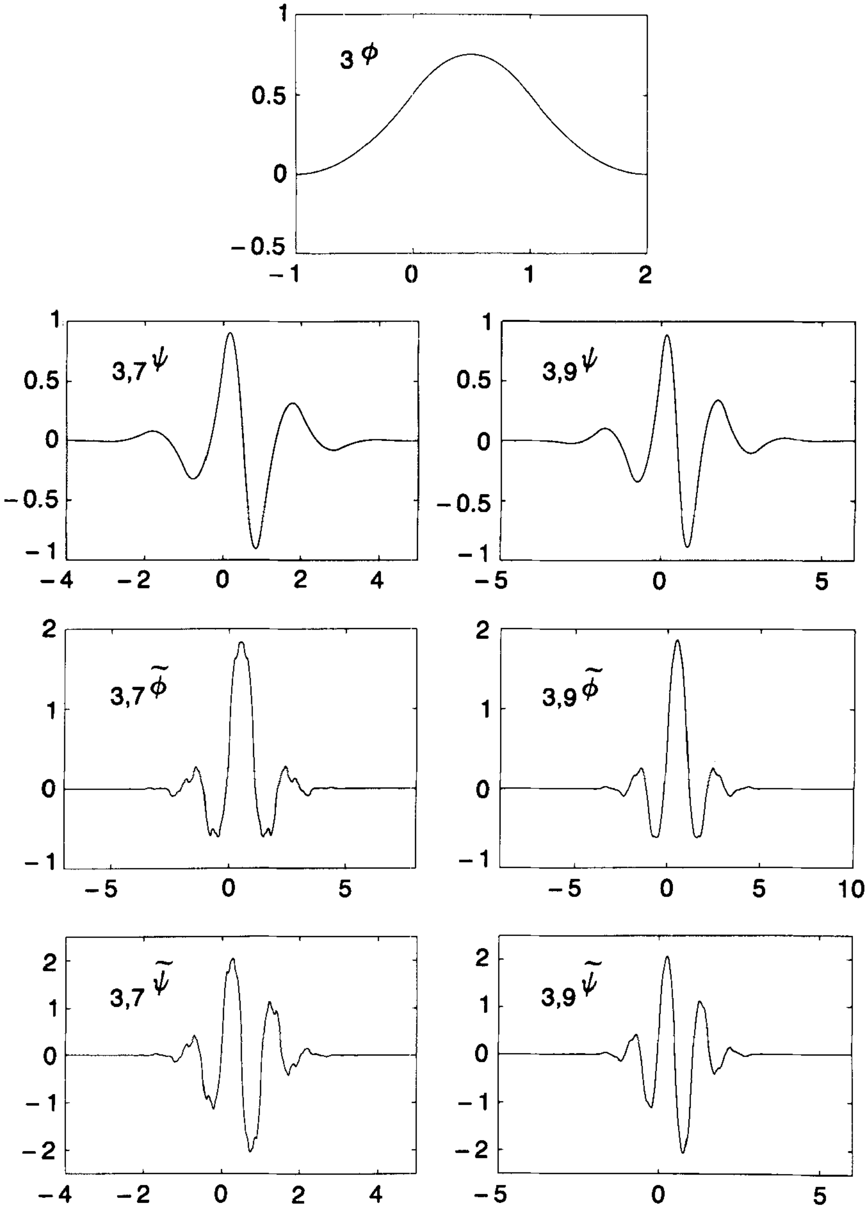

Figure 6.3 (continued).

variation on these spline examples which results in filters of less dissimilar lengths.

6.B. A Variation on the Spline Case: Filters with less Dissimilar Lengths

In the examples in this subsection we still choose $R \equiv 0$ in (6.14), as in the spline examples, but we shall determine the factors p_0 and $\tilde{p}_0$ so that the lengths of m_0 and $\widetilde{m}_0$ are very close, unlike the spline examples. For a fixed k, there is a limited number of possible factorizations. To find them, we determine the zeros (real and pairs of conjugated complex zeros) of $\sum_{n=0}^{k-1} \binom{k-1+n}{n} x^n$, so that we can write this polynomial as a product of real first and second order polynomials,

$$\sum_{n=0}^{k-1} \binom{k-1+n}{n} x^n = A \prod_{j=1}^{j_1} (x - x_j) \prod_{j'=1}^{j_2} (x^2 - 2 \text{ Re } z_{j'} x + |z_{j'}|^2) . \tag{6.20}$$

Regrouping of these factors leads to all the possibilities for p_0 and $\tilde{p}_0$. Table 6.2 gives the coefficients of m_0, $\widetilde{m}_0$ for three examples of this kind, for $k = 4$ and 5.

