Frames in the Bargmann Space of Entire Functions

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Abstract

We look at the decomposition of arbitrary f in $L^2(\mathbf{R})$ in terms of the family of functions $\varphi_{mn}(x) = \pi^{-1/4}\exp\{-\frac{1}{2}imnab + i\,max - \frac{1}{2}(x-nb)^2\}$, with $a, b > 0$. We derive bounds and explicit formulas for the minimal expansion coefficients in the case where $ab = 2\pi/N$, N an integer $\geqq 2$. Transported to the Hilbert space F of entire functions introduced by V. Bargmann, these results are expressed as inequalities of the form

$$\mathbf{m}\|f\|^2 \leqq \sum_{m,n\in\mathbf{Z}} |f(z_{mn})|^2 \exp\{-\tfrac{1}{2}|z_{mn}|^2\} \leqq \mathbf{M}\|f\|^2,$$

where $z_{mn} = ma + inb$, $\mathbf{m}, \mathbf{M} > 0$, and $\|\ \|$ is the norm in F,

$$\|f\|^2 = (2\pi)^{-1} \iint_{\mathbf{R}^2} dx\,dy\,|f(x+iy)|^2 \exp\{-\tfrac{1}{2}(x^2+y^2)\}.$$

We conjecture that these inequalities remain true for all a, b such that $ab < 2\pi$.

1. Introduction

In this paper we present some bounds on entire functions in the Bargmann space F. F is the complex vector space of entire analytic functions of one complex variable $z = x + iy$ such that

$$\|f\|^2 = \frac{1}{2\pi} \iint_{\mathbf{R}^2} dx\,dy\,|f(x+iy)|^2 \exp\{-(x^2+y^2)/2\} < \infty.$$

F is a Hilbert space with inner product

$$\langle f, g\rangle = \frac{1}{2\pi} \iint_{\mathbf{R}^2} dx\,dy\,\overline{f(x+iy)}\,g(x+iy) \exp\{-(x^2+y^2)/2\}.$$

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(We use the physicist's convention, where the inner product is anti-linear in the first and linear in the second argument.)

The space F has been discovered, re-discovered and studied by many authors. Its roots can be found in the search for a setting in which multiplication by z and differentiation with respect to z are each other's adjoint (see Fischer [9] and Fock [10]). The space itself made its full fledged appearance in Bargmann [3], [4], Segal [18] and Newman and Shapiro [15], [16]. The Hilbert space F has turned out to be useful in many different contexts. In mathematics, F has been used as a tool in the study of, for instance, the Fourier transform in [4] and convolution operators in [15], [16]. In physics, the space F arises naturally whenever the canonical coherent states (we shall give their definition below) are used in quantum mechanics (see e.g. Bargmann [3], or Chapter 7 in Klauder and Sudarshan [14]. For a recent review of the many applications in physics of canonical and other coherent states, see Klauder and Skagerstam [14]).

The space F has a reproducing kernel (see Aronszajn [1]) or, equivalently, a family of elements e_z that define evaluation functionals at a point. For any $z \in \mathbb{C}$, the function e_z is defined as

$$e_z(z') = \exp\{-\tfrac{1}{4}|z|^2 + \tfrac{1}{2}\bar{z} \cdot z'\}.$$

Then $e_z \in F$, $\|e_z\| = 1$, and, for every $f \in F$,

$$\langle e_z, f\rangle = \exp\{-\tfrac{1}{4}|z|^2\} f(z).$$

The vectors of the family $\{e_z;\, z \in \mathbb{C}\}$ are not mutually orthogonal; they satisfy

$$\begin{aligned} &\langle e_z, e_{z'}\rangle = \exp\{-\tfrac{1}{4}|z|^2 - \tfrac{1}{4}|z'|^2 + \tfrac{1}{2}z \cdot \bar{z}'\}, \\ &|\langle e_z, e_{z'}\rangle| = \exp\{-\tfrac{1}{4}|z - z'|^2\}. \end{aligned} \tag{1}$$

A unitary map from $L^2(\mathbb{R})$ onto F is given by the Bargmann transform U_B. It is defined by: for $\varphi \in L^2(\mathbb{R})$,

$$(U_B\varphi)(z) = \pi^{-1/4}\exp\{\tfrac{1}{4}z^2\}\int_{\mathbb{R}} dx \exp\{-ixz\}\exp\{-\tfrac{1}{2}x^2\}\varphi(x).$$

Its inverse is given by: for $f \in F$,

$$(U_B^{-1}f)(x) = \pi^{-1/4}\exp\{-\tfrac{1}{2}x^2\}\int d\mu(z) \exp\{ix\bar{z}\}\exp\{\tfrac{1}{4}\bar{z}^2\}f(z),$$

where

$$d\mu(z) = \frac{1}{2\pi}\, d(\mathcal{R}e\, z)\, d(\mathcal{I}m\, z)\exp\{-\tfrac{1}{2}|z|^2\}.$$

