An Iterative Thresholding Algorithm for Linear Inverse Problems with a Sparsity Constraint

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Abstract

We consider linear inverse problems where the solution is assumed to have a sparse expansion on an arbitrary preassigned orthonormal basis. We prove that replacing the usual quadratic regularizing penalties by weighted ℓ^p- penalties on the coefficients of such expansions, with $1 \le p \le 2$, still regularizes the problem. Use of such ℓ^p-penalized problems with $p < 2$ is often advocated when one expects the underlying ideal noiseless solution to have a sparse expansion with respect to the basis under consideration. To compute the corresponding regularized solutions, we analyze an iterative algorithm that amounts to a Landweber iteration with thresholding (or nonlinear shrinkage) applied at each iteration step. We prove that this algorithm converges in norm. © 2004 Wiley Periodicals, Inc.

1 Introduction

1.1 Linear Inverse Problems

In many practical problems in the sciences and applied sciences, the features of most interest cannot be observed directly, but have to be inferred from other, observable quantities. In the simplest approximation, which works surprisingly well in a wide range of cases and often suffices, there is a *linear* relationship between the features of interest and the observed quantities. If we model the *object* (the traditional shorthand for the features of interest) by a function f, and the derived quantities or *image* by another function h, we can cast the problem of inferring f from h as a *linear inverse problem*, the task of which is to solve the equation

$$Kf = h\,.$$

This equation and the task of solving it make sense only when placed in an appropriate framework. In this paper we shall assume that f and h belong to appropriate function spaces, typically Banach or Hilbert spaces, $f \in \mathcal{B}_{\mathrm{OBJECT}}$, $h \in \mathcal{B}_{\mathrm{IMAGE}}$,

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and K is a bounded operator from the space $\mathcal{B}_{\mathrm{OBJECT}}$ to $\mathcal{B}_{\mathrm{IMAGE}}$. The choice of the spaces must be appropriate for describing real-life situations.

The observations or *data*, which we shall model by yet another function, g, are typically not exactly equal to the image $h = Kf$, but rather to a distortion of h. This distortion is often modeled by an *additive noise* or *error term* e, i.e.,

$$g = h + e = Kf + e\,.$$

Moreover, one typically assumes that the "size" of the noise can be measured by its L^2-norm, $\|e\| = (\int_\Omega |e|^2)^{1/2}$ if e is a function on Ω. (In a finite-dimensional situation one uses $\|e\| = (\sum_{n=1}^N |e_n|^2)^{1/2}$ instead.) Our only "handle" on the image h is thus via the observed g, and we typically have little information on $e = g - h$ beyond an upper bound on its L^2-norm $\|e\|$. (We have here implicitly placed ourselves in the "deterministic setting" customary to the inverse problems community, of which this introduction is only a brief overview; see, e.g., [2, 3] or [23] for a comprehensive treatment. In the stochastic setting more familiar to statisticians, one assumes instead a bound on the variance of the components of e.) Therefore it is customary to take $\mathcal{B}_{\mathrm{IMAGE}} = L^2(\Omega)$; even if the "true images" h (i.e., the images Kf of the possible objects f) lie in a much smaller space, we can only know them up to some (hopefully) small L^2-distance.

We shall consider in this paper a whole family of possible choices for $\mathcal{B}_{\mathrm{OBJECT}}$, but we shall always assume that these spaces are subspaces of a basic Hilbert space $\mathcal{H}$ (often an L^2-space as well), and that K is a bounded operator from $\mathcal{H}$ to $L^2(\Omega)$. In many applications, K is an integral operator with a kernel representing the response of the imaging device; in the special case where this linear device is translation invariant, K reduces to a convolution operator.

To find an estimate of f from the observed g, one can minimize the *discrepancy* $\Delta(f)$,

$$\Delta(f) = \|Kf - g\|^2\,;$$

functions that minimize $\Delta(f)$ are called *pseudosolutions* of the inverse problem. If the operator K has a trivial null space, i.e., if $\mathrm{N}(K) = \{f \in \mathcal{H} : Kf = 0\} = \{0\}$, there is a unique minimizer, given by $\widetilde{f} = (K^*K)^{-1}K^*g$, where K^* is the adjoint operator. If $\mathrm{N}(K) \neq \{0\}$ it is customary to choose, among the set of pseudosolutions, the unique element $f^\dagger$ of minimal norm, i.e., $f^\dagger = \text{arg-min}\{\|f\| : f \text{ minimizes } \Delta(f)\}$. This function belongs to $\mathrm{N}(K)^\perp$ and is called the *generalized solution* of the inverse problem. In this case the map $K^\dagger : g \mapsto f^\dagger$ is called the *generalized inverse* of K. Even when K^*K is not invertible, $K^\dagger g$ is well-defined for all g such that $K^*g \in \mathrm{R}(K^*K)$. However, the generalized inverse operator may be unbounded (for so-called *ill-posed problems*) or have a very large norm (for *ill-conditioned problems*). In such instances, it has to be replaced by bounded approximants or approximants with smaller norm, so that numerically stable solutions can be defined and used as meaningful approximations of the true solution corresponding to the exact data. This is the issue of *regularization*.

1.2 Regularization by Imposing Additional Quadratic Constraints

The definition of a pseudosolution (or even, if one considers equivalence classes modulo $\mathrm{N}(K)$, of a generalized solution) makes use of the inverse of the operator K^*K; this inverse is well-defined on the range $\mathrm{R}(K^*)$ of K^* when K^*K is a strictly positive operator, i.e., when its spectrum is bounded below away from 0. When the spectrum of K^*K is not bounded below by a strictly positive constant, $\mathrm{R}(K^*K)$ is not closed, and not all elements of $\mathrm{R}(K^*)$ lie in $\mathrm{R}(K^*K)$. In this case there is no guarantee that $K^*g \in \mathrm{R}(K^*K)$; even if K^*g belongs to $\mathrm{R}(K^*K)$, the unboundedness of $(K^*K)^{-1}$ can cause severe numerical instabilities unless additional measures are taken.

This blowup or these numerical instabilities are regarded as "unphysical" in the sense that we typically know a priori that the true object would not have had a huge norm in $\mathcal{H}$ or other characteristics exhibited by the unconstrained "solutions." A standard procedure to avoid these instabilities or to *regularize* the inverse problem is to modify the functional to be minimized so that it incorporates not only the discrepancy but also the a priori knowledge one may have about the solution. For instance, if it is known that the object is of limited "size" in $\mathcal{H}$, i.e., if $\|f\|_{\mathcal{H}} \le \rho$, then the functional to be minimized can be chosen as

$$\Delta(f) + \mu\|f\|^2_{\mathcal{H}} = \|Kf - g\|^2_{L^2(\Omega)} + \mu\|f\|^2_{\mathcal{H}}$$

where μ is some positive constant called the *regularization parameter*. The minimizer is given by

$$f_\mu = (K^*K + \mu I)^{-1}K^*g \tag{1.1}$$

where I denotes the identity operator. The constant μ can then be chosen appropriately, depending on the application. If K is a compact operator, with singular value decomposition given by $Kf = \sum_{k=1}^{\infty} \sigma_k \langle f, v_k\rangle u_k$, where $(u_k)_{k\in\mathbb{N}}$ and $(v_k)_{k\in\mathbb{N}}$ are the orthonormal bases of eigenvectors of KK^* and K^*K, respectively, with corresponding eigenvalues σ_k^2, then (1.1) can be rewritten as

$$f_\mu = \sum_{k=1}^{\infty} \frac{\sigma_k}{\sigma_k^2 + \mu} \langle g, u_k\rangle v_k \,. \tag{1.2}$$

This formula shows explicitly how this regularization method reduces the importance of the eigenmodes of K^*K with small eigenvalues, which otherwise (if $\mu = 0$) lead to instabilities. If an estimate of the "noise" is known, i.e., if we know a priori that $g = Kf + e$ with $\|e\| \le \epsilon$, then one finds from (1.2) that

$$\|f - f_\mu\| \le \left\| \sum_{k=1}^{\infty} \frac{\mu\langle f, v_k\rangle}{\sigma_k^2 + \mu}\, v_k \right\| + \left\| \sum_{k=1}^{\infty} \frac{\sigma_k}{\sigma_k^2 + \mu} \langle e, u_k\rangle v_k \right\| \le \Gamma(\mu) + \frac{\epsilon}{\sqrt{\mu}} \,,$$

where $\Gamma(\mu) \to 0$ as $\mu \to 0$. This means that μ can be chosen appropriately, in an ϵ-dependent way, so that the error in estimation $\|f - f_\mu\|$ converges to 0 when ϵ (the estimation of the noise level) shrinks to 0. This feature of the method, usually called *stability*, is one that is required for any regularization method. It is similar to requiring that a statistical estimator be consistent.

Note that the "regularized estimate" f_μ of (1.2) is linear in g. This means that we have effectively defined a linear regularized estimation operator that is especially well adapted to the properties of the operator K; however, it proceeds with a "one method fits all" strategy, independent of the data. This may not always be the best approach. For instance, if $\mathcal{H}$ is an L^2-space itself, and K is an integral operator, the functions u_k and v_k are typically fairly smooth; if, on the other hand, the objects f are likely to have local singularities or discontinuities, an approximation of type (1.2) (effectively limiting the estimates f_μ to expansions in the first N v_k, with N determined by, say, $\sigma_k^2 < \mu/100$ for $k > N$) will of necessity be a smoothed version of f, without sharp features.

Other classical regularization methods with quadratic constraints may use quadratic Sobolev norms involving a few derivatives, as the "penalty" term added to the discrepancy. This introduces a penalization of the highly oscillating components, which are often the most sensitive to noise. This method is especially easy to use in the case where K is a convolution operator diagonal in the Fourier domain. In this case the regularization produces a smooth cutoff on the highest Fourier components, independently of the data. This works well for recovering smooth objects that have their relevant structure contained in the lower part of the spectrum and have spectral content homogeneously distributed across the space or time domain. However, the Fourier domain is clearly not the appropriate representation for expressing smoothness properties of objects that are either spatially inhomogeneous, with varying "local frequency" content, and/or present some discontinuities, because the frequency cutoff implies that the resolution with which the fine details of the solution can be stably retrieved is necessarily limited; it also implies that the achievable resolution is essentially the same at all points (see, e.g., the book [3] for an extensive discussion of these topics).

1.3 Regularization by Nonquadratic Constraints That Promote Sparsity

The problems with the standard regularization methods described above are well known and several approaches have been proposed for dealing with them. We discuss in this paper a regularization method that, like the classical methods just discussed, minimizes a functional obtained by adding a penalization term to the discrepancy; typically this penalization term will *not* be quadratic, but rather a weighted ℓ^p-norm of the coefficients of f with respect to a particular orthonormal basis in $\mathcal{H}$, with $1 \le p \le 2$. More precisely, given an orthonormal basis $(\varphi_\gamma)_{\gamma\in\Gamma}$ of $\mathcal{H}$, and given a sequence of strictly positive weights $\mathbf{w} = (w_\gamma)_{\gamma\in\Gamma}$, we define the functional $\Phi_{\mathbf{w},p}$ by

$$\Phi_{\mathbf{w},p}(f) = \Delta(f) + \sum_{\gamma\in\Gamma} w_\gamma |\langle f, \varphi_\gamma\rangle|^p = \|Kf - g\|^2 + \sum_{\gamma\in\Gamma} w_\gamma |\langle f, \varphi_\gamma\rangle|^p . \tag{1.3}$$

For the special case $p = 2$ and $w_\gamma = \mu$ for all $\gamma \in \Gamma$ (we shall write this as $\mathbf{w} = \mu\mathbf{w}_0$, where $\mathbf{w}_0$ is the sequence with all entries equal to 1), this reduces to the quadratic functional minimized by (1.1). If we consider the family of functionals $\Phi_{\mu\mathbf{w}_0,p}(f)$, keeping the weights fixed at μ, but decreasing p from 2 to 1, we gradually *increase* the penalization on "small" coefficients (those with $|\langle f, \varphi_\gamma\rangle| < 1$) while simultaneously *decreasing* the penalization on "large coefficients" (for which $|\langle f, \varphi_\gamma\rangle| > 1$). As far as the penalization term is concerned, we are thus putting a lesser penalty on functions f with large but few components with respect to the basis $(\varphi_\gamma)_{\gamma\in\Gamma}$, and a higher penalty on sums of many small components, when compared to the classical method of (1.1). This effect is the more pronounced the smaller p is. By taking $p < 2$, and especially for the limit value $p = 1$, the proposed minimization procedure thus promotes sparsity of the expansion of f with respect to the φ_γ.

This sparsity-promoting feature can also be interpreted in other ways. The minimization of the same variational functional (1.3) arises in the derivation of a penalized maximum-likelihood solution for problems with a Gaussian noise model and a Laplacian (or generalized Laplacian) prior; it is known that such priors promote sparsity. Moreover, natural image data when transferred on the wavelet domain are known to have a non-Gaussian distribution with density $c\exp(-|\langle f, \varphi_\gamma\rangle|^p)$ with p close to 1, so that this function is a natural prior for this application. The variational functional (1.3) also comes up in some formulations of independent component analysis in which the p^{th} power cost term is chosen when selecting for sparsity, such as in the FastICA and InfoMax algorithms [26]. Although some applications in signal analysis also use values of p with $0 < p < 1$, we shall restrict ourselves to $p \geq 1$ because the functional ceases to be convex if $p < 1$.

The bulk of this paper deals with an iterative algorithm to obtain minimizers f^* for the functional (1.3) for general operators K. In the special case where K happens to be diagonal in the φ_γ-basis, $K\varphi_\gamma = \kappa_\gamma\varphi_\gamma$, the analysis is easy and straightforward. Introducing the shorthand notation f_γ for $\langle f, \varphi_\gamma\rangle$ and g_γ for $\langle g, \varphi_\gamma\rangle$, we then have

$$\Phi_{\mathbf{w},p}(f) = \sum_{\gamma\in\Gamma} \left[|\kappa_\gamma f_\gamma - g_\gamma|^2 + w_\gamma |f_\gamma|^p\right] .$$

The minimization problem thus uncouples into a family of one-dimensional minimizations and is easily solved. Of particular interest is the almost trivial case where K is the identity operator, $\mathbf{w} = \mu\mathbf{w}_0$, and $p = 1$, which corresponds to the practical situation where the data g are equal to a noisy version of f itself, and we want to remove the noise (as much as possible); i.e., we wish to *denoise* g. In this case the minimizing $f^\star$ is given by [7]

$$f^\star = \sum_{\gamma\in\Gamma} f^\star_\gamma \varphi_\gamma = \sum_{\gamma\in\Gamma} S_\mu(g_\gamma)\varphi_\gamma \,, \tag{1.4}$$

where S_μ is the (nonlinear) thresholding function from $\mathbb{R}$ to $\mathbb{R}$ defined by

$$S_\mu(x) = \begin{cases} x + \frac{\mu}{2} & \text{if } x \le -\frac{\mu}{2} \\ 0 & \text{if } |x| < \frac{\mu}{2} \\ x - \frac{\mu}{2} & \text{if } x \ge \frac{\mu}{2}. \end{cases} \tag{1.5}$$

(We shall revisit the derivation of (1.4) below. For simplicity, we are assuming that all functions are real-valued. If the f_γ are complex, a derivation similar to that of (1.4) then leads to a complex thresholding operator, which is defined as $S_\mu(re^{i\theta}) = S_\mu(r)e^{i\theta}$; see Remark 2.5 below.)

In more general cases, especially when K is not diagonal with respect to the φ_γ-basis, it is not as straightforward to minimize (1.3). Minimizers can be obtained by an iterative procedure proposed and applied in, e.g., [11, 24, 35, 38, 39]. We shall discuss an appropriate functional framework in which this iterative approach arises naturally and use it to prove convergence in norm of the iterates. To our knowledge this is the first such proof.

An approach that promotes sparsity with respect to a particular basis makes sense only if we know that the objects f that we want to reconstruct do indeed have a sparse expansion with respect to this basis. In the next section we list some situations in which this is the case and to which the algorithm that we propose in this paper could be applied.

1.4 Possible Applications for Sparsity-Promoting Constraints

1.4.1 Sparse Wavelet Expansions

This is the application that was the primary motivation for this paper. Wavelets provide orthonormal bases of $L^2(\mathbb{R}^d)$ with localization in space and in scale; this makes them more suitable than, e.g., Fourier expansions for an efficient representation of functions that have space-varying smoothness properties. Appendix A gives a very succinct overview of wavelets and their link with a particular family of smoothness spaces, the Besov spaces. Essentially, the Besov space $B^s_{p,q}(\mathbb{R}^d)$ is a space of functions on $\mathbb{R}^d$ that "have s derivatives in $L^p(\mathbb{R}^d)$"; the index q provides some extra fine-tuning. The precise definition involves the moduli of continuity of the function, defined by finite differencing, instead of derivatives, and combines the behavior of these moduli at different scales. The Besov space $B^s_{p,q}(\mathbb{R}^d)$ is well-defined as a complete metric space even if the indices $p,\ q \in (0,\infty)$ are < 1, although it is no longer a Banach space in this case. Functions that are mostly smooth but that have a few local "irregularities" nevertheless can still belong to a

Besov space with high smoothness index. For instance, the one-dimensional function $F(x) = \operatorname{sign}(x)e^{-x^2}$ can belong to $B^s_{p,q}(\mathbb{R})$ for arbitrarily large s provided $0 < p < (s + \frac{1}{2})^{-1}$. (Note that this same example does not belong to any of the Sobolev spaces $W^s_p(\mathbb{R})$ with $s > 0$, mainly because these can be defined only for $p \geq 1$.) Wavelets provide unconditional bases for the Besov spaces, and one can express whether a function f on $\mathbb{R}^d$ belongs to a Besov space by a fairly simple and completely explicit requirement on the absolute values of the wavelet coefficients of f. This expression becomes particularly simple when $p = q$; as reviewed in Appendix A, $f \in B^s_{p,p}(\mathbb{R})$ if and only if

$$|||f|||_{s,p} = \Big(\sum_{\lambda\in\Lambda} 2^{\sigma p|\lambda|} |\langle f, \Psi_\lambda\rangle|^p\Big)^{\frac{1}{p}} < \infty\,,$$

where σ depends on s and p and is defined by $\sigma = s + d(\frac{1}{2} - \frac{1}{p})$, and where $|\lambda|$ stands for the scale of the wavelet Ψ_λ. (The $\frac{1}{2}$ in the formula for σ is due to the choice of normalization of the Ψ_λ, $\|\Psi_\lambda\|_{L^2} = 1$.) For $p = q \geq 1$, $|||\cdot|||_{s,p}$ is an equivalent norm to the standard Besov norm on $B^s_{p,q}(\mathbb{R}^d)$; we shall restrict ourselves to this case in this paper.