Note that $k = 4$ is the smallest value for which a nontrivial factorization of type (6.20) is possible, with real polynomials p_1 and p_2. For $k = 4$, the factorization of (6.20) is unique, for $k = 5$ there are two possibilities. In both cases we have then chosen N so as to obtain m_0 and $\widetilde{m}_0$ as filters with an odd number of taps (leading to symmetric 4, $\tilde{4}$) and a difference in length as small as possible. The corresponding functions ϕ, ψ, $\tilde{\phi}$ and $\tilde{\psi}$ are given in Figures 6.4 and 6.5. In all cases we can prove an estimate of type (4.16), proving that we do indeed have biorthogonal wavelet bases.

Remark. While it is important for many applications that the filter lengths of m_0 and $\widetilde{m}_0$ should be comparable, we feel that the examples in Table 6.2 have lost a very attractive feature present in Table 6.1: the entries are no longer dyadic fractions; they are not even rational.

6.C. Biorthogonal Bases Close to an Orthonormal Basis

6.C.1. Biorthogonal Bases Associated to Burt's Laplacian Pyramid

This first example was suggested by M. Barlaud, whose research group in vision analysis tried out the filters in Sections 6.A and 6.B for image coding; see [1]. Because of the popularity of the Laplacian pyramid scheme (see [6]), Barlaud wondered whether dual systems of wavelets could be constructed, using the Laplacian pyramid filter as either m_0 or $\widetilde{m}_0$. These filters are given explicitly by

$$-a\, e^{-2i\xi} + .25e^{-i\xi} + (.5 + 2a) + .25e^{i\xi} - a\, e^{2i\xi} . \tag{6.21}$$

Table 6.2. The coefficients of m_0, $\tilde{m}_0$ for three cases of "variations on the binomial case" with filters of similar length; corresponding to $k = 4$ and 5 (see text). For each filter we have also given the number of $(\cos\xi/2)$ factors (denoted $N, \tilde{N}$ — these determine the number of zero moments of $\psi, \tilde{\psi}$ — see Section 5). As in Table 6.1, multiplying the entries below with $\sqrt{2}$ gives the filter coefficients $h_n, \tilde{h}_n$.

$k, N, \tilde{N}$	n	Coefficient of $e^{-in\xi}$ in m_0	Coefficient of $e^{-in\xi}$ in $\tilde{m}_0$
$k=4$	0	.557543526229	.602949018236
$N=4$	1, −1	.295635881557	.266864118443
$\tilde{N}=4$	2, −2	−.028771763114	−.078223266529
	3, −3	−.045635881557	−.016864118443
	4, −4	0	.026748757411
$k=5$	0	.636046869922	.520897409718
$N=5$	1, −1	.337150822538	.244379838485
$\tilde{N}=5$	2, −2	−.066117805605	−.038511714155
	3, −3	−.096666153049	.005620161515
	4, −4	−.001905629356	.028063009296
	5, −5	.009515330511	0
$k=5$	0	.382638624101	.938348578330
$N=5$	1, −1	.242786343133	.333745161515
$\tilde{N}=5$	2, −2	.043244142922	−.257235611210
	3, −3	.000197904543	−.083745161515
	4, −4	.015436545027	.038061322045
	5, −5	.007015752324	0

For $a = -1/16$, this reduces to the spline filter ${}_4m_0$ as described in Section 6.A. For applications in vision, the choice $a = .05$ is especially popular: even though the corresponding filter has less regularity than ${}_4m_0$, it seems to lead to results that are better from the point of view of visual perception. At Barlaud's suggestion, we chose therefore $a = .05$ in (6.21), or

$$\begin{aligned} m_0(\xi) &= .6 + .5\cos\xi - .1\cos 2\xi \\ &= \left(\frac{\cos\xi}{2}\right)^2 \left(1 + \frac{4}{5}\sin^2\frac{\xi}{2}\right) . \end{aligned} \tag{6.22}$$