The image of e_z under U_B^{-1} is the function (with $z = p + iq$; $p, q \in \mathbb{R}$)

$$\varphi^{p,q}(x) = \left[U_B^{-1} e_{p+iq}\right](x) \tag{2}$$

$$= \pi^{-1/4}\exp\{-\tfrac{1}{2}ipq\}\exp\{ipx\}\exp\{-\tfrac{1}{2}(x-q)^2\};$$

$\varphi^{p,q}$ is in $L^2(\mathbb{R})$, $\|\varphi^{p,q)}\| = 1$. The absolute value of $\varphi^{p,q}(x)$ is a Gaussian, peaked at $x = q$; the absolute value of its Fourier transform is also a Gaussian, peaked at p. The function (2) is called by physicists a (canonical) coherent state, and by engineers a Gabor wavelet (see [11]).

The motivation for the present paper is the study of expansions of an arbitrary $f \in L^2(\mathbb{R})$ into discrete families of coherent states

$$\varphi_{mn} = \varphi^{ma,nb}, \qquad m, n \in \mathbb{Z},$$

$$\varphi_{mn}(x) = \pi^{-1/4}\exp\{-\tfrac{1}{2}imnab\}\exp\{imax\}\exp\{-\tfrac{1}{2}(x-nb)^2\}.$$

Here a, b are strictly positive; we shall discuss the families $(\varphi_{mn})_{m,n\in\mathbb{Z}}$ and expansions with respect to these families for various possible choices of a, b. Notice that the family $(\varphi_{mn})_{m,n\in\mathbb{Z}}$ corresponds to a doubly periodic lattice of points in the complex plane, generated by the numbers a and ib:

$$U_B^{-1}\varphi_{mn} = e_{z_{mn}},$$

$$z_{mn} = ma + inb.$$

It is well known (see Bargmann, Butero, Girardello and Klauder [5], Perelomov [17] and Bacry, Grossmann and Zak [2]) that the family $(\varphi_{mn})_{m,n\in\mathbb{Z}}$ is not complete in $L^2(\mathbb{R})$ if $ab > 2\pi$; only if $ab \leqq 2\pi$ can the φ_{mn} give rise to an expansion formula for arbitrary f in $L^2(\mathbb{R})$. It is important at this point to remark that, for $ab \leqq 2\pi$, the vectors φ_{mn} are not "ω-independent", in the sense that one vector of the family lies in the closed linear span of the other vectors. If $ab = 2\pi$, then removing one φ_{mn} turns the remaining family into an ω-independent set. If $ab < 2\pi$, then the family $(\varphi_{mn})_{m,n\in\mathbb{Z}}$ remains ω-dependent even after removal of any finite number of $\varphi_{mn} - s$.

We are interested in expansions of the form

$$f = \sum_{m,n\in\mathbb{Z}} c_{mn}\varphi_{mn}, \tag{3}$$

for arbitrary $f \in L^2(\mathbb{R})$. Expansions of this type were first suggested by Gabor [11], with a view of applications in electric engineering. For a discussion of Gabor expansions, see Janssen [12].

The family of vectors $(\varphi_{mn})_{m,n\in\mathbf{Z}}$ defines a map T from vectors in $L^2(\mathbb{R})$ to (m, n)-sequences of complex numbers, by

$$(Tf)_{mn} = \langle \varphi_{mn}, f\rangle, \qquad f \in L^2(\mathbb{R}). \tag{4}$$

It follows from standard arguments (see Section 2) that this map is bounded from $L^2(\mathbb{R})$ to $l^2(\mathbf{Z}\times\mathbf{Z})$, for all $a, b > 0$. That is, for all $f\in L^2(\mathbb{R})$,

$$\sum_{m,n\in\mathbf{Z}} |\langle \varphi_{mn}, f\rangle|^2 \leqq \mathbf{M}(a,b)\|f\|^2. \tag{5}$$

The adjoint of T is the map from l^2 to $L^2(\mathbb{R})$ defined by

$$T^*c = \sum_{m,n\in\mathbf{Z}} c_{mn}\varphi_{mn},$$

for all $c = (c_{mn})_{m,n\in\mathbf{Z}} \in l^2(\mathbf{Z}\times\mathbf{Z})$. One has thus, for all $f\in L^2(\mathbb{R})$,

$$T^*Tf = \sum_{m,n\in\mathbf{Z}} \varphi_{mn}\langle\varphi_{mn}, f\rangle. \tag{6}$$