It follows that minimizing the variational functional for an inverse problem with a Besov space prior constraint falls exactly within the category of problems studied in this paper: for such an inverse problem, with operator K and with the a priori knowledge that the object lies in some $B^s_{p,p}$, it is natural to define the variational functional to be minimized by

$$\Delta(f) + |||f|||^p_{s,p} = \|Kf - g\|^2 + \sum_{\lambda\in\Lambda} 2^{\sigma p|\lambda|} |\langle f, \Psi_\lambda\rangle|^p\,,$$

which is exactly of the type $\Phi_{\mathbf{w},p}(f)$, as defined in (1.3). For the case where K is the identity operator, it was noted already in [7] that the wavelet-based algorithm for the denoising of data with a Besov prior, derived earlier in [19], amounts exactly to the minimization of $\Phi_{\mu\mathbf{w}_0,1}(f)$, where K is the identity operator and the φ_γ-basis is a wavelet basis; the denoised approximant given in [19] then coincides exactly with (1.4)–(1.5).

It should be noted that if $d > 1$, and if we are interested in functions that are mostly smooth, with possible jump discontinuities (or other "irregularities") on smooth manifolds of dimension 1 or higher (i.e., not point irregularities), then the Besov spaces do not constitute the optimal smoothness space hierarchy. For $d = 2$, for instance, functions f that are C^∞ on the square $[0, 1]^2$, except on a finite set of smooth curves, belong to $B^1_{1,1}([0, 1]^2)$ but not to $B^s_{1,1}([0, 1]^2)$ for $s > 1$. In order to obtain more efficient (sparser) expansions of this type of function, other expansions have to be used, employing, e.g., ridgelets or curvelets [6, 18]. One can then again use the iterative approach discussed in this paper with respect to these more adapted bases [38].

1.4.2 Other Orthogonal Expansions

The framework of this paper applies to enforcing sparsity of the expansion of the solution on any orthonormal basis. We provide here three (nonwavelet) examples that are particularly relevant for applications, but this is of course not limitative.

The first example is the case where it is known a priori that the object to be recovered is sparse in the Fourier domain; i.e., f has only a few nonzero Fourier components. It then makes sense to choose a standard Fourier basis for the φ_γ and to apply the algorithms explained later in this paper. (They would have to be adapted to deal with complex functions, but this is easily done; see Remark 2.5 below.) In the case where K is the identity operator, this is a classical problem, sometimes referred to as "tracking sinusoids drowned in noise," for which many other algorithms have been developed.

For other applications, objects are naturally sparse in the original (space or time) domain. Then the same framework can be used again if we expand such objects in a basis formed by the characteristic functions of pixels or voxels. Once the inverse problem is discretized in pixel space, it is regularized by penalizing the ℓ^p-norm of the object with $1 \leq p \leq 2$. Possible applications include the restoration of astronomical images with scattered stars on a flat background. Objects formed by a few spikes are also typical of some inverse problems arising in spectroscopy or in particle sizing. In medical imaging, ℓ^p-norm penalization with p larger than but close to 1 has been used for the imaging of tiny blood vessels [31].

The third example refers to the case where K is compact and the use of SVD expansions is a viable computational approach, such as for solving relatively small-scale problems or for operators that can be diagonalized in an analytic way. As already stressed above, the linear regularization methods as, for example, the one given by (1.2) have the drawback that the penalization or cutoff eliminates the components corresponding to the smallest singular values independently of the type of data. In some instances, the most significant coefficients of the object may not correspond to the largest singular values; it may then happen that the object possesses significant coefficients beyond the cutoff imposed by linear methods. In order to avoid the elimination of such coefficients, it is preferable to use instead a nonlinear regularization analogous to (1.4)–(1.5), with basis functions φ_γ replaced by the singular vectors v_k. The theorems in this paper show that the *thresholded SVD expansion*

$$f^* = \sum_{k=1}^{+\infty} S_{\mu/\sigma_k^2}\left(\frac{\langle g, u_k\rangle}{\sigma_k}\right) v_k = \sum_{k=1}^{+\infty} \frac{1}{\sigma_k^2} S_\mu(\sigma_k \langle g, u_k\rangle) v_k\,,$$

which is the minimizer of the functional (1.3) with $\mathbf{w} = \mu \mathbf{w}_0$ and $p = 1$, provides a regularized solution that is better adapted to these situations.

1.4.3 Frame Expansions

In a Hilbert space $\mathcal{H}$, a frame $\{\psi_n\}_{n\in\mathbb{N}}$ is a set of vectors for which there exist constants $A, B > 0$ so that, for all $v \in \mathcal{H}$,

$$B^{-1}\sum_{n\in\mathbb{N}} |\langle v, \psi_n\rangle|^2 \leq \|v\|^2 \leq A^{-1}\sum_{n\in\mathbb{N}} |\langle v, \psi_n\rangle|^2 .$$

Frames always span the whole space $\mathcal{H}$, but the frame vectors ψ_n are typically not linearly independent. Frames were first proposed by Duffin and Schaeffer in [21]; they are now used in a wide range of applications. For particular choices of the frame vectors, the two frame bounds A and B are equal; one then has, for all $v \in \mathcal{H}$,

$$v = A^{-1}\sum_{n\in\mathbb{N}} \langle v, \psi_n\rangle \psi_n . \tag{1.6}$$

In this case, the frame is called *tight*. An easy example of a frame is given by taking the union of two (or more) different orthonormal bases in $\mathcal{H}$; these unions always constitute tight frames, with $A = B$ equal to the number of orthonormal bases used in the union.

Frames are typically "overcomplete"; i.e., they still span all of $\mathcal{H}$ even if some frame vectors are removed. It follows that, given a vector v in $\mathcal{H}$, one can find many different sequences of coefficients such that

$$v = \sum_{n\in\mathbb{N}} z_n \psi_n . \tag{1.7}$$

Among these sequences, some have special properties for which they are preferred. There is, for instance, a standard procedure to find the unique sequence with minimal ℓ^2-norm; if the frame is tight, then this sequence is given by $z_n = A^{-1}\langle v, \psi_n\rangle$, as in (1.6).

The problem of finding sequences $\mathbf{z} = (z_n)_{n\in\mathbb{N}}$ that satisfy (1.7) can be considered as an inverse problem. Let us define the operator K from $\ell^2(\mathbb{N})$ to $\mathcal{H}$ that maps a sequence $\mathbf{z} = (z_n)_{n\in\mathbb{N}}$ to the element $K\mathbf{z}$ of $\mathcal{H}$ by

$$K\mathbf{z} = \sum_{n\in\mathbb{N}} z_n \psi_n .$$

Then solving (1.7) amounts to solving $K\mathbf{z} = v$. Note that this operator K is associated with, but not identical to, what is often called the "frame operator." One has, for $v \in \mathcal{H}$,

$$KK^*v = \sum_{n\in\mathbb{N}} \langle v, \psi_n\rangle \psi_n ;$$

for $\mathbf{z} \in \ell^2$, the sequence $K^*K\mathbf{z}$ is given by

$$(K^*K\mathbf{z})_k = \sum_{l\in\mathbb{N}} z_l \langle \psi_l, \psi_k\rangle .$$

In this framework, the sequence $\mathbf{z}$ of minimum ℓ^2-norm that satisfies (1.7) is given simply by $\mathbf{z}^\dagger = K^\dagger v$. The standard procedure in frame lore for constructing $\mathbf{z}^\dagger$ can be rewritten as $\mathbf{z}^\dagger = K^*(KK^*)^{-1}v$, so that $K^\dagger = K^*(KK^*)^{-1}$ in this case. This last formula holds because this inverse problem is in fact well-posed: even though $\mathrm{N}(K) \neq \{0\}$, there is a gap in the spectrum of K^*K between the eigenvalue 0 and the remainder of the spectrum, which is contained in the interval $[A, B]$; the operator KK^* has its spectrum completely within $[A, B]$. In practice, one always works with frames for which the ratio B/A is reasonably close to 1, so that the problem is not only well-posed but also well-conditioned.

It is often of interest, however, to find sequences that are sparser than $\mathbf{z}^\dagger$. For instance, one may know a priori that v is a "noisy" version of a linear combination of ψ_n with a coefficient sequence of small ℓ^1-norm. In this case, it makes sense to determine a sequence $\mathbf{z}_\mu$ that minimizes

$$\|K\mathbf{z} - v\|^2_{\mathcal{H}} + \mu\|\mathbf{z}\|_{\ell^1}\,, \tag{1.8}$$

a problem that falls exactly in the category of problems described in Section 1.3. Note that although the inverse problem for K from $\ell^2(\mathbb{N})$ to $\mathcal{H}$ is well-defined, this need not be the case with the restriction $K|_{\ell^1}$ from $\ell^1(\mathbb{N})$ to $\mathcal{H}$. One can indeed find tight frames for which $\sup\{\|\mathbf{z}\|_{\ell^1} : \mathbf{z} \in \ell^1 \text{ and } \|K\mathbf{z}\| \leq 1\} = \infty$, so that for arbitrarily large R and arbitrarily small ϵ, one can find $\tilde{v} \in \mathcal{H}$, $\tilde{\mathbf{z}} \in \ell^1$, with $\|\tilde{v} - K\tilde{\mathbf{z}}\| = \epsilon$, yet $\inf\{\|\mathbf{z}\|_{\ell^1} : \|\tilde{v} - K\mathbf{z}\| \leq \frac{\epsilon}{2}\} \geq R\|\tilde{\mathbf{z}}\|_{\ell^1}$. In a noisy situation it therefore may not make sense to search for the sequence with minimal ℓ^1-norm that is "closest" to v; to find an estimate of the ℓ^1-sequences of which a given v is known to be a small perturbation, a better strategy is to compute the minimizer z_μ of (1.8).

Minimizing the functional (1.8) as an approach to obtain sequences that provide sparse approximations $K\mathbf{z}$ to v was proposed and applied to various problems by Chen, Donoho, and Saunders [8]; in the statistical literature, least-squares regression with ℓ^1-penalty has become known as the "lasso" [40]. Note that whereas [8] used quadratic programming methods to minimize (1.3), [38] attacks the problem by the iterative approach for which we prove convergence in this paper. Another block iterative technique has been proposed in [37] for a dictionary formed by the union of several orthonormal bases.

1.5 Summary of Our Approach and Results

Given an operator K from $\mathcal{H}$ to itself (or, more generally, from $\mathcal{H}$ to $\mathcal{H}'$) and an orthonormal basis $(\varphi_\gamma)_{\gamma\in\Gamma}$, our goal is to find minimizing $f^\star$ for the functionals $\Phi_{\mathbf{w},p}$ defined in Section 1.3. The corresponding variational equations are

$$\forall\gamma \in \Gamma, \quad \langle K^*Kf, \varphi_\gamma\rangle - \langle K^*g, \varphi_\gamma\rangle + \frac{w_\gamma p}{2}|\langle f, \varphi_\gamma\rangle|^{p-1}\mathrm{sign}(\langle f, \varphi_\gamma\rangle) = 0\,. \tag{1.9}$$

When $p \neq 2$ and K is not diagonal in the φ_γ-basis, this gives a coupled system of nonlinear equations for the $\langle f, \varphi_\gamma\rangle$. To solve this system, we introduce in Section 2 a sequence of "surrogate" functionals that are all easy to minimize, and for which we expect, by a heuristic argument, that the successive minimizers have our desired $f^\star$ as a limit. The formula that gives each minimizer as a function of the previous one is the same as the algorithm proposed in [11, 38, 39] and derived in the framework of an expectation-maximization (EM) approach to a maximum penalized likelihood solution in [24, 35].

The two main contributions of this paper are presented in Sections 3 and 4. In Section 3 we show that the successive minimizers of the "surrogate" functionals defined in Section 2 do indeed converge to $f^\star$; we first establish weak convergence, but conclude the section by proving that the convergence also holds in norm. Next, in Section 4, we show that this iterative method leads to a *stable* solution in the sense given in Section 1.2: if we apply the algorithm to data that are a small perturbation of a "true image" Kf_o, then the algorithm will produce $f^\star$ that converge to f_o as the norm of the perturbation tends to 0.

These results will be established for the general case $1 \le p \le 2$, but it may be enlightening to summarize here the two main theorems in the specific case $p = 1$. Let K be a bounded linear operator from $\mathcal{H}$ to $\mathcal{H}'$, with norm strictly bounded by 1 and with a trivial null space, let g be an element of $\mathcal{H}'$, and consider the functional

$$\Phi_\mu(f) = \|Kf - g\|^2 + \mu \sum_{\gamma\in\Gamma} |\langle f, \varphi_\gamma\rangle| \,. \tag{1.10}$$

Then:

(1) Consider the sequence of iterates

$$f^n = \mathbf{S}_\mu(f^{n-1} + K^*(g - Kf^{n-1}))\,, \quad n = 1, 2, \dots,$$

where the nonlinear operator $\mathbf{S}_\mu$ is defined componentwise by

$$\mathbf{S}_\mu(g) = \sum_{\gamma\in\Gamma} S_\mu(\langle g, \varphi_\gamma\rangle)\varphi_\gamma \tag{1.11}$$

with S_μ the soft-thresholding function defined in (1.5), and with f^0 arbitrarily chosen in $\mathcal{H}$. Then f^n converges strongly to the unique minimizer $f^\star_\mu$ of the functional Φ_μ.

(2) If $\mu = \mu(\epsilon)$ satisfies the requirements

$$\lim_{\epsilon\to 0} \mu(\epsilon) = 0 \quad \text{and} \quad \lim_{\epsilon\to 0} \frac{\epsilon^2}{\mu(\epsilon)} = 0\,, \tag{1.12}$$

then we have, for any $f_o \in \mathcal{H}$,

$$\lim_{\epsilon\to 0} \Big[\sup_{\|g - Kf_o\| \le \epsilon} \|f^\star_{\mu(\epsilon)} - f_o\| \Big] = 0\,. \tag{1.13}$$

1.6 Related Work

Exploiting the sparsity of the expansion on a given basis of an unknown signal, in order to assist in the estimation or approximation of the signal from noisy data, is an idea that has arisen in many different approaches and applications. The key role played by sparsity to achieve superresolution in diffraction-limited imaging was already emphasized by Donoho [15] more than a decade ago. Since the seminal paper by Donoho and Johnstone [19], the use of thresholding techniques for sparsifying the wavelet expansions of noisy signals in order to remove the noise (the so-called "denoising" problem) has been abundantly discussed in the literature, mainly in a statistical framework (see, e.g., the book [33]). Of particular importance for the background of this paper is the article by Chambolle et al. [7], which provides a variational formulation for denoising through the use of penalties on a Besov norm of the signal; this is the perspective adopted in the present paper.

Several approaches have been proposed to generalize the denoising framework to solve inverse problems. A first approach was to construct wavelet or "wavelet-inspired" bases that are in some sense adapted to the operator to be inverted. The wavelet-vaguelette decomposition (WVD) proposed by Donoho [17], as well as the twin vaguelette-wavelet decomposition method [1], and also the deconvolution in mirror wavelet bases [27, 33] can all be viewed as examples of this strategy. For the inversion of the Radon transform, Lee and Lucier [30] formulated a generalization of the WVD decomposition that uses a variational approach to set thresholding levels. A drawback of these methods is that they are limited to special types of operators K (essentially convolution-type operators under some additional technical assumptions).

Other papers have explored the application of Galerkin-type methods to inverse problems, using an appropriate but fixed wavelet basis [10, 14, 32]. The underlying intuition is again that if the operator lends itself to a fairly sparse representation in wavelets, e.g., if it is an operator of the type considered in [5], and if the object is mostly smooth with some singularities, then the inversion of the truncated operator will not be too onerous, and the approximate representation of the object will do a good job of capturing the singularities. In [10] the method is made adaptive, so that the finer-scale wavelets are used where lower scales indicate the presence of singularities.

The mathematical framework in this paper has the advantage of not presupposing any particular properties for the operator K (other than boundedness) or the basis $(\varphi_\gamma)_{\gamma\in\Gamma}$ (other than its orthonormality). We prove, in complete generality, that generalizing Tikhonov's regularization method from the ℓ^2-penalty case to a ℓ^1-penalty (or, more generally, a weighted ℓ^p-penalty with $1 \le p \le 2$) provides a proper regularization method for ill-posed problems in a Hilbert space $\mathcal{H}$, with estimates that are independent of the dimension of $\mathcal{H}$ (and are thus valid for infinite-dimensional separable $\mathcal{H}$). To our knowledge, this is the first proof of this fact. Moreover, we show that the Landweber-type iterative algorithm used in [11, 24, 35, 38, 39], which involves a denoising procedure at each iteration step, provides a sequence of approximations converging in norm to the variational minimizer, and we give estimates of the rate of convergence in particular cases.