Candidates for $\tilde{m}_0$ dual to this m_0 have to satisfy

$$m_0(\xi)\,\overline{\tilde{m}_0(\xi)} + m_0(\xi+\pi)\,\overline{\tilde{m}_0(\xi+\pi)} = 1 .$$

By Proposition 6.1, we know that such $\tilde{m}_0$ can be chosen to be symmetric (since m_0 is symmetric); we also opt for $\tilde{m}_0$ divisible by $(\cos\xi/2)^2$ (so that the corresponding ψ, $\tilde{\psi}$ both have two zero moments). In other words,

$$\tilde{m}_0(\xi) = \left(\cos\frac{\xi}{2}\right)^2 P\left(\sin^2\frac{\xi}{2}\right)$$

where

$$(1-x)^2\left(1+\frac{4}{5}x\right)P(x)+x^2\left(\frac{9}{5}-\frac{4}{5}x\right)P(1-x)=1\,.$$

By Theorem 6.3, together with the symmetry of this equation for substitution of x by $1-x$, this equation has a unique solution P of degree 2, which is easily found to be

$$P(x)=1+\frac{6}{5}x-\frac{24}{35}x^2\,.$$

This leads to

$$\tilde{m}_0(\xi)=\left(\frac{\cos\xi}{2}\right)^2\left(1+\frac{6}{5}\sin^2\frac{\xi}{2}-\frac{24}{35}\sin^4\frac{\xi}{2}\right) \tag{6.23}$$

$$\begin{aligned}&=-\frac{3}{280}e^{-3i\xi}-\frac{3}{56}e^{-2i\xi}+\frac{73}{280}e^{-i\xi}+\frac{17}{28}+\frac{73}{280}e^{i\xi}\\&\quad-\frac{3}{56}e^{2i\xi}-\frac{3}{280}e^{3i\xi}\,.\end{aligned} \tag{6.24}$$

One can check that both (6.22) and (6.23) satisfy estimates of type (4.13). It follows that these m_0 and $\tilde{m}_0$ do indeed correspond to a pair of biorthogonal wavelet bases. Figure 6.6 shows graphs of the corresponding ϕ, $\tilde{\phi}$, ψ and $\tilde{\psi}$. All four functions are continuous but not differentiable. It is very striking how similar $\tilde{\phi}$ and ϕ are, or ψ and $\tilde{\psi}$. This can be traced back to a similarity of m_0 and $\tilde{m}_0$, which is not immediately obvious from (6.22) and (6.23), but becomes apparent by comparison of the explicit numerical values of the filter coefficients, as in Table 6.3. In fact, both filters are very close to the (necessarily nonsymmetric) filter corresponding to one of the orthonormal wavelet bases constructed in [12], Section 4, listed in the third column in Table 6.3. This proximity of m_0 to an orthonormal wavelet filter explains why the $\tilde{m}_0$ dual to m_0 is so close to m_0 itself. A first application to image analysis of these biorthogonal bases associated to the Laplacian pyramid is given in [1].