If T^*T is bounded below away from zero, i.e., if there exists a constant $\mathbf{m}(a,b) > 0$ such that, for all $f\in L^2(\mathbb{R})$,

$$\sum_{m,n\in\mathbf{Z}} |\langle \varphi_{mn}, f\rangle|^2 \geqq \mathbf{m}(a,b)\|f\|^2, \tag{7}$$

then the range of T^*T is all of $L^2(\mathbb{R})$, and T^*T has a bounded inverse. For any $g\in L^2(\mathbb{R})$, $g = T^*Tf$, we obtain then from (6) that

$$g = \sum_{m,n\in\mathbf{Z}} \varphi_{mn}\langle\varphi_{mn}, (T^*T)^{-1}g\rangle. \tag{8}$$

This gives an expression of g in terms of the φ_{mn}. Notice that (7) is not only a sufficient, but also a necessary condition for the existence of an expansion of type (3), with square summable coefficients c_{mn}, for arbitrary $f\in L^2(\mathbb{R})$.

Provided (7) holds, our analyzing and reconstruction procedure is the following. To every $f\in L^2(\mathbb{R})$ we associate a well-defined set of coefficients $c_{mn}(f)$ defined by

$$c_{mn}(f) = \langle\varphi_{mn}, (T^*T)^{-1}f\rangle. \tag{9}$$

The sequence $c(f) = (c_{mn}(f))_{m,n\in\mathbf{Z}}$ is in l^2; the function f can be reconstructed from this sequence by

$$f = \sum_{m,n\in\mathbf{Z}} c_{mn}(f)\varphi_{mn}. \tag{10}$$

In general the sequence $(c_{mn}(f))_{m,n\in\mathbf{Z}}$, defined by (9), is not the only sequence satisfying (3) for a given $f \in L^2(\mathbb{R})$. This is because the range of T may be a closed proper subspace of l^2. Among all the sequences $(c_{mn})_{m,n\in\mathbf{Z}}$ satisfying (3) (f fixed), the sequence $(c_{mn}(f))_{m,n\in\mathbf{Z}}$ has minimal norm. Because of the non-zero lower bound (7) the whole procedure (9) + (10) is stable.

We shall prove in Section 2 that (7) holds, with $m > 0$, for $ab = 2\pi/N$, N an integer, $N \geqq 2$. We shall give below an explicit formula for $(T^*T)^{-1}$ in these cases (see (23)). The expansion (9) + (10) has been mainly studied in the case $ab = 2\pi$, where, as we shall prove below, (7) does *not* hold. As a result the expansion (9) + (10) is then a purely formal expression in general, and the coefficients $c_{mn}(f)$ may be rather ill-behaved. For this reason the Gabor expansion (9) + (10), with $ab = 2\pi$, is considered to be less useful than e.g. the Wigner distribution method (see Janssen [12] for a comparison of the two methods). If one takes however $ab = \pi$ (for instance), then the Gabor expansion is perfectly well-defined and presents no convergence problems; it should prove to be a useful tool. The fact that the φ_{mn} are not ω-independent does not pose any problem, as we saw above, even though this ω-dependence causes the c_{mn} in (3) to be non-unique. For $ab = \pi$, the φ_{mn} are basically an oversampling sequence of Gabor wavelets.

A family of vectors $(\varphi_j)_{j\in J}$ in a Hilbert space $\mathscr{H}$ for which there exist constants $\mathbf{m}, \mathbf{M} > 0$ such that, for all $f \in \mathscr{H}$,

$$\text{(11)} \qquad \mathbf{m}\|f\|^2 \leqq \sum_{j\in J} |\langle \varphi_j, f\rangle|^2 \leqq \mathbf{M}\|f\|^2,$$

is called a frame (see Duffin and Schaffer [8], Young [21]). It is clear that (5) + (7) are nothing else than (11). We believe that frames can be very important in signal analysis, as illustrated by the Gabor expansion example above. We have studied other examples elsewhere (see Daubechies, Grossmann and Meyer [7]); we even constructed explicit examples of frames, which we called tight frames, for which $\mathbf{m} = \mathbf{M}$, or equivalent $T^*T = \mathbf{m}\mathbf{1}$. The expansion (9) + (10) becomes then particularly simple (see [7]).

Let us now see what all this has to do with entire functions. We can transpose all the $L^2(\mathbb{R})$-statements to the Hilbert space F of entire functions, by means of the unitary map U_B^{-1}. In the theorem below we state our various results in terms of explicit estimates for functions in F. We give explicit expressions for all the constants involved, and list all the cases we are able to treat. Proofs are given in Section 2.