2 The Iterative Algorithm: A Derivation from Surrogate Functionals

It is the combined presence of K^*Kf (which couples all the equations) and the nonlinearity of the equations that makes system (1.9) unpleasant. For this reason, we borrow a technique of optimization transfer (see, e.g., [12, 29]) and construct surrogate functionals that effectively remove the term K^*Kf. We first pick a constant C so that $\|K^*K\| < C$, and then we define the functional $\Xi(f;a) = C\|f-a\|^2 - \|Kf - Ka\|^2$, which depends on an auxiliary element a of $\mathcal{H}$. Because $CI - K^*K$ is a strictly positive operator, $\Xi(f;a)$ is strictly convex in f for any choice of a. If $\|K\| < 1$, we are allowed to set $C = 1$; for simplicity, we will restrict ourselves to this case, without loss of generality since K can always be renormalized. We then add $\Xi(f;a)$ to $\Phi_{\mathbf{w},p}(f)$ to form the following "surrogate functional":

$$
\begin{aligned}
\Phi^{\mathrm{SUR}}_{\mathbf{w},p}(f;a) &= \Phi_{\mathbf{w},p}(f) - \|Kf - Ka\|^2 + \|f-a\|^2 \\
&= \|Kf - g\|^2 \\
&\quad + \sum_{\gamma\in\Gamma} w_\gamma |\langle f, \varphi_\gamma\rangle|^p - \|Kf - Ka\|^2 + \|f-a\|^2 \\
&= \|f\|^2 - 2\langle f, a + K^*g - K^*Ka\rangle \\
&\quad + \sum_{\gamma} w_\gamma |\langle f, \varphi_\gamma\rangle|^p + \|g\|^2 + \|a\|^2 - \|Ka\|^2 \\
&= \sum_{\gamma} \left[f_\gamma^2 - 2f_\gamma (a + K^*g - K^*Ka)_\gamma + w_\gamma |f_\gamma|^p \right] \\
&\quad + \|g\|^2 + \|a\|^2 - \|Ka\|^2
\end{aligned}
\tag{2.1}
$$

where we have again used the shorthand v_γ for $\langle v, \varphi_\gamma\rangle$ and implicitly assumed that we are dealing with real functions only. Since $\Xi(f;a)$ is strictly convex in f, $\Phi^{\mathrm{SUR}}_{\mathbf{w},p}(f;a)$ is also strictly convex in f and has a unique minimizer for any choice of a. The advantage of minimizing (2.1) in place of (1.9) is that the variational equations for the f_γ decouple. We can then try to approach the minimizer of $\Phi_{\mathbf{w},p}(f)$ by an iterative process that goes as follows: starting from an arbitrarily chosen f^0, we determine the minimizer f^1 of (2.1) for $a = f^0$; each successive iterate f^n is then the minimizer for f of the surrogate functional (2.1) anchored at the previous iterate, i.e., for $a = f^{n-1}$. The iterative algorithm thus goes as follows:

$$
f^0 \text{ arbitrary; } f^n = \text{arg-min}\left(\Phi^{\mathrm{SUR}}_{\mathbf{w},p}(f; f^{n-1})\right), \quad n = 1, 2, \ldots. \tag{2.2}
$$

To gain some insight into this iteration, let us first focus on two special cases.

In the case where $\mathbf{w} = \mathbf{0}$ (i.e., the functional $\Phi_{\mathbf{w},p}$ reduces to the discrepancy only), one needs to minimize

$$\Phi^{\mathrm{SUR}}_{\mathbf{0},p}(f; f^{n-1}) = \|f\|^2 - 2\langle f, f^{n-1} + K^*(g - Kf^{n-1})\rangle + \|g\|^2 + \|f^{n-1}\|^2 - \|Kf^{n-1}\|^2 ;$$

this leads to

$$f^n = f^{n-1} + K^*(g - Kf^{n-1}) .$$

This is nothing else than the standard Landweber iterative method, the convergence of which to the (generalized) solution of $Kf = g$ is well-known ([28]; see also [3, 23]).

In the case where $\mathbf{w} = \mu \mathbf{w_0}$ and $p = 2$, the n^{th} surrogate functional reduces to

$$\Phi^{\mathrm{SUR}}_{\mathbf{w},2}(f; f^{n-1}) = (1+\mu)\|f\|^2 - 2\langle f, f^{n-1} + K^*(g - Kf^{n-1})\rangle + \|g\|^2 + \|f^{n-1}\|^2 - \|Kf^{n-1}\|^2 ;$$

the minimizer is now

$$f^n = \frac{1}{1+\mu}[f^{n-1} + K^*(g - Kf^{n-1})] , \tag{2.3}$$

i.e., we obtain a damped or regularized Landweber iteration (see, e.g., [3]). The convergence of the function f^n defined by (2.3) follows immediately from the estimate $\|f^{n+1} - f^n\| = (1+\mu)^{-1}\|(I - K^*K)(f^n - f^{n-1})\| \le (1+\mu)^{-1}\|f^n - f^{n-1}\|$, showing that we have a contractive mapping, even if $\mathrm{N}(K) \neq \{0\}$.

In these two special cases we thus find that the f^n converge as $n \to \infty$. This permits one to hope that the f^n will converge for general $\mathbf{w}$ and p as well; whenever this is the case, the difference $\|f^n - f^{n-1}\|^2 - \|K(f^n - f^{n-1})\|^2$ between $\Phi^{\mathrm{SUR}}_{\mathbf{w},p}(f^n; f^{n-1})$ and $\Phi_{\mathbf{w},p}(f^n)$ tends to 0 as $n \to \infty$, suggesting that the minimizer f^n for the first functional could well tend to a minimizer $f^\star$ of the second. In Section 3 we shall see that all this is more than a pipedream; i.e., we shall prove that the f^n do indeed converge to a minimizer of $\Phi_{\mathbf{w},p}$.

In the remainder of this section we derive an explicit formula for the computation of the successive f^n. We first discuss the minimization of the functional (2.1) for a generic $a \in \mathcal{H}$. As noted above, the variational equations for the f_γ decouple. For $p > 1$, the summand in (2.1) is differentiable in f_γ, and the minimization reduces to solving the variational equation

$$2f_\gamma + pw_\gamma \operatorname{sign}(f_\gamma)|f_\gamma|^{p-1} = 2(a_\gamma + [K^*(g - Ka)]_\gamma) ;$$

since for any $w \ge 0$ and any $p > 1$, the real function

$$F_{w,p}(x) = x + \frac{wp}{2}\operatorname{sign}(x)|x|^{p-1} \tag{2.4}$$

is a one-to-one map from $\mathbb{R}$ to itself, we thus find that the minimizer of (2.1) satisfies

$$f_\gamma = S_{w_\gamma,p}(a_\gamma + [K^*(g - Ka)]_\gamma) , \tag{2.5}$$

where $S_{w,p}$ is defined by

$$S_{w,p} = (F_{w,p})^{-1} \quad \text{for } p > 1\,. \tag{2.6}$$

When $p = 1$, the summand of (2.1) is differentiable in f_γ only if $f_\gamma \neq 0$; except at the point of nondifferentiability, the variational equation now reduces to

$$2f_\gamma + w_\gamma \operatorname{sign}(f_\gamma) = 2(a_\gamma + [K^*(g - Ka)]_\gamma)\,.$$

For $f_\gamma > 0$, this leads to $f_\gamma = a_\gamma + [K^*(g - Ka)]_\gamma - w_\gamma/2$; for consistency we must impose in this case that $a_\gamma + [K^*(g - Ka)]_\gamma > w_\gamma/2$. For $f_\gamma < 0$, we obtain $f_\gamma = a_\gamma + [K^*(g-Ka)]_\gamma + w_\gamma/2$, valid only when $a_\gamma + [K^*(g-Ka)]_\gamma < -w_\gamma/2$. When $a_\gamma + [K^*(g - Ka)]_\gamma$ does not satisfy either of the two conditions, i.e., when $|a_\gamma + [K^*(g - Ka)]_\gamma| \le w_\gamma/2$, we put $f_\gamma = 0$. Summarizing,

$$f_\gamma = S_{w_\gamma,1}(a_\gamma + [K^*(g - Ka)]_\gamma)\,, \tag{2.7}$$

where the function $S_{w,1}$ from $\mathbb{R}$ to itself is defined by

$$S_{w,1}(x) = \begin{cases} x - \frac{w}{2} & \text{if } x \ge \frac{w}{2} \\ 0 & \text{if } |x| < \frac{w}{2} \\ x + \frac{w}{2} & \text{if } x \le -\frac{w}{2}. \end{cases} \tag{2.8}$$

(Note that this is the same nonlinear function encountered earlier in Section 1.3, in definition (1.5).)

The following proposition summarizes our findings and proves (the case $p = 1$ is not conclusively proven by the variational equations above) that we have indeed found the minimizer of $\Phi^{\mathrm{SUR}}_{\mathbf{w},p}(f; a)$:

PROPOSITION 2.1 *Suppose the operator K maps a Hilbert space $\mathcal{H}$ to another Hilbert space $\mathcal{H}'$, with $\|K^*K\| < 1$, and suppose g is an element of $\mathcal{H}'$. Let $(\varphi_\gamma)_{\gamma\in\Gamma}$ be an orthonormal basis for $\mathcal{H}$, and let $\mathbf{w} = (w_\gamma)_{\gamma\in\Gamma}$ be a sequence of strictly positive numbers. Pick arbitrary $p \ge 1$ and $a \in \mathcal{H}$. Define the functional $\Phi^{\mathrm{SUR}}_{\mathbf{w},p}(f; a)$ on $\mathcal{H}$ by*

$$\Phi^{\mathrm{SUR}}_{\mathbf{w},p}(f; a) = \|Kf - g\|^2 + \sum_{\gamma\in\Gamma} w_\gamma |f_\gamma|^p + \|f - a\|^2 - \|K(f - a)\|^2\,.$$

Then $\Phi^{\mathrm{SUR}}_{\mathbf{w},p}(f; a)$ has a unique minimizer in $\mathcal{H}$. This minimizer is given by $f = \mathbf{S}_{\mathbf{w},p}(a + K^*(g - Ka))$, where the operators $\mathbf{S}_{\mathbf{w},p}$ are defined by

$$\mathbf{S}_{\mathbf{w},p}(h) = \sum_\gamma S_{w_\gamma,p}(h_\gamma)\varphi_\gamma\,, \tag{2.9}$$

with the functions $S_{w,p}$ from $\mathbb{R}$ to itself given by (2.6) *and* (2.8). *For all $h \in \mathcal{H}$, one has*

$$\Phi^{\mathrm{SUR}}_{\mathbf{w},p}(f + h; a) \ge \Phi^{\mathrm{SUR}}_{\mathbf{w},p}(f; a) + \|h\|^2\,.$$

PROOF: The cases $p > 1$ and $p = 1$ should be treated slightly differently. We discuss here only the case $p = 1$; the simpler case $p > 1$ is left to the reader.

Take $f' = f + h$, where f is as defined in the proposition, and $h \in \mathcal{H}$ is arbitrary. Then

$$\Phi^{\text{SUR}}_{\mathbf{w},1}(f + h; a) = \Phi^{\text{SUR}}_{\mathbf{w},1}(f; a) + 2\langle h, f - a - K^*(g - Ka)\rangle + \sum_{\gamma\in\Gamma} w_\gamma(|f_\gamma + h_\gamma| - |f_\gamma|) + \|h\|^2 .$$

Define now $\Gamma_0 = \{\gamma \in \Gamma : f_\gamma = 0\}$, and $\Gamma_1 = \Gamma \setminus \Gamma_0$. Substituting the explicit expression (2.7) for the f_γ, we then have

$$\begin{aligned}\Phi^{\text{SUR}}_{\mathbf{w},1}&(f + h; a) - \Phi^{\text{SUR}}_{\mathbf{w},1}(f; a)\\ &= \|h\|^2 + \sum_{\gamma\in\Gamma_0}\left[w_\gamma|h_\gamma| - 2h_\gamma(a_\gamma + [K^*(g - Ka)]_\gamma)\right]\\ &\quad + \sum_{\gamma\in\Gamma_1}\left(w_\gamma|f_\gamma + h_\gamma| - w_\gamma|f_\gamma| + h_\gamma[-w_\gamma \operatorname{sign}(f_\gamma)]\right).\end{aligned}$$

For $\gamma \in \Gamma_0$, $2|a_\gamma + [K^*(g - Ka)]_\gamma| \le w_\gamma$, so that $w_\gamma|h_\gamma| - 2h_\gamma(a_\gamma + [K^*(g - Ka)]_\gamma) \ge 0$.

If $\gamma \in \Gamma_1$, we distinguish two cases, according to the sign of f_γ. If $f_\gamma > 0$, then

$$w_\gamma|f_\gamma + h_\gamma| - w_\gamma|f_\gamma| + h_\gamma[-w_\gamma \operatorname{sign}(f_\gamma)] = w_\gamma[|f_\gamma + h_\gamma| - (f_\gamma + h_\gamma)] \ge 0 .$$

If $f_\gamma < 0$, then

$$w_\gamma|f_\gamma + h_\gamma| - w_\gamma|f_\gamma| + h_\gamma[-w_\gamma \operatorname{sign}(f_\gamma)] = w_\gamma[|f_\gamma + h_\gamma| + (f_\gamma + h_\gamma)] \ge 0 .$$

It follows that $\Phi^{\text{SUR}}_{\mathbf{w},1}(f+h; a)-\Phi^{\text{SUR}}_{\mathbf{w},1}(f; a) \ge \|h\|^2$, which proves the proposition. □

For later reference it is useful to point out the following:

LEMMA 2.2 *The operators $\mathbf{S}_{\mathbf{w},p}$ are nonexpansive, i.e.,*

$$\forall v, v' \in \mathcal{H}, \quad \|\mathbf{S}_{\mathbf{w},p}v - \mathbf{S}_{\mathbf{w},p}v'\| \le \|v - v'\| .$$

PROOF: As shown by (2.9),

$$\|\mathbf{S}_{\mathbf{w},p}v - \mathbf{S}_{\mathbf{w},p}v'\|^2 = \sum_{\gamma\in\Gamma} |S_{w_\gamma,p}(v_\gamma) - S_{w_\gamma,p}(v'_\gamma)|^2 ,$$

which means that it suffices to show that, $\forall x, x' \in \mathbb{R}$, and all $w \ge 0$,

$$|S_{w,p}(x) - S_{w,p}(x')| \le |x - x'| . \tag{2.10}$$

If $p > 1$, then $S_{w,p}$ is the inverse of the function $F_{w,p}$; since $F_{w,p}$ is differentiable with derivative uniformly bounded below by 1, (2.10) follows immediately in this case.

If $p = 1$, then $S_{w,1}$ is not differentiable in $x = \frac{w}{2}$ or $x = -\frac{w}{2}$, and another argument must be used. For the sake of definiteness, let us assume $x \geq x'$. We will just check all the possible cases. If x and x' have the same sign and $|x|,\ |x'| \geq \frac{w}{2}$, then $|S_{w,p}(x) - S_{w,p}(x')| = |x - x'|$. If $x' \leq -\frac{w}{2}$ and $x \geq \frac{w}{2}$, then $|S_{w,p}(x) - S_{w,p}(x')| = x + |x'| - w < |x - x'|$. If $x \geq \frac{w}{2}$ and $|x'| < \frac{w}{2}$, then $|S_{w,p}(x) - S_{w,p}(x')| = x - \frac{w}{2} < |x - x'|$. A symmetric argument applies to the case $|x| < \frac{w}{2}$ and $x' \leq -\frac{w}{2}$. Finally, if both $|x|$ and $|x'|$ are less than $\frac{w}{2}$, we have $|S_{w,p}(x) - S_{w,p}(x')| = 0 \leq |x - x'|$. This establishes (2.10) in all cases. □

Having found the minimizer of a generic $\Phi^{\mathrm{SUR}}_{\mathbf{w},p}(f; a)$, we can apply this to the iteration (2.2), leading to the following:

COROLLARY 2.3 *Let $\mathcal{H}$, $\mathcal{H}'$, K, g, $\mathbf{w}$, and $(\varphi_\gamma)_{\gamma\in\Gamma}$ be as in Proposition* 2.1. *Pick f^0 in $\mathcal{H}$, and define the functions f^n recursively by the algorithm* (2.2). *Then*

$$f^n = \mathbf{S}_{\mathbf{w},p}(f^{n-1} + K^*(g - Kf^{n-1}))\,. \tag{2.11}$$

PROOF: This follows immediately from Proposition 2.1. □

Remark 2.4. In the argument above, we used essentially only two ingredients: the (strict) convexity of $\|f - a\|^2 - \|K(f - a)\|^2$ and the presence of the negative $-\|Kf\|^2$ term in this expression, canceling the $\|Kf\|^2$ in the original functional. We can use this observation to present a slight generalization, in which the identity operator used to upper bound K^*K is replaced by a more general operator D that is diagonal in the φ_γ-basis,

$$D\varphi_\gamma = d_\gamma \varphi_\gamma\,,$$

and that still gives a strict upper bound for K^*K, i.e., satisfies

$$D \geq K^*K + \eta I \quad \text{for some } \eta > 0\,.$$

In this case, the whole construction still carries through, with slight modifications; the successive f^n are now given by

$$f^n_\gamma = S_{w_\gamma/d_\gamma,p}\left(f^{n-1}_\gamma + \frac{[K^*(g - Kf^{n-1})]_\gamma}{d_\gamma}\right).$$

Introducing the notation $\mathbf{w}/\mathbf{d}$ for the sequence $(w_\gamma/d_\gamma)_\gamma$, we can rewrite this as

$$f^n = \mathbf{S}_{\mathbf{w}/\mathbf{d},p}\left(f^{n-1} + D^{-1}[K^*(g - Kf^{n-1})]\right).$$

For the sake of simplicity of notation, we shall restrict ourselves to the case $D = I$.

Remark 2.5. If we deal with complex rather than real functions, and the f_γ, $(K^*g)_\gamma, \ldots,$ are complex quantities, then the derivation of the minimizer of $\Phi^{\mathrm{SUR}}_{\mathbf{w},1}(f; a)$ has to be adapted somewhat. Writing $f_\gamma = r_\gamma e^{i\theta_\gamma}$, with $r_\gamma \geq 0$,

$\theta_\gamma \in [0, 2\pi)$, and likewise $(a + K^*g - K^*Ka)_\gamma = R_\gamma e^{i\Theta_\gamma}$, we find, instead of expression (2.1),

$$\Phi^{\mathrm{SUR}}_{\mathbf{w},p}(f; a) = \sum_\gamma \left[r_\gamma^2 + w_\gamma r_\gamma^p - 2r_\gamma R_\gamma \cos(\theta_\gamma - \Theta_\gamma)\right] + \|g\|^2 + \|a\|^2 - \|Ka\|^2 .$$

Minimizing over $r_\gamma \in [0, \infty)$ and $\theta_\gamma \in [0, 2\pi)$ leads to $\theta_\gamma = \Theta_\gamma$ and $r_\gamma = S_{w_\gamma,p}(R_\gamma)$. If we extend the definition of $S_{w,p}$ to complex arguments by setting $S_{w,p}(re^{i\theta}) = S_{w,p}(r)e^{i\theta}$, then this still leads to $f_\gamma = S_{w_\gamma,p}\left(a_\gamma + [K^*(g - Ka)]_\gamma\right)$, as in (2.5) and (2.7). The arguments of the different proofs still hold for this complex version after minor and straightforward modifications.

Remark 2.6. Iterative formula (2.11) is the same as (17) in [38] or (26) in [24].