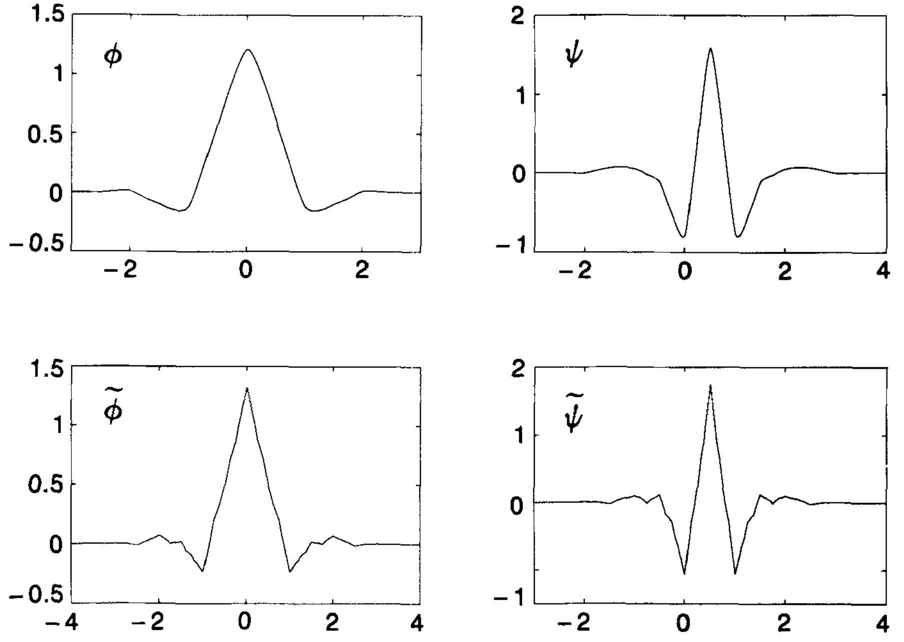

Figure 6.4. The functions ϕ, $\tilde{\phi}$, ψ, $\tilde{\psi}$ for the less asymmetric variant on the binomial examples, for $k = 4$ (see text). In this case support $\phi = [-3, 3]$, support $\tilde{\phi} = [-4, 4]$, support ψ = support $\tilde{\psi} = [-3, 4]$; ϕ and $\tilde{\phi}$ are symmetric around $x = 0$, ψ and $\tilde{\psi}$ around $x = 1/2$.

6.C.2. More Examples

M. Barlaud's suggestion led to the accidental discovery that the Burt filter is very close to an orthonormal wavelet filter. (One wonders whether this closeness makes the filter so effective in applications?) This example suggests that maybe other biorthogonal bases, with symmetric filters and rational filter coefficients, can be constructed by approximating and "symmetrizing" existing orthonormal wavelet filters, and computing the corresponding dual filter. The coiflet coefficients listed in Table 3 in [12] were obtained via a construction method that naturally led to close to symmetric filters (the aim was to obtain orthonormal bases for which both ψ and ϕ would have a prescribed number of zero moments); it is natural therefore to expect that symmetric biorthogonal filters close to an orthonormal basis will in fact be close to these coiflet bases. The analysis in [12], Section 4, suggests then

$$m_0(\xi) = (\cos\xi/2)^{2K} \left[\sum_{k=0}^{K-1} \binom{K-1+k}{k} (\sin\xi/2)^{2k} + O((\sin\xi/2)^{2K}) \right] .$$