THEOREM. *Define* $z_{mn} = ma + inb$, *where* $a, b > 0$,

$$e_{mn} = e_{z_{mn}},$$

i.e.,

$$e_{mn}(z) = \exp\{-\tfrac{1}{4}|z_{mn}|^2 + \tfrac{1}{2}z\cdot\overline{z_{mn}}\}.$$

For N an integer, $N \geqq 1$, *and* $t, s \in [-\frac{1}{2}, \frac{1}{2}]$, *let*

$$S(b, N; t, s) = \frac{b}{\sqrt{\pi}} \exp\{-s^2 b^2\} \sum_{r=0}^{N-1} \left| \theta_3\left(t - \frac{r}{N} + i\frac{b^2}{2\pi} s \,\middle|\, \frac{ib^2}{2\pi} \right) \right|^2$$

$$= \frac{b}{\sqrt{\pi}} \sum_{r=0}^{N-1} \left| \sum_{l \in \mathbf{Z}} \exp\{2\pi i l(t - r/N)\} \exp\left\{-\tfrac{1}{2} b^2 (l+s)^2\right\} \right|^2,$$

where θ_3 *is one of Jacobi's theta-functions, i.e., with the notations of e.g. Bateman* [6],

$$\theta_3(z|\tau) = 1 + 2\sum_{l=1}^{\infty} \cos 2\pi l z \exp\{i\pi\tau l^2\}.$$

Then:

(i) *For all* $a, b > 0$, *for all* $f \in F$,

$$\sum_{m,n \in \mathbf{Z}} |\langle e_{mn}, f\rangle|^2 \leqq \mathbf{M}_1(a, b)\|f\|^2, \tag{12}$$

or, equivalently,

$$\sum_{m,n \in \mathbf{Z}} |f(z_{mn})|^2 \exp\{-\tfrac{1}{2}|z_{mn}|^2\} \leqq \mathbf{M}_1(a, b) \int\int_{\mathbf{R}^2} \frac{d(\mathcal{R}e\, z)\, d(\mathcal{I}m\, z)}{2\pi} |f(z)|^2 \exp\{-\tfrac{1}{2}|z|^2\}, \tag{13}$$

with

$$\mathbf{M}_1(a, b) = \theta_3\left(0 \,\middle|\, \frac{ia^2}{4\pi}\right) \theta_3\left(0 \,\middle|\, \frac{ib^2}{4\pi}\right) = \left(1 + 2\sum_{l=1}^{\infty} \exp\{-\tfrac{1}{4}a^2 l^2\}\right)\left(1 + 2\sum_{l=1}^{\infty} \exp\{-\tfrac{1}{4}b^2 l^2\}\right). \tag{14}$$

(ii) *For* $ab = 2\pi/N$, *with N an integer,* $N \geqq 1$, *the above inequality can be sharpened. That is,* (12) *and* (13) *still hold if we replace* $\mathbf{M}_1(2\pi/Nb, b)$ *by the smaller constant*

$$\mathbf{M}_2\left(\frac{2\pi}{Nb}, b\right) = \sup_{t, s \in [-\frac{1}{2}, \frac{1}{2}]} S(b, N; t, s). \tag{15}$$

For the cases $ab = 2\pi/N$, (15) *is the best constant possible, i.e.*,

$$\sup_{f \in F,\, f \neq 0} \left(\|f\|^{-2} \sum_{m,n \in \mathbf{Z}} |\langle e_{mn}, f\rangle|^2 \right) = \mathbf{M}_2\left(\frac{2\pi}{Nb}, b\right). \tag{16}$$

(iii) *For* $ab = 2\pi/N$, *with* N *an integer*, $N \geqq 2$, *we have furthermore that*

$$\sum_{m,n\in\mathbf{Z}} |\langle e_{mn}, f\rangle|^2 \geqq \mathbf{m}\left(\frac{2\pi}{Nb}, b\right)\|f\|^2, \tag{17}$$

or equivalently,

$$\begin{aligned}&\sum_{m,n\in\mathbf{Z}} |f(z_{mn})|^2 \exp\{-\tfrac{1}{2}|z_{mn}|^2\}\\ &\quad\geqq \mathbf{m}\left(\frac{2\pi}{Nb}, b\right)\int\int_{\mathbf{R}^2} \frac{d(\mathscr{R}e\, z)\, d(\mathscr{I}m\, z)}{2\pi} |f(z)|^2 \exp\{-\tfrac{1}{2}|z|^2\},\end{aligned} \tag{18}$$

with

$$\mathbf{m}\left(\frac{2\pi}{Nb}, b\right) = \inf_{t,s\in[-\frac{1}{2},\frac{1}{2}]} S(b, N; t, s) > 0. \tag{19}$$

Again this is the best constant possible in the inequalities (17) *or* (18), *i.e.*,

$$\inf_{f\in F,\, f\neq 0}\left(\|f\|^{-2} \sum_{m,n\in\mathbf{Z}} |\langle e_{mn}, f\rangle|^2\right) = \mathbf{m}\left(\frac{2\pi}{Nb}, b\right). \tag{20}$$