3 Convergence of the Iterative Algorithm

In this section we discuss the convergence of the sequence $(f^n)_{n\in\mathbb{N}}$ defined by (2.11). The main result of this section is the following theorem:

THEOREM 3.1 *Let K be a bounded linear operator from $\mathcal{H}$ to $\mathcal{H}'$, with norm strictly bounded by* 1. *Take $p \in [1, 2]$, and let $\mathbf{S}_{\mathbf{w},p}$ be the shrinkage operator defined by* (2.9)*, where the sequence $\mathbf{w} = (w_\gamma)_{\gamma\in\Gamma}$ is uniformly bounded below away from* 0; *i.e., there exists a constant $c > 0$ such that $\forall\gamma \in \Gamma : w_\gamma \geq c$. Then the sequence of iterates*

$$f^n = \mathbf{S}_{\mathbf{w},p}(f^{n-1} + K^*(g - Kf^{n-1})) , \quad n = 1, 2, \ldots ,$$

with f^0 arbitrarily chosen in $\mathcal{H}$, converges strongly to a minimizer of the functional

$$\Phi_{\mathbf{w},p}(f) = \|Kf - g\|^2 + |||f|||^p_{\mathbf{w},p} ,$$

where $|||f|||_{\mathbf{w},p}$ denotes the norm

$$|||f|||_{\mathbf{w},p} = \Big[\sum_{\gamma\in\Gamma} w_\gamma |\langle f, \varphi_\gamma\rangle|^p\Big]^{\frac{1}{p}} , \quad 1 \leq p \leq 2 . \tag{3.1}$$

If either $p > 1$ or $\mathrm{N}(K) = \{0\}$*, then the minimizer $f^\star$ of $\Phi_{\mathbf{w},p}$ is unique, and every sequence of iterates f^n converges strongly to $f^\star$* (*i.e.*, $\|f^n - f^\star\| \to 0$)*, regardless of the choice of f^0.*

This theorem will be proven in several stages. To start, we prove weak convergence, and we establish that the weak limit is indeed a minimizer of $\Phi_{\mathbf{w},p}$. Next, we prove that the convergence holds in the norm topology as well as in the weak topology. To lighten our formulae, we introduce the shorthand notation

$$\mathbf{T}f = \mathbf{S}_{\mathbf{w},p}(f + K^*(g - Kf)) ;$$

with this new notation we have $f^n = \mathbf{T}^n f^0$.

3.1 Weak Convergence of the f^n

To prove weak convergence of the $f^n = \mathbf{T}^n f^0$, we apply the following theorem, due to Opial [36]:

THEOREM 3.2 *Let the mapping* $\mathbf{A}$ *from* $\mathcal{H}$ *to* $\mathcal{H}$ *satisfy the following conditions*:

(i) $\mathbf{A}$ *is nonexpansive*: $\forall v, v' \in \mathcal{H}$, $\|\mathbf{A}v - \mathbf{A}v'\| \le \|v - v'\|$,
(ii) $\mathbf{A}$ *is asymptotically regular*: $\forall v \in \mathcal{H}$, $\|\mathbf{A}^{n+1}v - \mathbf{A}^n v\| \xrightarrow[n\to\infty]{} 0\,$,
(iii) *the set* $\mathcal{F}$ *of the fixed points of* $\mathbf{A}$ *in* $\mathcal{H}$ *is not empty.*

Then, $\forall v \in \mathcal{H}$, *the sequence* $(\mathbf{A}^n v)_{n\in\mathbb{N}}$ *converges weakly to a fixed point in* $\mathcal{F}$.

Opial's original proof can be simplified; we provide the simplified proof (still mainly following Opial's approach) in Appendix B. (The theorem is slightly more general than what is stated in Theorem 3.2 in that the mapping $\mathbf{A}$ need not be defined on all of space; it suffices that it map a closed convex subset of $\mathcal{H}$ to itself; see Appendix B. Reference [36] also contains additional refinements, which we shall not need here.) One of the lemmas stated and proven in the appendix will be invoked in its own right, further below in this section; for the reader's convenience, we state it here in full as well:

LEMMA 3.3 *Suppose the mapping* $\mathbf{A}$ *from* $\mathcal{H}$ *to* $\mathcal{H}$ *satisfies conditions* (i) *and* (ii) *in Theorem* 3.2. *If a subsequence of* $(\mathbf{A}^n v)_{n\in\mathbb{N}}$ *converges weakly in* $\mathcal{H}$, *then its limit is a fixed point of* $\mathbf{A}$.

In order to apply Opial's theorem to our nonlinear operator $\mathbf{T}$, we need to verify that it satisfies the three conditions in Theorem 3.2. We do this in the following series of lemmas. We first have the following:

LEMMA 3.4 *The mapping* $\mathbf{T}$ *is nonexpansive, i.e.,* $\forall v, v' \in \mathcal{H}$,

$$\|\mathbf{T}v - \mathbf{T}v'\| \le \|v - v'\|\,.$$

PROOF: It follows from Lemma 2.2 that the shrinkage operator $\mathbf{S}_{\mathbf{w},p}$ is nonexpansive. Hence we have the following:

$$\begin{aligned}\|\mathbf{T}v - \mathbf{T}v'\| &\le \|(I - K^*K)v - (I - K^*K)v'\| \\ &\le \|I - K^*K\|\;\|v - v'\| \le \|v - v'\|\end{aligned}$$

because we assumed $\|K\| < 1$. □

This verifies that $\mathbf{T}$ satisfies the first condition (i) in Theorem 3.2. To verify the second condition, we first prove some auxiliary lemmas.

LEMMA 3.5 *Both* $(\Phi_{\mathbf{w},p}(f^n))_{n\in\mathbb{N}}$ *and* $(\Phi^{\mathrm{SUR}}_{\mathbf{w},p}(f^{n+1}; f^n))_{n\in\mathbb{N}}$ *are nonincreasing sequences.*

PROOF: For the sake of convenience, we introduce the operator $L = \sqrt{I - K^*K}$, so that $\|h\|^2 - \|Kh\|^2 = \|Lh\|^2$. Because f^{n+1} is the minimizer of the functional $\Phi^{\mathrm{SUR}}_{\mathbf{w},p}(f; f^n)$ and therefore

$$\begin{aligned}\Phi_{\mathbf{w},p}(f^{n+1}) + \|L(f^{n+1} - f^n)\|^2 &= \Phi^{\mathrm{SUR}}_{\mathbf{w},p}(f^{n+1}; f^n)\\ &\leq \Phi^{\mathrm{SUR}}_{\mathbf{w},p}(f^n; f^n) = \Phi_{\mathbf{w},p}(f^n)\,,\end{aligned}$$

we obtain

$$\Phi_{\mathbf{w},p}(f^{n+1}) \leq \Phi_{\mathbf{w},p}(f^n)\,.$$

On the other hand,

$$\begin{aligned}\Phi^{\mathrm{SUR}}_{\mathbf{w},p}(f^{n+2}; f^{n+1}) \leq \Phi_{\mathbf{w},p}(f^{n+1}) &\leq \Phi_{\mathbf{w},p}(f^{n+1}) + \|L(f^{n+1} - f^n)\|^2\\ &= \Phi^{\mathrm{SUR}}_{\mathbf{w},p}(f^{n+1}; f^n)\,.\end{aligned}$$

□

LEMMA 3.6 *Suppose the sequence* $\mathbf{w} = (w_\gamma)_{\gamma\in\Gamma}$ *is uniformly bounded below by a strictly positive number. Then the* $\|f^n\|$ *are bounded uniformly in* n.

PROOF: By Lemma 3.5 we have

$$|||f^n|||^p_{\mathbf{w},p} \leq \Phi_{\mathbf{w},p}(f^n) \leq \Phi_{\mathbf{w},p}(f^0)\,.$$

Hence the f^n are bounded uniformly in the $|||\cdot|||_{\mathbf{w},p}$-norm. Moreover, since $w_\gamma \geq c$, uniformly in γ, for some $c > 0$,

$$\begin{aligned}\|f\|^2 \leq c^{-\frac{2}{p}} \max_{\gamma\in\Gamma}\left[w_\gamma^{(2-p)/p}|f_\gamma|^{2-p}\right]|||f|||^p_{\mathbf{w},p} &\leq c^{-\frac{2}{p}}|||f|||^{2-p}_{\mathbf{w},p}\,|||f|||^p_{\mathbf{w},p}\\ &= c^{-\frac{2}{p}}|||f|||^2_{\mathbf{w},p}\,,\end{aligned}\tag{3.2}$$

and we also have a uniform bound on the $\|f^n\|$. □

LEMMA 3.7 *The series* $\sum_{n=0}^{\infty}\|f^{n+1} - f^n\|^2$ *is convergent.*

PROOF: This is a consequence of the strict positive-definiteness of L, which holds because $\|K\| < 1$. We have, for any $N \in \mathbb{N}$,

$$\sum_{n=0}^{N}\|f^{n+1} - f^n\|^2 \leq \frac{1}{A}\sum_{n=0}^{N}\|L(f^{n+1} - f^n)\|^2$$

where A is a strictly positive lower bound for the spectrum of L^*L. By Lemma 3.5,

$$\begin{aligned}\sum_{n=0}^{N}\|L(f^{n+1} - f^n)\|^2 &\leq \sum_{n=0}^{N}[\Phi_{\mathbf{w},p}(f^n) - \Phi_{\mathbf{w},p}(f^{n+1})]\\ &= \Phi_{\mathbf{w},p}(f^0) - \Phi_{\mathbf{w},p}(f^{N+1}) \leq \Phi_{\mathbf{w},p}(f^0)\,,\end{aligned}$$

where we have used that $(\Phi_{\mathbf{w},p}(f^n))_{n\in\mathbb{N}}$ is a nonincreasing sequence. It follows that $\sum_{n=0}^{N}\|f^{n+1} - f^n\|^2$ is bounded uniformly in N, so that the infinite series converges. □

As an immediate consequence, we have the following:

LEMMA 3.8 *The mapping* $\mathbf{T}$ *is asymptotically regular, i.e.,*

$$\|\mathbf{T}^{n+1} f^0 - \mathbf{T}^n f^0\| = \|f^{n+1} - f^n\| \to 0 \quad \text{for } n \to \infty .$$

We can now establish the following:

PROPOSITION 3.9 *The sequence* $f^n = \mathbf{T}^n f^0$, $n = 1, 2, \dots$, *converges weakly, and its limit is a fixed point for* $\mathbf{T}$.

PROOF: Since, by Lemma 3.6, the $f^n = \mathbf{T}^n f^0$ are uniformly bounded in n, it follows from the Banach-Alaoglu theorem that they have a weak accumulation point. By Lemma 3.3, this weak accumulation point is a fixed point for $\mathbf{T}$. It follows that the set of fixed points of $\mathbf{T}$ is not empty. Since $\mathbf{T}$ is also nonexpansive (by Lemma 3.4) and asymptotically regular (by Lemma 3.8), we can apply Opial's theorem (Theorem 3.1 above), and the conclusion of the proposition follows. □

By the following proposition this fixed point is also a minimizer for the functional $\Phi_{\mathbf{w},p}$.

PROPOSITION 3.10 *A fixed point for* $\mathbf{T}$ *is a minimizer for the functional* $\Phi_{\mathbf{w},p}$.

PROOF: If $f^\star = \mathbf{T} f^\star$, then by Proposition 2.1, we know that $f^\star$ is a minimizer for the surrogate functional $\Phi^{\mathrm{SUR}}_{\mathbf{w},p}(f; f^\star)$, and that $\forall h \in \mathcal{H}$,

$$\Phi^{\mathrm{SUR}}_{\mathbf{w},p}(f^\star + h; f^\star) \geq \Phi^{\mathrm{SUR}}_{\mathbf{w},p}(f^\star; f^\star) + \|h\|^2 .$$

Observing that $\Phi^{\mathrm{SUR}}_{\mathbf{w},p}(f^\star; f^\star) = \Phi_{\mathbf{w},p}(f^\star)$ and

$$\Phi^{\mathrm{SUR}}_{\mathbf{w},p}(f^\star + h; f^\star) = \Phi_{\mathbf{w},p}(f^\star + h) + \|h\|^2 - \|Kh\|^2 ,$$

we conclude that $\forall h \in \mathcal{H}$, $\Phi_{\mathbf{w},p}(f^\star + h) \geq \Phi_{\mathbf{w},p}(f^\star) + \|Kh\|^2$, which shows that $f^\star$ is a minimizer for $\Phi_{\mathbf{w},p}(f)$. □

The following proposition summarizes this subsection.

PROPOSITION 3.11 (Weak Convergence) *Make the same assumptions as in Theorem* 3.1. *Then, for any choice of the initial* f^0, *the sequence* $f^n = \mathbf{T}^n f^0$, $n = 1, 2, \dots$, *converges weakly to a minimizer for* $\Phi_{\mathbf{w},p}$. *If either* $\mathrm{N}(K) = \{0\}$ *or* $p > 1$, *then* $\Phi_{\mathbf{w},p}$ *has a unique minimizer* $f^\star$, *and all the sequences* $(f^n)_{n\in\mathbb{N}}$ *converge weakly to* $f^\star$, *regardless of the choice of* f^0.

PROOF: The only thing that hasn't been proven yet above is the uniqueness of the minimizer if $\mathrm{N}(K) = \{0\}$ or $p > 1$. This uniqueness follows from the observation that $|||f|||_{\mathbf{w},p}$ is strictly convex in f if $p > 1$, and that $\|Kf - g\|^2$ is strictly convex in f if $\mathrm{N}(K) = \{0\}$. In both these cases $\Phi_{\mathbf{w},p}$ is thus strictly convex, so that it has a unique minimizer. □

Remark 3.12. If one has the additional prior information that the object lies in some closed convex subset $\mathcal{C}$ of the Hilbert space $\mathcal{H}$, then the iterative procedure can be adapted to take this into account by replacing the shrinkage operator $\mathbf{S}_{\mathbf{w},p}$ by $\mathbf{P}_\mathcal{C}\mathbf{S}_{\mathbf{w},p}$, where $\mathbf{P}_\mathcal{C}$ is the projector on $\mathcal{C}$. For example, if $\mathcal{H} = L^2$, then $\mathcal{C}$ could be the cone of functions that are positive almost everywhere. The results in this section can be extended to this case; a more general version of Theorem 3.2 can be applied, in which $\mathbf{A}$ need not be defined on all of $\mathcal{H}$, but only on $\mathcal{C} \subset \mathcal{H}$; see Appendix B. We would, however, need to use other tools to ensure or assume outright that the set of fixed points of $\mathbf{T} = \mathbf{P}_\mathcal{C}\mathbf{S}_{\mathbf{w},p}$ is not empty; see also [22].

Remark 3.13. If $\Phi_{\mathbf{w},p}$ is strictly convex, then one can prove the weak convergence more directly, as follows: By the boundedness of the f^n (Lemma 3.6), we must have a weakly convergent subsequence $(f^{n_k})_{k\in\mathbb{N}}$. By Lemma 3.8, the sequence $(f^{n_k+1})_{k\in\mathbb{N}}$ must then also be weakly convergent, with the same weak limit $\widetilde{f}$. It then follows from the equation

$$f_\gamma^{n_k+1} = S_{w_\gamma,p}\left(f_\gamma^{n_k} + [K^*(g - Kf^{n_k})]_\gamma\right),$$

together with $\lim_{k\to\infty} f_\gamma^{n_k} = \lim_{k\to\infty} f_\gamma^{n_k+1} = \widetilde{f}_\gamma$, that $\widetilde{f}$ must be the fixed point $f^\star$ of $\mathbf{T}$. Since this holds for any weak accumulation point of $(f^n)_{n\in\mathbb{N}}$, the weak convergence of $(f^n)_{n\in\mathbb{N}}$ to $f^\star$ follows.

Remark 3.14. The proof of Lemma 3.6 is the only place, so far, where we have explicitly used $p \le 2$. If it were possible to establish a uniform bound on the $\|f^n\|$ by some other means (e.g., by showing that the $\|\mathbf{T}^n f^0\|$ are bounded uniformly in n), then we could dispense with the restriction $p \le 2$, and Proposition 3.11 would hold for all $p \ge 1$.

3.2 Strong Convergence of the f^n

In this section we shall prove that the convergence of the successive iterates $\{f^n\}$ holds not only in the weak topology, but also in the Hilbert space norm. Again, we break up the proof into several lemmas. For the sake of convenience, we introduce the following notation:

$$f^\star = \operatorname*{w-lim}_{n\to\infty} f^n\,, \quad u^n = f^n - f^\star\,, \quad h = f^\star + K^*(g - Kf^\star)\,. \tag{3.3}$$

Here and below, we use the notation w-lim as a shorthand for *weak limit.*

LEMMA 3.15 *$\|Ku^n\| \to 0$ for $n \to \infty$.*

PROOF: Since

$$u^{n+1} - u^n = \mathbf{S}_{\mathbf{w},p}(h + (I - K^*K)u^n) - \mathbf{S}_{\mathbf{w},p}(h) - u^n$$

and $\|u^{n+1} - u^n\| = \|f^{n+1} - f^n\| \to 0$ for $n \to \infty$ by Lemma 3.8, we have

$$\|\mathbf{S}_{\mathbf{w},p}(h + (I - K^*K)u^n) - \mathbf{S}_{\mathbf{w},p}(h) - u^n\| \to 0 \quad \text{for } n \to \infty\,, \tag{3.4}$$

and hence also

$$\max\left(0, \|u^n\| - \|\mathbf{S}_{\mathbf{w},p}(h + (I - K^*K)u^n) - \mathbf{S}_{\mathbf{w},p}(h)\|\right) \to 0 \quad \text{for } n \to \infty\,. \tag{3.5}$$

Since $\mathbf{S}_{\mathbf{w},p}$ is nonexpansive (Lemma 2.2), we have

$$\|\mathbf{S}_{\mathbf{w},p}(h + (I - K^*K)u^n) - \mathbf{S}_{\mathbf{w},p}(h)\| \leq \|(I - K^*K)u^n\| \leq \|u^n\|\,;$$

therefore the "max" in (3.5) can be dropped, and it follows that

$$\|u^n\| - \|(I - K^*K)u^n\| \to 0 \quad \text{for } n \to \infty\,. \tag{3.6}$$

Because

$$\begin{aligned}\|u^n\| + \|(I - K^*K)u^n\| &\leq 2\|u^n\| = 2\|f^n - f^\star\| \\ &\leq 2\big(\|f^\star\| + \sup_k \|f^k\|\big) = C\end{aligned}$$

where C is a finite constant (by Lemma 3.6), we obtain

$$0 \leq \|u^n\|^2 - \|(I - K^*K)u^n\|^2 \leq C(\|u^n\| - \|(I - K^*K)u^n\|)\,,$$

which tends to 0 by (3.6). The inequality

$$\|u^n\|^2 - \|(I - K^*K)u^n\|^2 = 2\|Ku^n\|^2 - \|K^*Ku^n\|^2 \geq \|Ku^n\|^2$$

then implies that $\|Ku^n\|^2 \to 0$ for $n \to \infty$. □

Remark 3.16. Note that if K is a compact operator, the weak convergence to 0 of the u_n automatically implies that $\|Ku_n\|$ tends to 0 as n tends to ∞, so that we don't need Lemma 3.15 in this case.