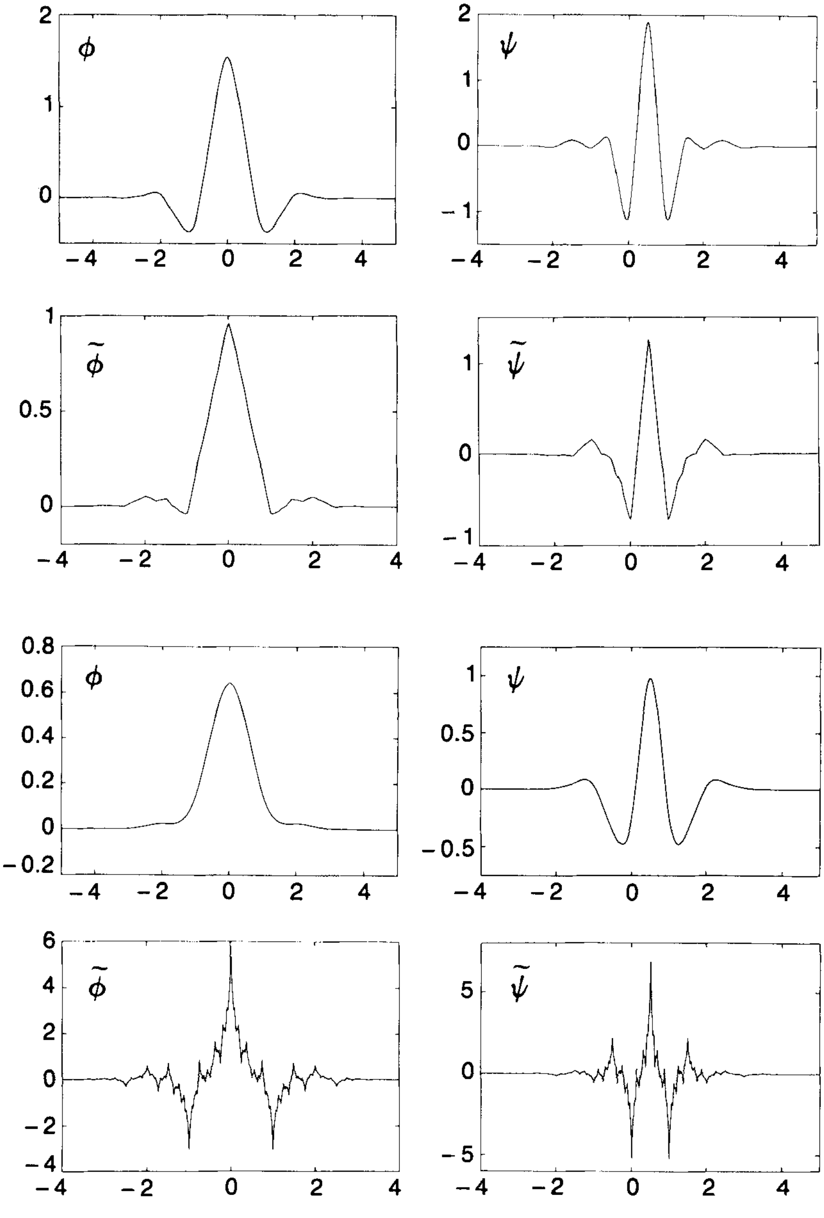

Figure 6.5. The functions ϕ, $\tilde{\phi}$, ψ, $\tilde{\psi}$ for the less asymmetric variant on the binomial examples for $k = 5$. In both cases support $\phi = [-5, 5]$, support $\tilde{\phi} = [-4, 4]$, and support ψ = support $\tilde{\psi} = [-4, 5]$.

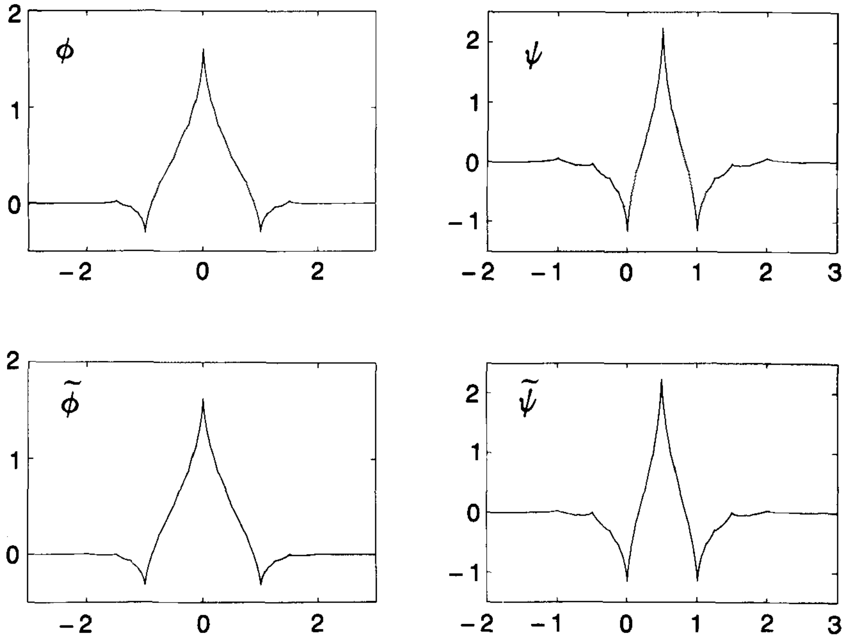

Figure 6.6. The functions ϕ, $\tilde{\phi}$, ψ, $\tilde{\psi}$ if m_0 is the Laplacian pyramid filter. In order to bring out the similarity between ϕ, $\tilde{\phi}$ better, we have chosen the same scale for their plots, even though support $\phi = [-2, 2]$, support $\tilde{\phi} = [-3, 3]$.

In the examples below we have chosen in particular

$$m_0(\xi) = (\cos\xi/2)^{2K} \left[\sum_{k=0}^{K-1} \binom{K-1+k}{k} (\sin\xi/2)^{2k} + a(\sin\xi/2)^{2K}\right]$$

and we have then followed the following procedure:

1. Find a such that $|\int_{-\pi}^{\pi} d\xi\, [1 - |m_0(\xi)|^2 - |m_0(\xi+\pi)|^2]|$ is minimal (zero in the examples we looked at). This optimization criterion can of course be replaced by other criteria (e.g., least sum of squares of *all* the Fourier coefficients of $1-|m_0(\xi)|^2-|m_0(\xi+\pi)|^2$ instead of only the coefficient of $e^{i\ell\xi}$ with $\ell = 0$). We checked the cases $K = 1, 2, 3$, where the smallest root for a was .861001748086, 3.328450120793, 13.113494845221 respectively.
2. Replace this (irrational) "optimal" value for a by a close value expressible as a simple fraction. For our examples we chose $a = .8 = 4/5$ for $K = 1$, $a = 3.2 = 16/5$ for $K = 2$, and $a = 13$ for $K = 3$. For $K = 1$, this reduces then to the example in Section 6.C.1.