(iv) *In the case* $ab \geqq 2\pi$, *there is* <u>*no*</u> *constant* $\mathbf{m}(a, b) > 0$ *for which* (17) *would hold for all* $f \in F$. *In other words*,

$$\inf_{f\in F,\, f\neq 0}\left(\|f\|^{-2} \sum_{m,n\in\mathbf{Z}} |\langle e_{mn}, f\rangle|^2\right) = 0 \quad \textit{if} \quad ab \geqq 2\pi. \tag{21}$$

Remarks. 1. If $ab > 2\pi$, one can explicitly construct non-zero functions f in F vanishing at all the z_{mn}. One such example is $f(z) = \theta_1(a^{-1}z|ia^{-1}b)$ where θ_1 is another of Jacobi's theta-functions (see [6]). For these functions $\langle e_{mn}, f\rangle = 0$ for all $m, n \in \mathbf{Z}$, while nevertheless $f \in F$. This implies (21) for $ab > 2\pi$.

2. Relations (16) and (17) follow from an explicit construction (see Section 2), showing that the map $T^*T \in \mathscr{B}(L^2(\mathbf{R}))$ defined by

$$T^*T = U_B^{-1}\left(\sum_{m,n\in\mathbf{Z}} e_{mn}\langle e_{mn}, \cdot\rangle\right)U_B = \sum_{m,n\in\mathbf{Z}} \varphi_{mn}\langle\varphi_{mn}, \cdot\rangle$$

is unitarily equivalent to a multiplication operator. The unitary transform used in this construction maps $L^2(\mathbf{R})$ onto $L^2([-\frac{1}{2}, \frac{1}{2}] \times [-\frac{1}{2}, \frac{1}{2}])$; it is defined by

$$(V_{Z,b}f)(t, s) = \sqrt{b}\sum_{l\in\mathbf{Z}} \exp\{2\pi i t l\} f(b(s - l)),$$

with inverse

$$\left(V_{Z,b}^{-1}\varphi\right)(x) = \sqrt{b}\int_{-1/2}^{1/2} dt\, \exp\left\{2\pi i t\left\lfloor \frac{x}{b} + \frac{1}{2}\right\rfloor\right\}\varphi\left(t, \frac{x}{b} - \left\lfloor \frac{x}{b} + \frac{1}{2}\right\rfloor\right),$$

where the notation $\lfloor y \rfloor$ denotes the largest integer not exceeding y. $V_{Z,b}$ is the Zak transform (see [22]; we give a few more details in Section 2). We prove in Section 2 that, for $ab = 2\pi/N$, N integer, $N \geqq 1$,

$$V_{Z,b}T^*TV_{Z,b}^{-1} = \text{multiplication by } S(b, N; t, s). \tag{22}$$

Consequently,

$$\inf_{t,s\in[-\frac{1}{2},\frac{1}{2}]} S(b, N; t, s) \leqq T^*T \leqq \sup_{t,s\in[-\frac{1}{2},\frac{1}{2}]} S(b, N; t, s),$$

which proves (15), (16) and (17), (18). The continuity of the function $S(b, N; \cdot, \cdot)$ moreover implies that these are the best possible upper and lower bounds.

3. For $ab = 2\pi$, T^*T is unitarily equivalent to multiplication by $S(b, 1; t, s)$. Since $\theta_3(\frac{1}{2} + ib^2/4\pi | ib^2/2\pi) = 0$, we find $S(b, 1; \frac{1}{2}, \frac{1}{2}) = 0$. This implies that the spectrum of T^*T is of the form $[0, \Lambda]$. Hence (21) follows.

4. If however $ab = 2\pi/N$ with N an integer, $N \geqq 2$, then $(\sqrt{\pi}/b)S(b, N; t, s)$ is the sum of at least two squares of absolute values of θ_3-functions with non-coinciding zeros. This implies $\inf_{t,s\in[-\frac{1}{2},\frac{1}{2}]} S(b, N; t, s) > 0$, or $\mathbf{m}(2\pi/Nb, b) > 0$.