If K had a bounded inverse, we could conclude from $\|Ku_n\| \to 0$ that $\|u_n\| \to 0$ for $n \to \infty$. However, when K has no bounded inverse (and therefore for all ill-posed linear inverse problems), we need some extra work to show the norm convergence of f^n to $f^\star$.

LEMMA 3.17 *For h given by* (3.3), $\|\mathbf{S}_{\mathbf{w},p}(h+u^n) - \mathbf{S}_{\mathbf{w},p}(h) - u^n\| \to 0$ *for* $n \to \infty$.

PROOF: We have

$$\begin{aligned}&\|\mathbf{S}_{\mathbf{w},p}(h + u^n) - \mathbf{S}_{\mathbf{w},p}(h) - u^n\| \\ &\quad\leq \|\mathbf{S}_{\mathbf{w},p}(h + u^n - K^*Ku^n) - \mathbf{S}_{\mathbf{w},p}(h) - u^n\| \\ &\qquad + \|\mathbf{S}_{\mathbf{w},p}(h + u^n) - \mathbf{S}_{\mathbf{w},p}(h + u^n - K^*Ku^n)\| \\ &\quad\leq \|\mathbf{S}_{\mathbf{w},p}(h + u^n - K^*Ku^n) - \mathbf{S}_{\mathbf{w},p}(h) - u^n\| + \|K^*Ku^n\|\,,\end{aligned}$$

where we used the nonexpansivity of $\mathbf{S}_{\mathbf{w},p}$ (Lemma 2.2). The result follows since both terms in this last bound tend to 0 for $n \to \infty$ because of Lemma 3.15 and statement (3.4). □

LEMMA 3.18 *If for some $a \in \mathcal{H}$ and some sequence* $(v^n)_{n\in\mathbb{N}}$, $\text{w-lim}_{n\to\infty} v^n = 0$ *and* $\lim_{n\to\infty} \|\mathbf{S}_{\mathbf{w},p}(a + v^n) - \mathbf{S}_{\mathbf{w},p}(a) - v^n\| = 0$, *then* $\|v^n\| \to 0$ *for* $n \to \infty$.

PROOF: The argument of the proof is slightly different for the cases $p = 1$ and $p > 1$, and we treat the two cases separately.

We start with $p > 1$. Since the sequence $\{v^n\}$ is weakly convergent, it has to be bounded: there is a constant B such that $\forall n$, $\|v^n\| \leq B$, and hence also $\forall n$, $\forall \gamma \in \Gamma$,

$|v_\gamma^n| \le B$. Next, we define the set $\Gamma_0 = \{\gamma \in \Gamma : |a_\gamma| \ge B\}$; since $a \in \mathcal{H}$, this is a finite set. We then have $\forall \gamma \in \Gamma_1 = \Gamma \setminus \Gamma_0$, that $|a_\gamma|$ and $|a_\gamma + v_\gamma^n|$ are bounded above by $2B$. Recalling the definition of $S_{w_\gamma,p} = (F_{w_\gamma,p})^{-1}$, we observe that, because $p \le 2$,

$$F'_{w_\gamma,p}(x) = 1 + w_\gamma \frac{p(p-1)|x|^{p-2}}{2} \ge 1 + w_\gamma \frac{p(p-1)}{2(2B)^{2-p}}$$

if $|x| \le 2B$. Hence we have

$$\begin{aligned}\left|S_{w_\gamma,p}(a_\gamma + v_\gamma^n) - S_{w_\gamma,p}(a_\gamma)\right| &\le \big(\max_{|x|\le 2B} |S'_{w_\gamma,p}(x)|\big)\left|v_\gamma^n\right| \\ &\le \left(1 + w_\gamma \frac{p(p-1)}{2(2B)^{2-p}}\right)^{-1}\left|v_\gamma^n\right| \\ &\le \left(1 + c\frac{p(p-1)}{2(2B)^{2-p}}\right)^{-1}\left|v_\gamma^n\right|;\end{aligned}$$

in the second inequality, we have used that $|S_{w_\gamma,p}(x)| \le |x|$, a consequence of the nonexpansivity of $S_{w_\gamma,p}$ (see Lemma 2.2) to upper bound the derivative $S'_{w_\gamma,p}$ on the interval $[-2B, 2B]$ by the inverse of the lower bound for $F'_{w_\gamma,p}$ on the same interval; in the last inequality we used the uniform lower bound on the w_γ, i.e., $\forall\gamma$, $w_\gamma \ge c > 0$. Rewriting $(1 + cp(p-1)/[2(2B)^{2-p}])^{-1} = C' < 1$, we have thus, $\forall \gamma \in \Gamma_1$, $C'|v_\gamma^n| \ge |S_{w_\gamma,p}(a_\gamma + v_\gamma^n) - S_{w_\gamma,p}(a_\gamma)|$, which implies

$$\begin{aligned}\sum_{\gamma\in\Gamma_1} \left|v_\gamma^n\right|^2 &\le \frac{1}{(1-C')^2} \sum_{\gamma\in\Gamma_1} \left|v_\gamma^n - S_{w_\gamma,p}(a_\gamma + v_\gamma^n) + S_{w_\gamma,p}(a_\gamma)\right|^2 \\ &\to 0 \quad \text{as } n \to \infty\,.\end{aligned}$$

On the other hand, since Γ_0 is a finite set, and the v^n tend to 0 weakly as n tends to ∞, we also have

$$\sum_{\gamma\in\Gamma_0} \left|v_\gamma^n\right|^2 \to 0 \quad \text{as } n \to \infty\,.$$

This proves the proposition for the case $p > 1$.

For $p = 1$, we define a finite set $\Gamma_0 \subset \Gamma$ so that $\sum_{\gamma\in\Gamma\setminus\Gamma_0} |a_\gamma|^2 \le (\frac{c}{4})^2$, where c is again the uniform lower bound on the w_γ. Because this is a finite set, the weak convergence of the v^n implies that $\sum_{\gamma\in\Gamma_0} |v_\gamma^n|^2 \xrightarrow[n\to\infty]{} 0$, so that we can concentrate on $\sum_{\gamma\in\Gamma\setminus\Gamma_0} |v_\gamma^n|^2$ only.

For each n, we split $\Gamma_1 = \Gamma \setminus \Gamma_0$ into two subsets: $\Gamma_{1,n} = \{\gamma \in \Gamma_1 : |v_\gamma^n + a_\gamma| < w_\gamma/2\}$ and $\widetilde{\Gamma}_{1,n} = \Gamma_1 \setminus \Gamma_{1,n}$. If $\gamma \in \Gamma_{1,n}$, then $S_{w_\gamma,1}(a_\gamma + v_\gamma^n) = S_{w_\gamma,1}(a_\gamma) = 0$ (since $|a_\gamma| \le \frac{c}{4} \le w_\gamma/2$), so that $|v_\gamma^n - S_{w_\gamma,1}(a_\gamma + v_\gamma^n) + S_{w_\gamma,1}(a_\gamma)| = |v_\gamma^n|$. It follows that

$$\sum_{\gamma\in\Gamma_{1,n}} \left|v_\gamma^n\right|^2 \le \sum_{\gamma\in\Gamma} \left|v_\gamma^n - S_{w_\gamma,1}(a_\gamma + v_\gamma^n) + S_{w_\gamma,1}(a_\gamma)\right|^2 \to 0 \quad \text{as } n \to \infty\,.$$

It remains to prove only that the remaining sum, $\sum_{\gamma\in\widetilde{\Gamma}_{1,n}} |v_\gamma^n|^2$, also tends to 0 as $n \to \infty$.

If $\gamma \in \Gamma_1$ and $|v_\gamma^n + a_\gamma| \geq w_\gamma/2$, then $|v_\gamma^n| \geq |v_\gamma^n + a_\gamma| - |a_\gamma| \geq w_\gamma/2 - \frac{c}{4} \geq \frac{c}{4} \geq |a_\gamma|$, so that $v_\gamma^n + a_\gamma$ and v_γ^n have the same sign; it then follows that

$$\begin{aligned}\left|v_\gamma^n - S_{w_\gamma,1}(a_\gamma + v_\gamma^n) + S_{w_\gamma,1}(a_\gamma)\right| &= \left|v_\gamma^n - S_{w_\gamma,1}(a_\gamma + v_\gamma^n)\right| \\ &= \left|v_\gamma^n - (a_\gamma + v_\gamma^n) + \frac{w_\gamma}{2}\operatorname{sign}(v_\gamma^n)\right| \\ &\geq \frac{w_\gamma}{2} - |a_\gamma| \geq \frac{c}{4}\,.\end{aligned}$$

This implies that

$$\sum_{\gamma\in\widetilde{\Gamma}_{1,n}} \left|v_\gamma^n - S_{w_\gamma,1}(a_\gamma + v_\gamma^n) + S_{w_\gamma,1}(a_\gamma)\right|^2 \geq \left(\frac{c}{4}\right)^2 \#\widetilde{\Gamma}_{1,n}\,;$$

since $\|v^n - \mathbf{S}_{\mathbf{w},1}(a + v^n) + \mathbf{S}_{\mathbf{w},1}(a)\| \xrightarrow[n\to\infty]{} 0$, we know on the other hand that

$$\sum_{\gamma\in\widetilde{\Gamma}_{1,n}} \left|v_\gamma^n - S_{w_\gamma,1}(a_\gamma + v_\gamma^n) + S_{w_\gamma,1}(a_\gamma)\right|^2 < \left(\frac{c}{4}\right)^2$$

when n exceeds some threshold N, which implies that $\widetilde{\Gamma}_{1,n}$ is empty when $n > N$. Consequently, $\sum_{\gamma\in\widetilde{\Gamma}_{1,n}} |v_\gamma^n|^2 = 0$ for $n > N$. This completes the proof for the case $p = 1$. □

Combining the lemmas in this section with the results of Section 3.1 gives a complete proof of Theorem 3.1 as stated at the start of Section 3.

4 Regularization Properties and Stability Estimates

In the preceding section we discussed the iterative algorithm (2.11) that converges towards a minimizer of the functional

$$\Phi_{\mathbf{w},p}(f) = \|Kf - g\|^2 + |||f|||_{\mathbf{w},p}^p\,. \tag{4.1}$$

For simplicity, let us assume, until further notice, that either $p > 1$ or $\mathrm{N}(K) = \{0\}$, so that there is a unique minimizer.

In this section we shall discuss to what extent this minimizer is acceptable as a *regularized solution* of the (possibly ill-posed) inverse problem $Kf = g$. Of particular interest to us is the *stability* of the estimate. For instance, if $\mathrm{N}(K) = \{0\}$, we would like to know to what extent the proposed solution, in this case the minimizer of $\Phi_{\mathbf{w},p}$, deviates from the ideal solution f_o if the data are a (small) perturbation of the image Kf_o of f_o. (If $\mathrm{N}(K) \neq \{0\}$, then there exist other f that have the same image as f_o, and the algorithm might choose one of those; see below.) In this discussion both the "size" of the perturbation and the weight of the penalty term in the variational functional, given by the coefficients $(w_\gamma)_{\gamma\in\Gamma}$, play a role. We argued earlier that we need $\mathbf{w} \neq \mathbf{0}$ in order to provide a meaningful estimate if, e.g., K is a compact operator; on the other hand, if $g = Kf_o$, then the presence of the penalty term will cause the minimizer of $\Phi_{\mathbf{w},p}$ to be different from f_o. We therefore need to strike a balance between the respective weights of the perturbation $g - Kf_o$ and the penalty term. Let us first define a framework in which we can make this statement more precise.

Because we shall deal in this section with data functions g that are not fixed, we adjust our notation for the variational functional to make the dependence on g explicit where appropriate; with this more elaborate notation, the right-hand side of, for instance, (4.1) is now $\Phi_{\mathbf{w},p;g}(f)$. (Because we work with one fixed operator K, the dependence of the functional on K remains "silent.") In order to make it possible to vary the weight of the penalty term in the functional, we introduce an extra parameter μ. We shall thus consider the functional

$$\Phi_{\mu,\mathbf{w},p;g}(f) = \|Kf - g\|^2 + \mu |||f|||^p_{\mathbf{w},p} \,. \tag{4.2}$$

Its minimizer will likewise depend on all these parameters. In its full glory, we denote it by $f^\star_{\mu,\mathbf{w},p;g}$; when confusion is impossible we abbreviate this notation. In particular, since $\mathbf{w}$ and p typically will not vary in the limit procedure that defines stability, we may omit them in the heat of the discussion. Notice that the dependence on $\mathbf{w}$ and μ arises only through the product $\mu\mathbf{w}$.

As mentioned above, if the "error" $e = g - Kf_o$ tends to 0, we would like to see our estimate for the solution of the inverse problem tend to f_o; since the minimizer of $\Phi_{\mu,\mathbf{w},p;g}(f)$ differs from f_o if $\mu \neq 0$, this means that we shall have to consider simultaneously a limit for $\mu \to 0$. More precisely, we want to find a functional dependence of μ on the noise level ϵ, $\mu = \mu(\epsilon)$, such that

$$\mu(\epsilon) \underset{\epsilon\to 0}{\longrightarrow} 0 \quad \text{and} \quad \sup_{\|g-Kf_o\|\le\epsilon} \|f^\star_{\mu(\epsilon),\mathbf{w},p;g} - f_o\| \underset{\epsilon\to 0}{\longrightarrow} 0 \tag{4.3}$$

for each f_o in a certain class of functions. If we can achieve this, then the ill-posed inverse problem will be *regularized* (in norm or "strongly") by our iterative method, and $f^\star_{\mu,\mathbf{w},p;g}$ will be called a *regularized solution*. One also says in this case that the minimization of the penalized least-squares functional (4.1) provides us with a *regularizing algorithm* or *regularization method*.

4.1 A General Regularization Theorem

If the w_γ tend to ∞, or more precisely, if

$$\forall C > 0, \quad \#\{\gamma \in \Gamma : w_\gamma \le C\} < \infty \,, \tag{4.4}$$

then the embedding of $\mathcal{B}_{\mathbf{w},p} = \{f \in \mathcal{H} : \sum_{\gamma\in\Gamma} w_\gamma |f_\gamma|^p < \infty\}$ in $\mathcal{H}$ is compact. (This is because the identity operator from $\mathcal{B}_{\mathbf{w},p}$ to $\mathcal{H}$ is then the norm-limit in $\mathcal{L}(\mathcal{B}_{\mathbf{w},p}, \mathcal{H})$, as $C \to \infty$, of the finite rank operators P_C defined by $P_C f = \sum_{\gamma\in\Gamma_C} w_\gamma \langle f, \varphi_\gamma\rangle \varphi_\gamma$, where $\Gamma_C = \{\gamma \in \Gamma : w_\gamma \le C\}$.) In this case, general compactness arguments can be used to show that (4.3) can be achieved. (See also further below.) We are, however, also interested in the general case, where the w_γ need not grow unboundedly. The following theorem proves that we can then nevertheless choose the dependence $\mu(\epsilon)$ so that (4.3) holds:

THEOREM 4.1 *Assume that K is a bounded operator from $\mathcal{H}$ to $\mathcal{H}'$ with $\|K\| < 1$, that $1 \le p \le 2$, and that the entries in the sequence $\mathbf{w} = (w_\gamma)_{\gamma\in\Gamma}$ are bounded below uniformly by a strictly positive number c. Assume that either $p > 1$ or* $\mathrm{N}(K) = \{0\}$. *For any $g \in \mathcal{H}'$ and any $\mu > 0$, define $f^\star_{\mu,\mathbf{w},p;g}$ to be the minimizer of $\Phi_{\mu,\mathbf{w},p;g}(f)$. If $\mu = \mu(\epsilon)$ satisfies the requirements*

$$\lim_{\epsilon\to 0}\mu(\epsilon) = 0 \quad \textit{and} \quad \lim_{\epsilon\to 0}\frac{\epsilon^2}{\mu(\epsilon)} = 0\,, \tag{4.5}$$

then we have, for any $f_o \in \mathcal{H}$,

$$\lim_{\epsilon\to 0}\Big[\sup_{\|g-Kf_o\|\le\epsilon}\|f^\star_{\mu(\epsilon),\mathbf{w},p;g} - f^\dagger\|\Big] = 0\,,$$

where $f^\dagger$ is the unique element of minimum $|\!|\!|\cdot|\!|\!|_{\mathbf{w},p}$-norm in $\mathcal{S} = \mathrm{N}(K) + f_o = \{f : Kf = Kf_o\}$.

Note that under the conditions of Theorem 4.1, $f^\dagger$ must indeed be unique: if $p > 1$, then the $|\!|\!|\cdot|\!|\!|_{\mathbf{w},p}$-norm is strictly convex, so that there is a unique minimizer for this norm in the hyperspace $\mathrm{N}(K) + f_o$; if $p = 1$, our assumptions require $\mathrm{N}(K) = \{0\}$. Note also that if $\mathrm{N}(K) = \{0\}$ (whether or not $p = 1$), then necessarily $f^\dagger = f_o$.

To prove Theorem 4.1, we will need the following two lemmas:

LEMMA 4.2 *The functions $S_{w,p}$ from $\mathbb{R}$ to itself, defined by* (2.6) *and* (2.8) *for $p > 1$ and $p = 1$, respectively, satisfy*

$$|S_{w,p}(x) - x| \le \frac{wp}{2}\,|x|^{p-1}\,.$$

PROOF: For $p = 1$, the definition (2.8) implies immediately that $|x - S_{w,1}(x)| = \min(\frac{w}{2}, |x|) \le \frac{w}{2}$, so that the proposition holds for $x \neq 0$. For $x = 0$, $S_{w,1}(x) = 0$.