Table 6.3. Filter coefficients for $(m_0)_{\text{Burt}}$, for the dual filter $(\widetilde{m}_0)_{\text{Burt}}$ computed in this section, and for a very close filter $(m_0)_{\text{coiflet}}$ corresponding to a special orthonormal basis of wavelets ("coiflets") constructed in [12].

n	$(m_0)_{\text{Burt}}$	$(\widetilde{m}_0)_{\text{Burt}}$	$(m_0)_{\text{coiflet}}$
−3	0.	.010714285714	0
−2	−.05	−.053571428571	−.051429728471
−1	.25	.260714285714	.238929728471
0	.6	.607142857143	.602859456942
1	.25	.260714285714	.272140543058
2	−.05	−.053571428571	−.051429972847
3	0.	−.010714285714	−.011070271529

3. Since m_0 is now fixed, we can compute $\widetilde{m}_0$. We require that $\widetilde{m}_0$ be also divisible by $(\cos\xi/2)^{2K}$, so that

$$\widetilde{m}_0(\xi) = (\cos\xi/2)^{2K}\, P_K((\sin\xi/2)^2)\ , \tag{6.25}$$

where P_K is a polynomial of degree $3K-1$. The same analysis as in [12], Section 4, shows that

$$P_K(x) = \sum_{k=0}^{K-1} \binom{K-1+k}{k} x^k + O(x^K)\ ,$$

thereby determining already K of the $3K$ coefficients of P_K. The others can be computed easily. For $K=1$, P_1 was already computed in Section 6.C.1; for $K=2$ and 3 we find

$$P_2(x) = 1 + 2x + \frac{14}{5}x^2 + 8x^3 - \frac{8024}{455}x^4 + \frac{3776}{455}x^5 \tag{6.26}$$

Table 6.4. Numerical values for the filters m_0, $\widetilde{m}_0$ for the cases $K = 2$ and 3 (see text). The third column lists the coefficients of an orthonormal wavelet filter to which m_0 and $\widetilde{m}_0$ are very close. In order to compare the different coefficients more easily, we have expressed everything in decimal notation; in fact the coefficients of m_0 and $\widetilde{m}_0$ are rational (see (6.26) and (6.27)).

K	n	Coefficients of m_0	Coefficients of $\widetilde{m}_0$	Coefficients of $(m_0)_{\text{coiflet}}$ $n \leqq 0$	$n \geqq 0$
2	0	.575	.575291895604	.574682393857	
	±1	.28125	.286392513736	.273021046535	.294867193696
	±2	−.05	−.052305116758	−.047639590310	−.054085607092
	±3	−.03125	−.039723557692	−.029320137980	−.042026480461
	±4	.0125	.015925480769	.011587596739	.016744410163
	±5	0	.003837568681	0	.003967883613
	±6	0	−.001266311813	0	−.001289203356
	±7	0	−.000506524725	0	−.000509505399
3	0	.5634765625	.560116167736	.561285256870	
	±1	.29296875	.296144908701	.286503335274	.302983571773
	±2	−.047607421875	−.047005100329	−.043220763560	−.050770140755
	±3	−.048828125	−.055220135661	−.046507764479	−.058196250762
	±4	.01904296875	.021983637555	.016583560479	.024434094321
	±5	.005859375	.010536373594	.005503126709	.011229240962
	±6	−.003173828125	−.005725661541	−.002682418671	−.006369601011
	±7	0	−.001774953991	0	−.001820458916
	±8	0	.000736056355	0	−.000790205101
	±9	0	.000339274308	0	−.000329665174
	±10	0	−.000047015908	0	−.000050192775
	±11	0	−.000025466950	0	−.000024465734

$$P_3(x) = 1 + 3x + 6x^2 + 7x^3 + 30x^4 + 42x^5 - \frac{1721516}{6075}x^6 + \frac{1921766}{6075}x^7 - \frac{648908}{6075}x^8 . \tag{6.27}$$