5. The unitary equivalence (22) also enables us to make the analyzing and reconstruction procedure (9) + (10) completely explicit. Choose $a, b > 0$ such that $2\pi/ab$ is an integer $N \geqq 2$. Let $\varphi_{mn} \in L^2(\mathbb{R})$ be the associated discrete family of coherent states,

$$\varphi_{mn}(x) = \pi^{-1/4}\exp\{-\pi i m n/N\}\exp\{2\pi i m x/Nb\}\exp\left\{-\tfrac{1}{2}(x - nb)^2\right\}.$$

Then $f \in L^2(\mathbb{R})$ can be written as

$$f = \sum_{m,n\in\mathbf{Z}} c_{mn}(f)\varphi_{mn}$$

with, $m = Nk + r$, $k, r \in \mathbf{Z}$, $0 \leqq r < N$,

$$\begin{aligned} c_{mn}(f) &= c_{Nk+r\,n}(f) = \left\langle V_{Z,b}\varphi_{Nk+r\,n}, \frac{1}{S(b,N)}V_{Z,b}f\right\rangle \\ &= \sqrt{b}\,\pi^{-1/4}(-1)^{kn}\exp\{i\pi n r/N\} \\ &\quad\cdot\int_{-1/2}^{1/2} dt \int_{-1/2}^{1/2} ds \left(\exp\{-2\pi i s(k + r/N)\}\right. \\ &\quad\left.\cdot\exp\{2\pi i t n\}\exp\left\{-\tfrac{1}{2}b^2s^2\right\}\theta_3\left(t - \frac{r}{N} - \frac{ib^2}{2\pi}s \middle| \frac{ib^2}{2\pi}\right)\frac{(V_{Z,b}f)(t,s)}{S(b, N; t, s)}\right). \end{aligned} \tag{23}$$

6. In the following two tables we give numerical values of the constants $\mathbf{M}_1, \mathbf{M}_2$ for $ab = 2\pi$, and of $\mathbf{M}_1, \mathbf{M}_2$, m for $ab = \pi$.

Table I.

b	$\mathbf{M}_1(2\pi/b, b)$	$\mathbf{M}_2(2\pi/b, b)$
.5	7.090	7.090
1.0	3.545	3.545
1.5	2.422	2.365
2.0	2.073	1.824
2.5	2.015	1.669
3.0	2.050	1.769
3.5	2.159	1.992
4.0	2.339	2.260

Table II.

b	$\mathbf{M}_1(\pi/b, b)$	$\mathbf{M}_2(\pi/b, b)$	$\mathbf{m}(\pi/b, b)$
.5	7.091	7.090	.0007
1.0	4.146	3.546	.601
1.5	4.001	2.482	1.540
2.0	4.000	2.425	1.600
2.5	4.014	2.843	1.178
3.0	4.100	3.387	.713
3.5	4.319	3.949	.369
4.0	4.679	4.514	.165

In [6] we studied the expansion problem (3) for a different family of φ_{mn}, based on a continuous function with compact support rather than on a Gaussian as used here (see (2)). In that case we could prove the lower bound (7), with non-zero $\mathbf{m}$, for all a, b with $ab < 2\pi$ ($ab = 2\pi$ being excluded there too). These lower bounds cannot be translated into estimates of type (18) for entire functions. It nevertheless seems reasonable to conjecture that also in the present setting (7) should hold, with $\mathbf{m} > 0$, for all $ab < 2\pi$. This would mean the existence of bounds of type (17) or (18) for all $ab < 2\pi$ (rather than only $ab = 2\pi/N$, N integer, $N \geq 2$, as proved here). We hope to have made a convincing case of the interest of such bounds. We would feel gratified if experts on entire functions (which we are not) were motivated by our results to study this matter.

Before we proceed to the proof of our theorem, let us make two more remarks. The first is that while we worked here with only one complex variable, the generalization to any finite number of variables is straightforward. The second is that we want to stress that the question we have raised here is different from the question whether the $(z_{mn})_{m,n \in \mathbf{Z}}$ constitute an interpolating sequence in the sense of Shapiro [19] or Young [21]. For ab large enough, the family $(z_{mn})_{m,n \in \mathbf{Z}}$ is a "universal interpolating sequence" (see Shapiro [19], page 96).

For $ab \leqq 2\pi$ however, the case of most interest to us, the family $(z_{mn})_{m,n\in\mathbf{Z}}$ is *not* an interpolating sequence. The difference can be easily explained in terms of the operator T defined by (4). A family of points $(z_j)_{j\in J}$ is an interpolating sequence (in the sense of [19] or [21]) if and only if the associated map $\tilde{T}$ from F to $l^2(J)$, defined by

$$(\tilde{T}f)_j = \langle e_{z_j}, f\rangle,$$

is bounded and surjective. Since the $e_{mn} \equiv e_{z_{mn}}$ are not ω-independent for $ab \leqq 2\pi$, the associated map $\tilde{T} = TU_B^{-1}$ is not surjective.