For $p > 1$, $S_{w,p} = (F_{w,p})^{-1}$, where $F_{w,p}(y) = y + \frac{wp}{2}|y|^{p-1}\operatorname{sign}(y)$ satisfies $|F_{w,p}(y)| \ge |y|$, and $|F_{w,p}(y) - y| \le \frac{wp}{2}|y|^{p-1}$. It follows that $|S_{w,p}(x)| \le |x|$ and $|x - S_{w,p}(x)| \le \frac{wp}{2}\,|S_{w,p}(x)|^{p-1} \le \frac{wp}{2}|x|^{p-1}$. □

LEMMA 4.3 *If the sequence of vectors $(v_k)_{k\in\mathbb{N}}$ converges weakly in $\mathcal{H}$ to v, and $\lim_{k\to\infty}|\!|\!|v_k|\!|\!|_{\mathbf{w},p} = |\!|\!|v|\!|\!|_{\mathbf{w},p}$, then $(v_k)_{k\in\mathbb{N}}$ converges to v in the $\mathcal{H}$-norm, i.e., $\lim_{k\to\infty}\|v - v_k\| = 0$.*

PROOF: It is a standard result that if $\text{w-}\lim_{k\to\infty} v_k = v$ and $\lim_{k\to\infty}\|v_k\| = \|v\|$, then it follows that $\lim_{k\to\infty}\|v - v_k\|^2 = \lim_{k\to\infty}(\|v\|^2 + \|v_k\|^2 - 2\langle v, v_k\rangle) = \|v\|^2 + \|v\|^2 - 2\langle v, v\rangle = 0$. We thus need to prove only that $\lim_{k\to\infty}\|v_k\| = \|v\|$.

Since the v_k converge weakly, they are uniformly bounded. It follows that the $|v_{k,\gamma}| = |\langle v_k, \varphi_\gamma\rangle|$ are bounded uniformly in k and γ by some finite number C. Define $r = 2/p$. Since, for $x, y > 0$, we have $|x^r - y^r| \le r|x - y|\max(x, y)^{r-1}$, it follows that $||v_{k,\gamma}|^2 - |v_\gamma|^2| \leq rC^{p(r-1)}||v_{k,\gamma}|^p - |v_\gamma|^p|$. Because the w_γ are uniformly bounded below by $c > 0$, we obtain

$$\begin{aligned}\left|\|v_k\|^2 - \|v\|^2\right| &\leq \sum_{\gamma\in\Gamma} \left||v_{k,\gamma}|^2 - |v_\gamma|^2\right| \\ &\leq \frac{2}{cp} C^{2-p} \sum_{\gamma\in\Gamma} w_\gamma \left||v_{k,\gamma}|^p - |v_\gamma|^p\right|,\end{aligned}$$

so that it suffices to prove that this last expression tends to 0 as k tends to ∞. Define now $u_{k,\gamma} = \min(|v_{k,\gamma}|, |v_\gamma|)$. Clearly $\forall\gamma \in \Gamma$, $\lim_{k\to\infty} u_{k,\gamma} = |v_\gamma|$; since $\sum_{\gamma\in\Gamma} w_\gamma |v_\gamma|^p < \infty$, it follows by the dominated convergence theorem that $\lim_{k\to\infty} \sum_{\gamma\in\Gamma} w_\gamma u_{k,\gamma}^p = \sum_{\gamma\in\Gamma} w_\gamma |v_\gamma|^p$. Since

$$\sum_{\gamma\in\Gamma} w_\gamma \left||v_{k,\gamma}|^p - |v_\gamma|^p\right| = \sum_{\gamma\in\Gamma} w_\gamma \left(|v_\gamma|^p + |v_{k,\gamma}|^p - 2u_{k,\gamma}^p\right) \xrightarrow[k\to\infty]{} 0\,,$$

the lemma follows. □

We are now ready to prove Theorem 4.1.

PROOF OF THEOREM 4.1: Let's assume that $\mu(\epsilon)$ satisfies the requirements in (4.5).

We first establish weak convergence. For this it is sufficient to prove that if $(g_n)_{n\in\mathbb{N}}$ is a sequence in $\mathcal{H}'$ such that $\|g_n - Kf_o\| \leq \epsilon_n$, where $(\epsilon_n)_{n\in\mathbb{N}}$ is a sequence of strictly positive numbers that converges to 0 as $n \to \infty$, then $\text{w-lim}_{n\to\infty} f^\star_{\mu(\epsilon_n);g_n} = f^\dagger$, where $f^\star_{\mu;g}$ is the unique minimizer of $\Phi_{\mu,\mathbf{w},p;g}(f)$. (As announced, we have dropped here the explicit indication of the dependence of $f^\star$ on $\mathbf{w}$ and p; these parameters will keep fixed values throughout this proof. We will take the liberty to drop them in our notation for Φ as well, when this is convenient.) For the sake of convenience, we abbreviate $\mu(\epsilon_n)$ as μ_n.

Then the $f^\star_{\mu_n;g_n}$ are uniformly bounded in $\mathcal{H}$ by the following argument:

$$\begin{aligned}\|f^\star_{\mu_n;g_n}\|^p \leq \frac{1}{c}|||f^\star_{\mu_n;g_n}|||^p_{\mathbf{w},p} &\leq \frac{1}{\mu_n c}\Phi_{\mu_n;g_n}(f^\star_{\mu_n;g_n}) \\ &\leq \frac{1}{\mu_n c}\Phi_{\mu_n;g_n}(f^\dagger) \\ &= \frac{1}{\mu_n c}\left[\|Kf_o - g_n\|^2 + \mu_n|||f^\dagger|||^p_{\mathbf{w},p}\right] \\ &\leq \frac{1}{c}\left(\frac{\epsilon_n^2}{\mu_n} + |||f^\dagger|||^p_{\mathbf{w},p}\right),\end{aligned} \tag{4.6}$$

where we have used, respectively, the bound (3.2), the fact that $f^\star_{\mu_n;g_n}$ minimizes $\Phi_{\mu_n;g_n}(f)$, $Kf^\dagger = Kf_o$, and the bound $\|Kf_o - g_n\|^2 \leq \epsilon_n^2$. By assumption (4.5), ϵ_n^2/μ_n tends to 0 for $n \to \infty$ and hence can be bounded by a constant independent of n.

It follows that the sequence $(f^\star_{\mu_n;g_n})_{n\in\mathbb{N}}$ has at least one weak accumulation point, i.e., there exists a subsequence $(f^\star_{\mu_{n_l};g_{n_l}})_{l\in\mathbb{N}}$ that has a weak limit. This sequence is bounded in the $|||\cdot|||$-norm, so that, by passing to a subsequence $(f^*_{\mu_{n_{l(k)}};g_{n_{l(k)}}})_{k\in\mathbb{N}}$, we can ensure that the $|||f^*_{\mu_{n_{l(k)}};g_{n_{l(k)}}}|||_{\mathbf{w},p}$ constitute a converging sequence. To simplify notation, we define $\widetilde{\mu}_k = \mu_{n_{l(k)}}$ and $\widetilde{f}_k = f^\star_{\mu_{n_{l(k)}};g_{n_{l(k)}}}$; the $\widetilde{f}_k$ have the same weak limit $\widetilde{f}$ as the $f^*_{\mu_{n_l};g_{n_l}}$. We also define $\widetilde{g}_k = g_{n_{l(k)}}$, $\widetilde{e}_k = \widetilde{g}_k - Kf_o$, and $\widetilde{\epsilon}_k = \epsilon_{n_{l(k)}}$. We shall show that $\widetilde{f} = f^\dagger$.

For each k, $\widetilde{f}_k$ is the minimizer of the functional $\Phi_{\widetilde{\mu}_k;\widetilde{g}_k}$. Under the conditions of Theorem 4.1 this minimizer is unique; on the other hand, every fixed point of the operator $\widetilde{\mathbf{T}}_k$ (defined as the nonlinear operator $\mathbf{T}$ corresponding to the functional $\Phi_{\widetilde{\mu}_k;\widetilde{g}_k}$) is a minimizer for $\Phi_{\widetilde{\mu}_k;\widetilde{g}_k}$, by Proposition 3.10. Since the set of fixed points of $\widetilde{\mathbf{T}}_k$ is nonempty by Proposition 3.9, it follows that each $\widetilde{f}_k$ is a fixed point of its corresponding $\widetilde{\mathbf{T}}_k$. Therefore, for any $\gamma\in\Gamma$, $\widetilde{f}_\gamma = \langle\widetilde{f},\varphi_\gamma\rangle$ satisfies

$$\widetilde{f}_\gamma = \lim_{k\to\infty}(\widetilde{f}_k)_\gamma = \lim_{k\to\infty} S_{\widetilde{\mu}_k w_\gamma,p}[(\widetilde{h}_k)_\gamma]$$

with $\widetilde{h}_k = \widetilde{f}_k + K^*(\widetilde{g}_k - K\widetilde{f}_k) = \widetilde{f}_k + K^*K(f_o - \widetilde{f}_k) + K^*\widetilde{e}_k$. We now rewrite this as

$$\widetilde{f}_\gamma = \lim_{k\to\infty}\big(S_{\widetilde{\mu}_k w_\gamma,p}[(\widetilde{h}_k)_\gamma] - (\widetilde{h}_k)_\gamma\big) + \lim_{k\to\infty}(\widetilde{h}_k)_\gamma\,. \tag{4.7}$$

By Lemma 4.2 the first limit in the right-hand side is 0, since

$$\begin{aligned}\big|S_{\widetilde{\mu}_k w_\gamma,p}[(\widetilde{h}_k)_\gamma] - (\widetilde{h}_k)_\gamma\big| &\le p w_\gamma \widetilde{\mu}_k \frac{|(\widetilde{h}_k)_\gamma|^{p-1}}{2}\\ &\le pC\widetilde{\mu}_k \frac{[3C+\widetilde{\epsilon}_k]^{p-1}}{2} \xrightarrow[k\to\infty]{} 0\,,\end{aligned}$$

where we have used $\|K\| < 1$ (C is some constant depending on w_γ). Because $\lim_{k\to\infty}\|\widetilde{e}_k\| = 0$, and $\text{w-lim}_{k\to\infty}\widetilde{f}_k = \widetilde{f}$, it then follows from (4.7) that

$$\widetilde{f}_\gamma = \lim_{k\to\infty}(\widetilde{h}_k)_\gamma = \widetilde{f}_\gamma + [K^*K(f^\dagger - \widetilde{f})]_\gamma\,.$$

Since this holds for all γ, it follows that $K^*K(f^\dagger - \widetilde{f}) = 0$. If $\mathrm{N}(K) = \{0\}$, then this allows us immediately to conclude that $\widetilde{f} = f^\dagger$. When $\mathrm{N}(K) \ne \{0\}$, we can conclude only that $f^\dagger - \widetilde{f} \in \mathrm{N}(K)$. Because $f^\dagger$ has the smallest $|||\cdot|||_{\mathbf{w},p}$-norm among all $f \in \mathcal{S} = \{f : Kf = Kf_o\}$, it follows that $|||\widetilde{f}|||_{\mathbf{w},p} \ge |||f^\dagger|||_{\mathbf{w},p}$. On the other hand, because the $\widetilde{f}_k$ weakly converge to $\widetilde{f}$, and therefore, for all γ, $(\widetilde{f}_k)_\gamma \to \widetilde{f}_\gamma$ as $k\to\infty$, we can use Fatou's lemma to obtain

$$\begin{aligned}|||\widetilde{f}|||^p_{\mathbf{w},p} = \sum_\gamma w_\gamma|\widetilde{f}_\gamma|^p &\le \limsup_{k\to\infty}\sum_\gamma w_\gamma|(\widetilde{f}_k)_\gamma|^p\\ &= \limsup_{k\to\infty}|||\widetilde{f}_k|||^p_{\mathbf{w},p} = \lim_{k\to\infty}|||\widetilde{f}_k|||^p_{\mathbf{w},p}\,.\end{aligned} \tag{4.8}$$

It then follows from (4.6) that

$$\lim_{k\to\infty} |||\widetilde{f}_k|||^p_{\mathbf{w},p} \le \lim_{k\to\infty}\left[\frac{\widetilde{\epsilon}_k^2}{\widetilde{\mu}_k} + |||f^\dagger|||^p_{\mathbf{w},p}\right] = |||f^\dagger|||^p_{\mathbf{w},p} \le |||\widetilde{f}|||^p_{\mathbf{w},p}\,. \tag{4.9}$$

Together, the inequalities (4.8) and (4.9) imply that

$$\lim_{k\to\infty} |||\widetilde{f}_k|||_{\mathbf{w},p} = |||f^\dagger|||_{\mathbf{w},p} = |||\widetilde{f}|||_{\mathbf{w},p}\,. \tag{4.10}$$

Since $f^\dagger$ is the unique element in $\mathcal{S}$ of minimal $|||\cdot|||_{\mathbf{w},p}$-norm, it follows that $\widetilde{f} = f^\dagger$. The same argument holds for any other weakly converging subsequence of $(f^\star_{\mu_n;g_n})_{n\in\mathbb{N}}$; it follows that the sequence itself converges weakly to $f^\dagger$. Similarly we conclude from (4.10) that $\lim_{n\to\infty} |||f^\star_{\mu_n;g_n}|||_{\mathbf{w},p} = |||f^\dagger_{\mu_n;g_n}|||_{\mathbf{w},p}$. It then follows from Lemma 4.3 that the $f^\star_{\mu_n;g_n}$ converge to $f^\dagger$ in the $\mathcal{H}$-norm. □

Remark 4.4. Even when $p = 1$ and $N(K) \neq \{0\}$, it may still be the case that, for any $f_o \in \mathcal{H}$, there is a unique element $f^\dagger$ of minimal norm in $\mathcal{S} = \{f \in \mathcal{H} : Kf = Kf_o\}$. (For instance, if K is diagonal in the φ_γ-basis with some zero eigenvalues, then the unique minimizer $f^\dagger$ in $\mathcal{S}$ is given by setting to 0 all the components of f_o corresponding to γ for which $K\varphi_\gamma = 0$.) In this case the proof still applies, and we still have norm-convergence of the $f^\star_{\mu(\epsilon),\mathbf{w},p;g}$ to $f^\dagger$ if $\mu(\epsilon)$ satisfies (4.5) and $\|g - Kf_o\| \le \epsilon \to 0$.

4.2 Stability Estimates

The regularization theorem of the previous subsection gives no information on the rate at which the regularized solution approaches the exact solution when the noise (as measured by ϵ) decreases to 0. Such rates are not available in the general case but can be derived under additional assumptions discussed below. For the remainder of this section we shall assume that the operator K is invertible on its range, i.e., that $\mathrm{N}(K) = \{0\}$. Suppose that the unknown exact solution of the problem, f_o, satisfies the constraint $|||f_o|||_{\mathbf{w},p} \le \rho$, where $\rho > 0$ is given; in other words, we know a priori that the unknown solution lies in the ball around the origin with radius ρ in the Banach space $\mathcal{B}_{\mathbf{w},p}$; we shall denote this ball by $\mathrm{B}_{\mathbf{w},\mathrm{p}}(0,\rho)$. If we also know that g lies within a distance ϵ of Kf_o in $\mathcal{H}'$, then we can localize the exact solution within the set

$$\mathcal{F}(\epsilon,\rho) = \{f \in \mathcal{H} : \|Kf - g\| \le \epsilon,\ |||f|||_{\mathbf{w},p} \le \rho\}\,.$$

The diameter of this set is a measure of the uncertainty of the solution for a given a priori constraint and a given noise level ϵ. The maximum diameter of $\mathcal{F}$, namely $\mathrm{diam}(\mathcal{F}) = \sup\{\|f - f'\| : f, f' \in \mathcal{F}\}$ is bounded by $2M(\epsilon,\rho)$, where $M(\epsilon,\rho)$, defined by

$$M(\epsilon,\rho) = \sup\{\|h\| : \|Kh\| \le \epsilon,\ |||h|||_{\mathbf{w},p} \le \rho\}\,, \tag{4.11}$$

is called the *modulus of continuity* of K^{-1} under the a priori constraint. (The introduction of the modulus of continuity is a standard technique for the derivation of convergence rates; see [2, 25]. Note that we have once more dropped the explicit reference in our notation to the dependence on $\mathbf{w}$ and p.) If (4.4) is satisfied, then the ball $\mathrm{B}_{\mathbf{w},\mathrm{p}}(0,\rho)$ is compact in $\mathcal{H}$, and it follows from a general topological lemma (see, e.g., [23]) that $M(\epsilon,\rho) \to 0$ when $\epsilon \to 0$; the uncertainty on the solution thus vanishes in this limit. However, this topological argument, which holds for any regularization method enforcing the a priori constraint $|||f_o|||_{\mathbf{w},p} \le \rho$, does not tell us anything about the rate of convergence of any specific method.

In what follows we shall systematically assume that (4.4) is satisfied. We shall also make additional assumptions that will make it possible to derive more precise convergence results. Our specific regularization method consists in taking the minimizer $f^*_{\mu;g}$ of the functional $\Phi_{\mu;g}(f)$ given by (4.2) as an estimate of the exact solution f_o, where we leave any links between μ and ϵ unspecified for the moment. (Because of the compactness argument above, we could conceivably dispense with (4.5); see below.) An upper bound on the reconstruction error $\|f^*_{\mu;g}-f_o\|$, valid for all g such that $\|g-Kf_o\|\le\epsilon$, as well as uniformly in f_o, is given by the following *modulus of convergence*:

$$M_\mu(\epsilon,\rho) = \sup\left\{\|f^*_{\mu;g}-f\| : f\in\mathcal{H},\ g\in\mathcal{H}',\ \|Kf-g\|\le\epsilon,\ |||f|||_{\mathbf{w},p}\le\rho\right\}. \tag{4.12}$$

The decay of this modulus of convergence as $\epsilon\to 0$ is governed by the decay of the modulus of continuity (4.11), as shown by the following proposition:

PROPOSITION 4.5 *The modulus of convergence* (4.12) *satisfies*

$$M(\epsilon,\rho) \le M_\mu(\epsilon,\rho) \le M(\epsilon+\epsilon',\rho+\rho') \tag{4.13}$$

where

$$\epsilon' = (\epsilon^2+\mu\rho^p)^{\frac{1}{2}}, \qquad \rho' = \left(\rho^p+\frac{\epsilon^2}{\mu}\right)^{\frac{1}{p}}, \tag{4.14}$$

and $M(\epsilon,\rho)$ is defined by (4.11).