In Table 6.4 we list the explicit numerical values of the filter coefficients for m_0, $\widetilde{m}_0$ and the closest coiflet, for $K = 2$ and 3. We have graphed ϕ, $\widetilde{\phi}$, ψ and $\widetilde{\psi}$ for both cases in Figure 6.7. In both cases, as in Section 6.C.1, the biorthogonal wavelet filters are very close to a nonsymmetric orthonormal filter (coefficients taken from Table 3 in [12]). It is worthwhile to note that

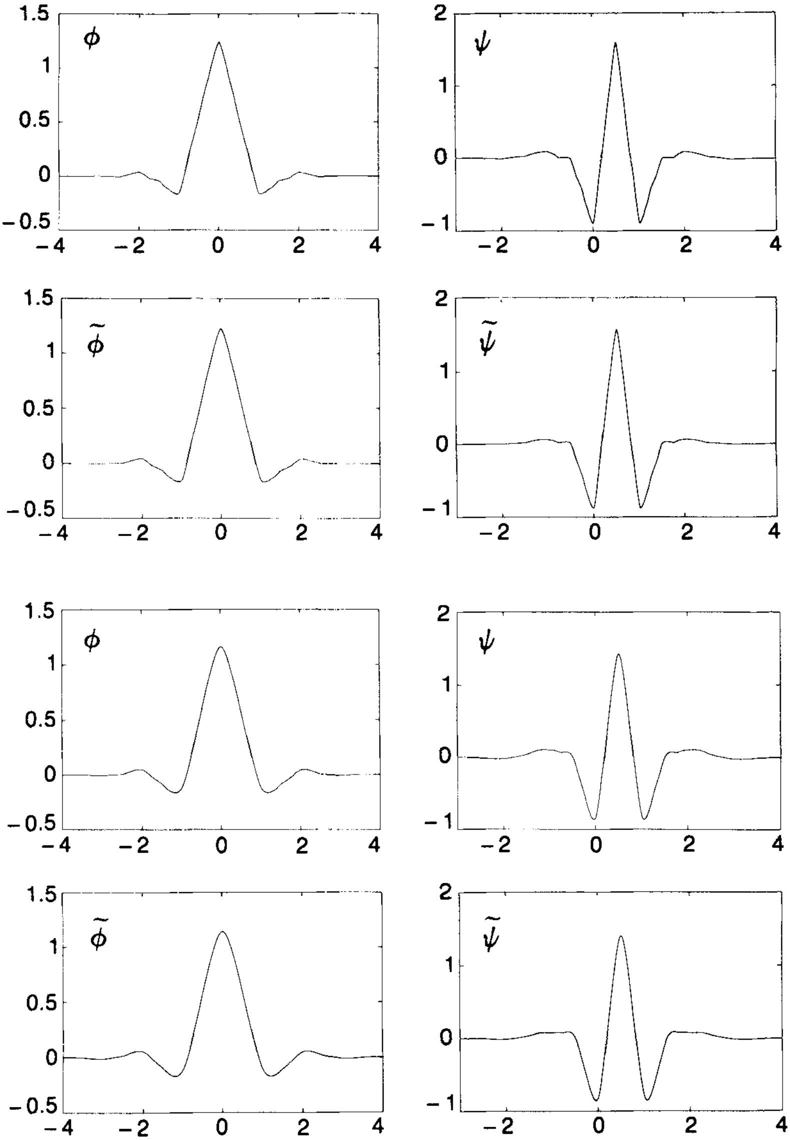

Figure 6.7. The functions ϕ, $\tilde{\phi}$, ψ, $\tilde{\psi}$ corresponding to the filters with rational coefficients close to orthonormal wavelet filters, as constructed in Section 6.C.2, for $K = 2$ and 3. We have only shown the parts of the graphs where the functions are significantly different from zero; in fact support $\phi = [-4, 4]$, support $\tilde{\phi} = [-7, 7]$, and support ψ = support $\tilde{\psi} = [-5, 6]$ for $K = 2$. For $K = 3$ these supports are $[-6, 6]$, $[-11, 11]$, $[-8, 9]$, respectively.

the computation of the biorthogonal filters m_0, $\tilde{m}_0$, as explained by the above procedure, is much simpler than the computation in [12] of the orthonormal coiflet filters! This illustrates the greater flexibility of the construction of biorthogonal wavelet bases versus orthonormal wavelet bases.

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