2. Proofs

We start by proving the bound (12) + (14), using standard arguments (see e.g. Shapiro [19], Section 6.2 or Young [21], Section 4.2). The bound (12) can be rewritten as

$$\|T\| \leqq \mathbf{M}_1(a,b)^{1/2}, \quad \text{or} \quad \|T^*T\| \leqq \mathbf{M}_1(a,b), \tag{24}$$

where T is the operator from $L^2(\mathbf{R})$ to l^2 defined by (4), i.e., $(Tf)_{mn} = \langle \varphi_{mn}, f\rangle$. The bound (24) is equivalent to

$$\|TT^*\| \leqq \mathbf{M}_1(a,b). \tag{25}$$

For $c = (c_{mn})_{m,n\in\mathbf{Z}} \in l^2$, TT^*c is given by

$$(TT^*c)_{mn} = \sum_{m',n'\in\mathbf{Z}} \langle \varphi_{mn}, \varphi_{m'n'}\rangle c_{m'n'}.$$

A sufficient condition for the infinite hermitian matrix $(\langle \varphi_{mn}, \varphi_{m',n'}\rangle)_{m,n;m',n'}$ to define a bounded operator on $l^2(\mathbf{Z}\times\mathbf{Z})$ is given by: for all $m', n' \in \mathbf{Z}$,

$$\sum_{m,n\in\mathbf{Z}} |\langle \varphi_{mn}, \varphi_{m'n'}\rangle| \leqq M. \tag{26}$$

If (26) is satisfied, one easily derives (25), with $\mathbf{M}_1(a,b) = M$, by using the Cauchy-Schwarz inequality (see [19], [21]). In our present case we have (use (1))

$$|\langle \varphi_{mn}, \varphi_{m',n'}\rangle| = |\langle e_{z_{mn}}, e_{z_{m'n'}}\rangle|$$

$$= \exp\{-\tfrac{1}{4}(m-m')^2a^2 - \tfrac{1}{4}(n-n')^2b^2\}.$$

This implies that the sufficient condition (26) is satisfied.

Indeed, for all $m', n' \in \mathbf{Z}$,

$$\sum_{m,n\in\mathbf{Z}} |\langle \varphi_{mn}, \varphi_{m'n'}\rangle| \leqq \left[\sum_{m\in\mathbf{Z}} \exp\{-\tfrac{1}{4}m^2a^2\}\right]\left[\sum_{n\in\mathbf{Z}} \exp\{-\tfrac{1}{4}n^2b^2\}\right]$$

$$= \theta_3(0|ia^2/4\pi)\theta_3(0|ib^2/4\pi).$$

This proves (12), (14).

As pointed out in the remarks following the theorem, all the other statements in the theorem follow from the unitary equivalence (22). We now proceed to prove this unitary equivalence. For $\alpha > 0$, we define the Zak transform $V_{Z,\alpha}$: $L^2(\mathbf{R}) \to L^2([-\frac{1}{2}, \frac{1}{2}] \times [-\frac{1}{2}, \frac{1}{2}])$ by

$$(V_{Z,\alpha}f)(t,s) = \sqrt{\alpha} \sum_{l\in\mathbf{Z}} \exp\{2\pi itl\} f[\alpha(s-l)]. \tag{27}$$

Strictly speaking, the series in the right-hand side need not converge for an arbitrary $f \in L^2(\mathbf{R})$. For $f \in C_0^\infty(\mathbf{R})$, the set of C^∞-functions with compact support, (27) is well defined however (only a finite number of terms contribute), and one easily checks $\|V_{Z,\alpha}f\| = \|f\|$. Since $C_0^\infty(\mathbf{R})$ is dense in $L^2(\mathbf{R})$ we can extend $V_{Z,\alpha}$ to all of $L^2(\mathbf{R})$. One finds that the range of $V_{Z,\alpha}$ is all of $L^2([-\frac{1}{2}, \frac{1}{2}] \times [-\frac{1}{2}, \frac{1}{2}])$ (see [2]; this follows also from our calculations below). Hence $V_{Z,\alpha}$ is unitary.

There exists a link between the Zak transform and an abelian subfamily of the Weyl operators. The Weyl operators are a family of unitary operators $W(p,q)$ of fundamental importance in quantum mechanics (see Weyl [20]). On $L^2(\mathbf{R})$ they are defined by

$$[W(p,q)f](x) = \exp\{-\tfrac{1}{2}ipq\}\exp\{ipx\}f(x-q).$$

The coherent states defined by (2) can be seen as an orbit of the family of Weyl operators,

$$\varphi^{p,q} = W(p,q)\varphi^{0,0} \quad \text{with} \quad \varphi^{0,0}(x) = \pi^{-1/4}\exp\{-\tfrac{1}{2}x^2\}.$$

For any $\alpha \in \mathbf{R}^*$, the subfamily of Weyl operators $W(m2\pi/\alpha, n\alpha)$, with $m, n \in \mathbf{Z}$, is abelian. The following simple relationship links this abelian subfamily with the Zak transform. For all $f \in L^2(\mathbf{R})$,

$$\left[V_{Z,\alpha}W\left(m\frac{2\pi}{\alpha}, n\alpha\right)f\right](t,s) = (-1)^{mn}\exp\{-2\pi itn\}\exp\{2\pi ims\}[V_{Z,\alpha}f](t,s).$$