PROOF: We first note that $\Phi_{\mu;g}(f^*_{\mu;g}) \le \Phi_{\mu;g}(f_o) \le \epsilon^2+\mu\rho^p$ because $f^*_{\mu;g}$ is the minimizer of $\Phi_{\mu;g}(f)$ and $f_o\in\mathcal{F}(\epsilon,\rho)$. It follows that

$$\|Kf^*_{\mu;g}-g\|^2 \le \Phi_{\mu;g}(f^*_{\mu;g}) \le \epsilon^2+\mu\rho^p$$

and

$$\mu|||f^*_{\mu;g}|||^p_{\mathbf{w},p} \le \Phi_{\mu;g}(f^*_{\mu;g}) \le \epsilon^2+\mu\rho^p$$

or, equivalently, $f^*_{\mu;g}\in\mathcal{F}(\epsilon',\rho')$ with ϵ' and ρ' given by (4.14). The modulus of convergence (4.12) can then be bounded as follows, using the triangle inequality. Indeed, for any $f\in\mathcal{F}(\epsilon,\rho)$ and $f'\in\mathcal{F}(\epsilon',\rho')$, we have $\|K(f-f')\|\le\epsilon+\epsilon'$ and $|||f-f'|||_{\mathbf{w},p}\le\rho+\rho'$, and we immediately obtain from the definition of (4.11) the upper bound in (4.13). To derive the lower bound, observe that for the particular choice $g=0$, the minimizer $f^*_{\mu;g}$ of the functional (4.2) is $f^*_{\mu;0}=0$. The desired lower bound then follows immediately upon inspection of the two definitions (4.11) and (4.12). □

Let us briefly discuss the meaning of the previous proposition. The modulus of continuity $M(\epsilon, \rho)$ yields the best possible convergence rate for any regularization method that enforces the error bound and the a priori constraint defined by (4.11). Proposition 4.5 provides a relation between the modulus of continuity and the convergence rate $M_\mu(\epsilon, \rho)$ of the specific regularization method considered in this paper, which is defined by the minimization of the functional (4.2). Optimizing the upper bound in (4.13) suggests the choice $\mu = \epsilon^2/\rho^p$, yielding $\epsilon' = \sqrt{2}\,\epsilon$ and $\rho' = 2^{1/p}\rho$. With these choices, we ensure that $f^*_{\mu;g} \to f_o$ when $\epsilon \to 0$, i.e., that the problem is *regularized*, provided we can show that the modulus of continuity tends to 0 with ϵ. Moreover, once we establish its rate of decay (see below), we know that our regularization method is (nearly) optimal in the sense that the modulus of convergence (4.12) will decay *at the same rate as the optimal rate* given by the modulus of continuity $M(\epsilon, \rho)$. (We call it *nearly* optimal because, although the rate of decay is optimal, the constant multiplier probably is not.) Note that because of the assumption of compactness of the ball $\mathrm{B}_{\mathbf{w},\mathrm{p}}(0, \rho)$ (which amounts to assuming that (4.4) is satisfied), we achieve regularization even in some cases where ϵ^2/μ does not tend to 0 for $\epsilon \to 0$, which is a case not covered by Theorem 4.1.

In order to derive upper or lower bounds on $M(\epsilon, \rho)$, we must know more information about the operator K. The following proposition illustrates how such information can be used.

PROPOSITION 4.6 *Suppose that there exist sequences* $\mathbf{b} = (b_\gamma)_{\gamma\in\Gamma}$ *and* $\mathbf{B} = (B_\gamma)_{\gamma\in\Gamma}$ *satisfying,* $\forall\gamma \in \Gamma$, $0 < b_\gamma, B_\gamma < \infty$, *and such that for all* h *in* $\mathcal{H}$,

$$\sum_{\gamma\in\Gamma} b_\gamma |h_\gamma|^2 \le \|Kh\|^2 \le \sum_{\gamma\in\Gamma} B_\gamma |h_\gamma|^2 . \tag{4.15}$$

Then the following upper and lower bounds hold for $M(\epsilon, \rho)$:

$$M(\epsilon, \rho) \ge \max_{\gamma\in\Gamma} \left[\min\left(\rho w_\gamma^{-1/p}, \epsilon B_\gamma^{-1/2}\right)\right], \tag{4.16}$$

$$M(\epsilon, \rho) \le \min_{\Gamma=\Gamma_1\cup\Gamma_2} \sqrt{\frac{\epsilon^2}{\min_{\gamma\in\Gamma_1} b_\gamma} + \frac{\rho^2}{\min_{\gamma\in\Gamma_2} w_\gamma^{2/p}}} . \tag{4.17}$$

PROOF: To prove the lower bound, we need only exhibit one particular h such that $\|Kh\| \le \epsilon$ and $|||h|||_{\mathbf{w},p} \le \rho$, for which $\|h\|$ is given by the right-hand side of (4.16). For this we need only identify the index γ_m such that $\forall\gamma \in \Gamma$,

$$\nu = \min\left(\rho w_{\gamma_m}^{-1/p}, \epsilon B_{\gamma_m}^{-1/2}\right) \ge \min\left(\rho w_\gamma^{-1/p}, \epsilon B_\gamma^{-1/2}\right),$$

and choose $h = \nu\varphi_{\gamma_m}$. Then $|||h|||_{\mathbf{w},p} = \nu w_{\gamma_m}^{1/p} \le \rho$ and $\|Kh\| \le \nu B_{\gamma_m}^{1/2} \le \epsilon$; on the other hand, ν equals the right-hand side of (4.16).

On the other hand, for any partition of Γ into two subsets, $\Gamma = \Gamma_1 \cup \Gamma_2$, and for any $h \in \{u : \|Ku\| \le \epsilon, |||u|||_{\mathbf{w},p} \le \rho\}$, we have

$$\begin{aligned}\sum_{\gamma\in\Gamma} |h_\gamma|^2 &= \sum_{\gamma\in\Gamma_1} |h_\gamma|^2 + \sum_{\gamma\in\Gamma_2} |h_\gamma|^2 \\ &\le \max_{\gamma'\in\Gamma_1} \big[b_{\gamma'}^{-1}\big] \sum_{\gamma\in\Gamma_1} b_\gamma |h_\gamma|^2 \\ &\quad + \max_{\gamma'\in\Gamma_2} \big[w_{\gamma'}^{-2/p}\big]\Big[\max_{\gamma''\in\Gamma_2} w_{\gamma''}|h_{\gamma''}|^p\Big]^{\frac{2}{p}-1} \sum_{\gamma\in\Gamma_2} w_\gamma |h_\gamma|^p \\ &\le \max_{\gamma'\in\Gamma_1} \big[b_{\gamma'}^{-1}\big]\epsilon^2 + \max_{\gamma'\in\Gamma_2} \big[w_{\gamma'}^{-2/p}\big]\rho^2 .\end{aligned}$$

Since this is true for any partition $\Gamma = \Gamma_1 \cup \Gamma_2$, we still have an upper bound, uniformly valid for all $h \in \{u : \|Ku\| \le \epsilon, |||u|||_{\mathbf{w},p} \le \rho\}$, if we take the minimum over all such partitions. The upper bound on $M(\epsilon, \rho)$ then follows upon taking the square root. □

To illustrate how Proposition 4.6 could be used, let us apply it to one particular example, in which we choose the (φ_γ)-basis with respect to which the $|||\cdot|||_{\mathbf{w},p}$-norm is defined to be a wavelet basis $(\Psi_\lambda)_{\lambda\in\Lambda}$. As already pointed out in Section 1.4.1 the Besov spaces $B^s_{p,p}(\mathbb{R}^d)$ can then be identified with the Banach spaces $\mathcal{B}_{\mathbf{w},p}$ for the particular choice $w_\lambda = 2^{\sigma p|\lambda|}$, where $\sigma = s + d(\frac{1}{2} - \frac{1}{p})$ is assumed to be nonnegative. For $f \in B^s_{p,p}(\mathbb{R}^d)$, the Banach norm $|||f|||_{\mathbf{w},p}$ then coincides with the Besov norm $[\![f]\!]_{s,p} = [\sum_{\lambda\in\Lambda} w_\lambda |\langle f, \Psi_\lambda\rangle|^p]^{1/p}$. Let us now consider an inverse problem for the operator K with such a Besov a priori constraint. If we assume that the operator K has particular smoothing properties, then we can use these to derive bounds on the corresponding modulus of continuity, and thus also on the rate of convergence for our regularization algorithm. In particular, let us assume that the operator K is a smoothing operator of order α, a property that can be formulated as an equivalence between the norm $\|Kh\|$ and the norm of h in a Sobolev space of negative order $H^{-\alpha}$, i.e., in a Besov space $B^{-\alpha}_{2,2}$; see, e.g., [10, 13, 23, 32]. In other words, we assume that for some $\alpha > 0$, there exist constants A_ℓ and A_u such that for all $h \in L^2(\mathbb{R}^d)$,

$$A_\ell^2 \sum_\lambda 2^{-2|\lambda|\alpha} |h_\lambda|^2 \le \|Kh\|^2 \le A_u^2 \sum_\lambda 2^{-2|\lambda|\alpha} |h_\lambda|^2 . \tag{4.18}$$

The decay rate of the modulus of continuity is then characterized as follows:

PROPOSITION 4.7 *If the operator K satisfies the smoothness condition* (4.18)*, then the modulus of continuity $M(\epsilon, \rho)$, defined by*

$$M(\epsilon, \rho) = \max\{\|h\| : \|Kh\| \le \epsilon,\ [\![h]\!]_{s,p} \le \rho\} ,$$

satisfies

$$c\left(\frac{\epsilon}{A_u}\right)^{\frac{\sigma}{\sigma+\alpha}}\rho^{\frac{\alpha}{\sigma+\alpha}}\le M(\epsilon,\rho)\le C\left(\frac{\epsilon}{A_\ell}\right)^{\frac{\sigma}{\sigma+\alpha}}\rho^{\frac{\alpha}{\sigma+\alpha}} \tag{4.19}$$

where $\sigma = s + d(\frac{1}{2}-\frac{1}{p}) \ge 0$ *and* c *and* C *are constants depending on* σ *and* α *only.*

PROOF: By (4.18), the operator K satisfies (4.15) with $b_\lambda = A_\ell^2 2^{-2|\lambda|\alpha}$ and $B_\lambda = A_u^2 2^{-2|\lambda|\alpha}$.

It then follows from (4.16) that

$$M(\epsilon,\rho)\ge\max_\lambda\left[\min\left(\rho 2^{-\sigma|\lambda|},\frac{\epsilon}{A_u}2^{\alpha|\lambda|}\right)\right];$$

if $x = |\lambda|$ could take on all positive real values, then one easily computes that this max-min would be given for $x = -[\log_2(\epsilon/\rho A_u)]/(\alpha+\sigma)$ and would be equal to $(\epsilon/A_u)^{\sigma/(\alpha+\sigma)}\rho^{\alpha/(\alpha+\sigma)}$. Because $|\lambda|$ is constrained to take only the values in $\mathbb{N}$, the max-min is guaranteed only to be within a constant of this bound (corresponding to an integer neighbor of the optimal x), which leads to the lower bound in (4.19).

For the upper bound (4.17), we must partition the index set. Splitting Λ into $\Lambda_1 = \{\lambda : |\lambda| < J\}$ and $\Lambda_2 = \{\lambda : |\lambda| \ge J\}$, we find that

$$\frac{\epsilon^2}{\min_{\lambda\in\Lambda_1} b_\lambda}+\frac{\rho^2}{\min_{\lambda\in\Lambda_2} w_\lambda^{2/p}}=\frac{\epsilon^2}{A_\ell^2}2^{2\alpha(J-1)}+\rho^2 2^{-2\sigma J}.$$

The minimizing partition for Λ thus corresponds with the minimizing J for the right-hand side of this expression. This value for J is an integer neighbor of $y = -[\log_2(\epsilon/\rho A_\ell)]/(\alpha+\sigma)$, which leads to the upper bound in (4.19). □

The stability estimates we have derived are standard in regularization theory for the special case $p = 2$. The case $p < 2$, $K = I$, was treated in [16, 20]; these two papers discuss a wider range of Besov spaces than considered here, as well as Triebel spaces; in addition, they treat other than L^2-norms for the discrepancy. In [11] the classical bounds for general K and $p = 2$ were extended to the case $1 \le p < 2$.

The bounds show the interplay between the smoothing order of the operator characterized by α and the assumed smoothness of the solutions characterized by $\sigma = s + d(\frac{1}{2}-\frac{1}{p})$ (for Besov spaces, we recall that this amounts to solutions having s derivatives in L^p). For $\sigma/(\sigma+\alpha)$ close to 1, the problem is mildly ill-posed, whereas the stability degrades for large α. Note that if the bound (4.18) were replaced by another one, in which the decay of the b_λ and B_λ was given by an exponential decay in $D = 2^{|\lambda|}$ (instead of the much slower decaying negative power $D^{-2\alpha}$ of (4.18)), then the modulus of continuity would tend to 0 only as an inverse power of $|\log\epsilon|$. This is the so-called *logarithmic continuity*, which has been extensively discussed in the case $p = 2$, and which extends, as shown by an easy application of Proposition 4.6, to $1 \le p < 2$.

5 Generalizations and Additional Comments

The algorithm discussed in this paper can be generalized in several directions, some of which we list here with brief comments.

The penalization functionals $|||f|||_{\mathbf{w},p}$ we have used are symmetric; i.e., they are invariant under the exchange of f for $-f$. We can equally well consider penalization functionals that treat positive and negative values of the f_γ differently. If $(w_\gamma^+)_{\gamma\in\Gamma}$ and $(w_\gamma^-)_{\gamma\in\Gamma}$ are two sequences of strictly positive numbers, then we can consider the problem of minimizing the functional

$$\Phi_{\mathbf{w}^+,\mathbf{w}^-,p}(f) = \|Kf-g\|^2 + \sum_{\gamma\in\Gamma}\left(w_\gamma^+[f_\gamma]_+^p + w_\gamma^-[f_\gamma]_-^p\right) \tag{5.1}$$

where, for $r\in\mathbb{R}$, $r_+ = \max(0,r)$ and $r_- = \max(0,-r)$. One easily checks that all the arguments in this paper can be applied equally well (after some straightforward modifications) to the general functional (5.1), provided we replace the thresholding functions $S_{w_\gamma,p}$ in the iterative algorithm by $S_{w_\gamma^+,w_\gamma^-,p}$, where, for $p>1$,

$$S_{w^+,w^-,p} = (F_{w^+,w^-,p})^{-1}$$

with

$$F_{w^+,w^-,p}(x) = x + \frac{p}{2}w^+[x]_+^{p-1} - \frac{p}{2}w^-[x]_-^{p-1}\,,$$

and for $p=1$,

$$S_{w^+,w^-,1} = \begin{cases} x+\frac{w^-}{2} & \text{if } x\le -\frac{w^-}{2} \\ 0 & \text{if } -\frac{w^-}{2} < x < \frac{w^+}{2} \\ x-\frac{w^+}{2} & \text{if } x\ge \frac{w^+}{2}. \end{cases}$$

The above applies when the f_γ are all real; a generalization to complex f_γ is not straightforward. When dealing with complex functions, one could generalize the penalization $\sum_{\gamma\in\Gamma} w_\gamma|f_\gamma|^p$ to $\sum_{\gamma\in\Gamma,|f_\gamma|\neq 0} w_\gamma(\arg f_\gamma)|f_\gamma|^p$, where the weight coefficients have been replaced by strictly positive 2π-periodic C^1-functions on the 1-torus $\mathbb{T}=\{x\in\mathbb{C},|x|=1\}$. It turns out, however, that the variational equation for $e^{i\arg f_\gamma} = f_\gamma|f_\gamma|^{-1}$ then no longer uncouples from that for $|f_\gamma|$ (as it does in the case where w_γ is a constant), leading to a more complicated "generalized thresholding" operation in which the absolute value and phase of the complex number $S_{w,p}(f_\gamma)$ are given by a system of two coupled nonlinear equations.

When the $(\varphi_\gamma)_{\gamma\in\Gamma}$-basis is chosen to be a wavelet basis, then we saw in Section 1.4.1 that is is possible to make the $|||\cdot|||_{\mathbf{w},p}$-norm equivalent to the Besov norm $|||\!|\cdot|||\!|_{s,p}$ by choosing the weight $w_\lambda = 2^{|\lambda|\sigma p}$ for $|\langle f,\Psi_\lambda\rangle|^p$, where $|\lambda|$ is the scale of wavelet Ψ_λ. The label λ contains much more information than just the scale, however, since it also indicates the location of the wavelet, as well as its "species" (i.e., exactly which combination of one-dimensional scaling functions and wavelets is used to construct the product function Ψ_λ). One could choose the w_λ so that certain regions in space are given extra weight, or on the contrary deemphasized, depending on prior information. In pixel space, prior information on the support of the object to be reconstructed can be easily enforced by simply setting the corresponding weights to very small values or by choosing very large weights outside the object support. This type of constraint is of uttermost importance to achieve superresolution in inverse problems in optics and imaging; see, e.g., [4]. When thresholding in the wavelet domain, a constraint on the object support can be enforced in a similar way due to the good spatial localization of the wavelets. If no a priori information is known, one could even imagine repeating the wavelet thresholding algorithm several times, adapting the weights w_λ after each pass, depending on the results of the previous pass; this could be used, e.g., to emphasize certain locations at fine scales if coarser scale coefficients indicate the possible existence of an edge. The results of this paper guarantee that each pass will converge.