This implies that, for all $f, g \in L^2(\mathbb{R})$,

$$\sum_{m,n\in\mathbb{Z}} \left| \left\langle W\left(m\frac{2\pi}{\alpha}, n\alpha\right) f, g \right\rangle \right|^2$$

$$= \sum_{m,n\in\mathbb{Z}} \left| \int_{-1/2}^{1/2} dt \int_{-1/2}^{1/2} ds \exp\{2\pi itn\}\exp\{-2\pi ims\} \cdot \overline{(V_{Z,\alpha}f)(t,s)}\,(V_{Z,\alpha}g)(t,s) \right|^2$$

$$= \int_{-1/2}^{1/2} dt \int_{-1/2}^{1/2} ds \, |(V_{Z,\alpha}f)(t,s)|^2 |(V_{Z,\alpha}g)(t,s)|^2 .$$

Let us now apply all this to the problem at hand. Suppose that $ab = 2\pi/N$ with N an integer, $N \geqq 1$. Take $\alpha = b$. For $m = Nk + r$, with $k, r \in \mathbb{Z}$, $0 \leqq r < N$,

$$\varphi_{mn} = W(ma, nb)\varphi_{00} = W\left(k\frac{2\pi}{b} + r\frac{2\pi}{Nb}, nb\right)\varphi_{00}$$

$$= \exp\{i\pi rn/N\} W\left(k\frac{2\pi}{b}, nb\right) W\left(\frac{2\pi}{Nb}r, 0\right)\varphi_{00}.$$

On the other hand,

$$\left[V_{Z,b} W\left(\frac{2\pi}{Nb}r, 0\right)\varphi_{00}\right](t,s)$$

$$= \sqrt{b}\,\pi^{-1/4} \sum_{l\in\mathbb{Z}} \exp\{2\pi itl\}\exp\{i2\pi r(s-l)/N\}\exp\left\{-\tfrac{1}{2}b^2(s-l)^2\right\}$$

$$= \sqrt{b}\,\pi^{-1/4}\exp\{2\pi irs/N\} \sum_{l\in\mathbb{Z}} \exp\{2\pi il(t-r/N)\}\exp\left\{-\tfrac{1}{2}b^2(s-l)^2\right\}.$$

Hence

$$\sum_{r=0}^{N-1} \left| \left[V_{Z,b} W\left(\frac{2\pi}{Nb}r, 0\right)\varphi_{00}\right](t,s) \right|^2$$

$$= \frac{b}{\sqrt{\pi}} \sum_{r=0}^{N-1} \left| \sum_{l\in\mathbb{Z}} \exp\{2\pi il(t-r/N)\}\exp\left\{-\tfrac{1}{2}b^2(s-l)^2\right\} \right|^2$$

$$= S(b, N; t, s).$$

Putting everything together, we find

$$\begin{aligned}\langle f, T^*Tf\rangle &= \sum_{m,n\in\mathbf{Z}} |\langle \varphi_{mn}, f\rangle|^2 \\ &= \sum_{k,n\in\mathbf{Z}} \sum_{r=0}^{N-1} \left|\langle W\left(k\frac{2\pi}{b}, nb\right) W\left(\frac{2\pi}{Nb}r, 0\right)\varphi_{00}, f\rangle\right|^2 \\ &= \sum_{r=0}^{N-1} \int_{-1/2}^{1/2} dt \int_{-1/2}^{1/2} ds \left|\left[V_{Z,b} W\left(\frac{2\pi}{Nb}r, 0\right)\varphi_{00}\right](t,s)\right|^2 |(V_{Z,b}f)(t,s)|^2 \\ &= \int_{-1/2}^{1/2} dt \int_{-1/2}^{1/2} ds\, S(b, N; t, s)|(V_{Z,b}f)(t,s)|^2.\end{aligned}$$

This shows that $V_{Z,b}T^*TV_{Z,b}^{-1}$ is multiplication by $S(b, N; t, s)$. Together with the remarks listed following the theorem, this proves the theorem.

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Note added in proof: After the submission of this paper, the conjecture that (17) should hold, $\mathbf{M} > 0$, for all $ab < 2\pi$, was partially proved by one of us. It turns out that the e_{mn} constitutes a frame for all $ab < 2\pi \times .996$. The full conjecture (I.E., for ab up to 2π) is still open. This new result will be published in *The wavelet transform, time-frequency and signal analysis*, by I. Daubechies, submitted to IEEE Trans. Inf. Theory.

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