In this paper we have restricted ourselves to penalty functions that are weighted ℓ^p-norms of the $f_\gamma = \langle f, \varphi_\gamma \rangle$. The approach can be extended naturally to include penalty functions that can be written as sums, over $\gamma \in \Gamma$, of more general functions of f_γ, so that the functional to be minimized is then written as

$$\widetilde{\Phi}_{\mathbf{W}}(f) = \|Kf - g\|^2 + \sum_{\gamma \in \Gamma} W_\gamma(|f_\gamma|)\,.$$

The arguments in this paper will still be applicable to this more general case if the functions $W_\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ are convex and satisfy some extra technical conditions, which ensure that the corresponding generalized component-shrinkage functions $\widetilde{S}_\gamma$ are still nonexpansive (used in several places), and that, for some $c > 0$,

$$\inf_{\|v\| \le 1} \inf_{\|a\| \le c} \|v\|^{-2} \sum_{\gamma \in \Gamma} \left| v_\gamma + \widetilde{S}_\gamma(a_\gamma) - \widetilde{S}_\gamma(v_\gamma + a_\gamma) \right|^2 > 0$$

(used in Lemma 3.18). To ensure that both conditions are satisfied, it is sufficient to choose functions W_γ that are convex, with a minimum at 0 and, e.g., twice differentiable, except possibly at 0 (where they should nevertheless still be left and right differentiable), and for which $W_\gamma'' > 1$ on $V \setminus \{0\}$, where V is a neighborhood of 0.

We conclude this section with some comments concerning the numerical complexity of the algorithm.

At each iteration step, we must compute the action of the operator K^*K on the current object estimate, expressed in the φ_γ-basis. In a finite-dimensional setting where the solution is represented by a vector of length N, this necessitates in principle a matrix multiplication of complexity $O(N^2)$ if we neglect the cost of the shrinkage operation in each iteration step. After sufficient accuracy is attained and the iterations are stopped, the resulting $(f^n)_\gamma$ must be transformed back into the standard representation domain of the object function except in the special case where the φ_γ are already the basis for the standard representation (e.g., if the φ_γ correspond to the pixel representation for images). This adds one final $O(N^2)$-matrix multiplication. In this scenario, the total cost equals that of the classical

Landweber algorithm on the basis of a comparable number of iterations. It follows that this method can become computationally competitive with the $O(N^3)$ SVD algorithms only when N is large compared to the number of iterations necessary; since it is known that Landweber's algorithm typically requires a substantial number of iterations, this will happen only for very large N.

Several methods have been proposed in the literature to accelerate the convergence of Landweber's iteration, which could be used for the present algorithm as well. For instance, one could use some form of preconditioning (using the operator D of Remark 2.4) or group together k Landweber iteration steps and apply thresholding only every k steps; see, e.g., the book [23].

Much more substantial gains can be obtained when the operator K^*K can be implemented via fast algorithms. In a first important class of applications, the matrix $(\langle K^*K\varphi_\gamma, \varphi_{\gamma'}\rangle)_{\gamma,\gamma'\in\Gamma}$ is sparse; if, for instance, there are only $O(N)$ nonvanishing entries in this matrix, then standard techniques to deal with the action of sparse matrices will reduce the cost of each iteration step to $O(N)$ instead of $O(N^2)$. If the φ_γ-basis is a wavelet basis, this is the case for a large class of integrodifferential operators of interest; see, e.g., [5]. Even if K^*K is sparse in the φ_γ-basis but has an even simpler expression in another basis, and if the transforms back and forth between the two bases can be carried out via fast algorithms, then it may be useful to implement the action of K^*K via these back-and-forth transformations. For instance, if the object is of a type that will have a sparse representation in a wavelet basis, and the operator K^*K is a convolution operator, then we can pick the φ_γ-basis to be a wavelet basis and implement the operator K^*K by doing, successively, a fast reconstruction from wavelet coefficients, followed by an FFT, a multiplication in the Fourier domain, an inverse FFT, and a wavelet transform, for a total complexity of $O(N\log N)$. One can obtain similar complexity estimates if the algorithm is modified to not only take the nonlinear thresholding into account, but also additional projections $P_{\mathcal{C}}$ on a convex set, such as the cone of functions that are a.e. positive; in this case, after the thresholding operation, one needs to carry out an additional fast reconstruction from, say, the wavelet domain, take the positive part, and then perform the fast transform back, without affecting the $O(N\log N)$ complexity estimate.

6 Conclusions

We have discussed in this paper the functional

$$\Phi_{\mathbf{w},p}(f) = \|Kf - g\|^2 + \sum_{\gamma\in\Gamma} w_\gamma |\langle f, \varphi_\gamma\rangle|^p ,$$

and we have shown that the iterative algorithm

$$f^n = \mathbf{S}_{\mathbf{w},p}(f^{n-1} + K^*(g - Kf^{n-1}))$$

with $\mathbf{S}_{\mathbf{w},p}$ as defined by (2.9) combined with (2.6) and (2.8) generates a sequence that converges in norm to a minimizer of $\Phi_{\mathbf{w},p}$. In the particular case $p = 1$, each iteration corresponds to a soft-thresholding of the standard Landweber iterate obtained by adding the backprojected residual error to the previous iterate. Such soft-thresholded iterative algorithms have been proposed by several authors [24, 35, 38, 39] with very good results for the simultaneous deblurring and denoising of images. It is gratifying that, as our mathematical analysis shows, the convergence "scales" well, i.e., holds irrespective of the dimension.

Unless the operator K can be implemented sparsely, the iteration may be too heavy, computationally, to compete effectively with, e.g., modern high-dimensional optimization methods applied to the minimization of $\Phi_{\mathbf{w},p}$ if the number of variables is extremely large. On the other hand, it is conceptually so simple, and so easy to implement, that it is an excellent way of testing whether, and for what parameters, the variational approach is sensible for a problem at hand. Moreover, helped by the present speed of computing, the convergence of the algorithm is still sufficiently fast to permit a high-quality convolutional deblurring combined with denoising of 256×256 images in a few seconds [35], so that even for moderate-sized problems it may be an attractive method.

Appendix A: Wavelets and Besov Spaces

We give a brief review of basic definitions of wavelets and their connection with Besov spaces. This will be a sketch only; for details, we direct the reader to, e.g., [9, 13, 33, 34].

For simplicity we start with dimension 1. Starting from a (very special) function ψ we define

$$\psi_{j,k}(x) = 2^{\frac{j}{2}}\ \psi(2^j x - k)\,, \quad j, k \in \mathbb{Z}\,,$$

and we assume that the collection $\{\psi_{j,k} : j,\ k \in \mathbb{Z}\}$ constitutes an orthonormal basis of $L^2(\mathbb{R})$. For all wavelet bases used in practical applications, there also exists an associated *scaling function* ϕ that is orthogonal to its translates by integers and such that, for all $j \in \mathbb{Z}$,

$$\overline{\operatorname{Span}\{\phi_{j,k};\, k \in \mathbb{Z}\}} \oplus \overline{\operatorname{Span}\{\psi_{j,k};\, k \in \mathbb{Z}\}} = \overline{\operatorname{Span}\{\phi_{j+1,k};\, k \in \mathbb{Z}\}}\,, \tag{A.1}$$

where the $\phi_{j,k}$ are defined analogously to the $\psi_{j,k}$. Typically, the functions ϕ and ψ are very well localized in the sense that $\forall N \in \mathbb{N}$, $\int_{\mathbb{R}}(1 + |x|)^N(|\phi(x)| + |\psi(x)|)dx < \infty$; one can even choose ϕ and ψ such that they are supported on a finite interval. This can be achieved with arbitrary finite smoothness; i.e., for any preassigned $L \in \mathbb{N}$, one can find such ϕ and ψ that are in $C^L(\mathbb{R})$. Because of (A.1), one can consider (inhomogeneous) wavelet expansions in which not all scales j are used, but a cutoff is introduced at some coarsest scale, often set at $j = 0$. More precisely, we shall use the following wavelet expansion of $f \in L^2$,

$$f = \sum_{k=-\infty}^{+\infty} \langle f, \phi_{0,k}\rangle \phi_{0,k} + \sum_{j=0}^{+\infty} \sum_{k=-\infty}^{+\infty} \langle f, \psi_{j,k}\rangle \psi_{j,k}\,. \tag{A.2}$$

Wavelet bases in higher dimensions can be built by taking appropriate products of one-dimensional wavelet and scaling functions. Such d-dimensional bases can be viewed as the result of translating (by elements k of $\mathbb{Z}^d$) and dilating (by integer powers j of 2) of not just one, but several (finite in number) "mother wavelets," typically numbered from 1 to $2^d - 1$. It will be convenient to abbreviate the full label (including j, k, and the number of the mother wavelet) to just λ, with the convention that $|\lambda| = j$. We shall again cut off at some coarsest scale, and we shall follow the convenient slight abuse of notation used in [9] that sweeps up the coarsest-j scaling functions (as in (A.2)) into the Ψ_λ as well. We thus denote the complete d-dimensional, inhomogeneous wavelet basis by $\{\Psi_\lambda : \lambda \in \Lambda\}$.

It turns out that $\{\Psi_\lambda : \lambda \in \Lambda\}$ is not only an orthonormal basis for $L^2(\mathbb{R}^d)$, but also an unconditional basis for a variety of other useful Banach spaces of functions, such as Hölder spaces, Sobolev spaces, and, more generally, Besov spaces. Again, we review only some basic facts; a full study can be found in [9, 13, 34]. The Besov spaces $B^s_{p,q}(\mathbb{R}^d)$ consist, basically, of functions that "have s derivatives in L^p"; the parameter q provides some additional fine-tuning to the definition of these spaces. The norm $\|f\|_{B^s_{p,q}}$ in a Besov space $B^s_{p,q}$ is traditionally defined via the *modulus of continuity* of f in $L^p(\mathbb{R})$, of which an additional weighted L^q-norm is then taken, in which the integral is over different scales. We shall not give its details here; for our purposes it suffices that this traditional Besov norm is equivalent with a norm that can be computed from wavelet coefficients. More precisely, let us assume that the original one-dimensional ϕ and ψ are in $C^L(\mathbb{R})$, with $L > s$, that $\sigma = s + d(\frac{1}{2} - \frac{1}{p}) \geq 0$, and define the norm $|||\cdot|||_{s;p,q}$ by

$$|||f|||_{s;p,q} = \left(\sum_{j=0}^{\infty} \left(2^{j\sigma p} \sum_{\lambda\in\Lambda, |\lambda|=j} |\langle f, \Psi_\lambda\rangle|^p \right)^{\frac{q}{p}} \right)^{\frac{1}{q}} . \tag{A.3}$$

Then this norm is equivalent to the traditional Besov norm, $|||f|||_{s;p,q} \sim \|f\|_{B^s_{p,q}}$, that is, there exist strictly positive constants A and B such that

$$A|||f|||_{s;p,q} \leq \|f\|_{B^s_{p,q}} \leq B|||f|||_{s;p,q} . \tag{A.4}$$

The condition that $\sigma \geq 0$ is imposed to ensure that $B^s_{p,q}(\mathbb{R}^d)$ is a subspace of $L^2(\mathbb{R}^d)$; we shall restrict ourselves to this case in this paper. From (A.3) one can gauge the fine-tuning role played by the parameter q in the definition of the Besov spaces. A particularly convenient choice, to which we shall stick in the remainder of this paper, is $q = p$, for which the expression simplifies to

$$|||f|||_{s,p} = \left(\sum_{\lambda\in\Lambda} 2^{\sigma p|\lambda|} |\langle f, \Psi_\lambda\rangle|^p \right)^{\frac{1}{p}} ;$$

to alleviate notation, we shall drop the extra index q wherever it normally occurs on the understanding that $q = p$ when we do so.

When $0 < p, q < 1$, the Besov spaces can still be defined as complete metric spaces, although they are no longer Banach spaces (because (A.3) no longer is a norm). This allows for more local variability in local smoothness than is typical for functions in the usual Hölder or Sobolev spaces. For instance, a real function f on $\mathbb{R}$ that is piecewise continuous, but for which each piece is locally in C^s, can be an element of $B^s_p(\mathbb{R})$, despite the possibility of discontinuities at the transition from one piece to the next, provided $p > 0$ is sufficiently small, and some technical conditions are met on the number and size of the discontinuities and on the decay at ∞ of f.

Wavelet bases are thus closely linked to a rich class of smoothness spaces; they also provide a good tool for high-accuracy, nonlinear approximation of a wide variety of functions. For instance, if the bounded function f on $[0, 1]$ has only finitely many discontinuities and is C^s elsewhere, then one can find a way of renumbering (dependent on f itself) the wavelets in the standard wavelet expansion of f, so that the distance in, say, $L^2([0, 1])$ between f and the first N terms of this reordered wavelet expansion decreases proportionally to N^{-s}. If s is large, it follows that a very accurate approximation to f can be obtained with relatively few wavelets; this is possible because the smooth patches of the piecewise continuous f will be well approximated by coarse scale wavelets, which are few in number; to capture the behavior of f near the discontinuities, much more localized finer scale wavelets are required, but only those wavelets located close to the discontinuities will be needed, which amounts again to a small number.

In higher dimensions, $d > 1$, the suitability of wavelets is influenced by the dimension of the manifolds on which singularities occur. If the singularities in the functions of interest are solely point singularities, then expansions using N wavelets can still approximate such functions with distances that decrease like N^{-s}, depending on their behavior away from the singularities. If, however, we are interested in f that may have, e.g., discontinuities along manifolds of dimension higher than 0, then such wavelet approximations are not optimal. For instance, if $f : \mathbb{R}^2 \to \mathbb{R}$ is piecewise C^L, with possible jumps across the boundaries of the smoothness domains, which are themselves smooth (say, C^L again) curves, then N-term wavelet approximations to f cannot achieve an error rate decay faster than $N^{-1/2}$, regardless of the value of $L > 1$.

It follows that whenever we are faced with an inverse problem that needs regularization, in which the objects to be restored are expected to be mostly smooth, with very localized lower-dimensional areas of singularities, we can expect that their expansions into wavelets will be sparse. This sparsity can be expressed by requiring that the wavelet coefficients (possibly with some scale-dependent weight) have a finite (or small) ℓ^p-norm, with $1 \leq p \leq 2$, or equivalently that the Besov-equivalent norm $|||f|||_{s,p}$ is finite (or small), where $|||f|||_{s,p}$ is exactly of the form defined in (1.3).

Appendix B: A Fixed-Point Theorem

We provide here the proof of the theorem needed to establish the weak convergence of the iterative algorithm. The theorem is given in [36]; we give a simplified proof here (see the remark at the end), which nevertheless still follows the main lines of Opial's paper.

THEOREM B.1 *Let $\mathcal{C}$ be a closed convex subset of the Hilbert space $\mathcal{H}$ and let the mapping* $\mathbf{A} : \mathcal{C} \to \mathcal{C}$ *satisfy the following conditions*:

(i) $\mathbf{A}$ *is nonexpansive*: $\|\mathbf{A}v - \mathbf{A}v'\| \le \|v - v'\| \; \forall v, v' \in \mathcal{C}$,
(ii) $\mathbf{A}$ *is asymptotically regular*: $\|\mathbf{A}^{n+1}v - \mathbf{A}^n v\| \xrightarrow[n\to\infty]{} 0 \; \forall v \in \mathcal{C}$,
(iii) *the set $\mathcal{F}$ of the fixed points of* $\mathbf{A}$ *in $\mathcal{C}$ is not empty.*

Then, $\forall v \in \mathcal{C}$, the sequence $(\mathbf{A}^n v)_{n\in\mathbb{N}}$ converges weakly to a fixed point in $\mathcal{F}$.

The proof of the main theorem will follow from a series of lemmas. As before, we use the notation w-lim to indicate a *weak* limit.

LEMMA B.2 *If $u, v \in \mathcal{H}$, and if $(v_n)_{n\in\mathbb{N}}$ is a sequence in $\mathcal{H}$ such that* $\text{w-lim}_{n\to\infty} v_n = v$, *and $u \neq v$, then* $\liminf_{n\to\infty} \|v_n - u\| > \liminf_{n\to\infty} \|v_n - v\|$.

PROOF: We have

$$\begin{aligned}\liminf_{n\to\infty} \|v_n - u\|^2 &= \liminf_{n\to\infty} \|v_n - v\|^2 + \|v - u\|^2 \\ &\quad + 2 \lim_{n\to\infty} \operatorname{Re}(v_n - v, v - u) \\ &= \liminf_{n\to\infty} \|v_n - v\|^2 + \|v - u\|^2 \,,\end{aligned}$$

whence the result. □

LEMMA B.3 *Suppose that* $\mathbf{A} : \mathcal{C} \to \mathcal{C}$ *satisfies condition* (i) *in Theorem* B.1. *If* $\text{w-lim}_{n\to\infty} u_n = u$ *and* $\lim_{n\to\infty} \|u_n - \mathbf{A}u_n - h\| = 0$, *then* $h = u - \mathbf{A}u$.

PROOF: Because of the nonexpansivity of $\mathbf{A}$ (assumption (i)), we have

$$\begin{aligned}\|u_n - (h + \mathbf{A}u)\| &\le \|u_n - h - \mathbf{A}u_n\| + \|\mathbf{A}u_n - \mathbf{A}u\| \\ &\le \|u_n - h - \mathbf{A}u_n\| + \|u_n - u\| \,.\end{aligned}$$

Hence,

$$\begin{aligned}\liminf_{n\to\infty} \|u_n - (h + \mathbf{A}u)\| &\le \lim_{n\to\infty} \|h - (u_n - \mathbf{A}u_n)\| + \liminf_{n\to\infty} \|u_n - u\| \\ &= \liminf_{n\to\infty} \|u_n - u\| \,.\end{aligned}$$

It then follows from Lemma B.2 that $u = h + \mathbf{A}u$ or $h = u - \mathbf{A}u$. □

LEMMA B.4 *Suppose that* $\mathbf{A} : \mathcal{C} \to \mathcal{C}$ *satisfies conditions* (i) *and* (ii) *in Theorem* B.1. *If a subsequence of* $(\mathbf{A}^n v)_{n\in\mathbb{N}}$, *with $v \in \mathcal{C}$, converges weakly in $\mathcal{C}$, then its limit is in $\mathcal{F}$.*

PROOF: Suppose $\text{w-lim}_{k\to\infty} \mathbf{A}^{n_k} v = u$. Since, by assumption (ii) of asymptotic regularity, $\lim_{n\to\infty} \|\mathbf{A}^n v - \mathbf{A}\mathbf{A}^n v\| = 0$, we have $\lim_{k\to\infty} \|\mathbf{A}^{n_k} v - \mathbf{A}\mathbf{A}^{n_k} v\| = 0$. By Lemma B.3, it follows that $u - \mathbf{A}u = 0$, i.e., that u is in $\mathcal{F}$. □

LEMMA B.5 *Suppose that* $\mathbf{A} : \mathcal{C} \to \mathcal{C}$ *satisfies conditions* (i) *and* (iii) *in Theorem* B.1. *Then, for all* $h \in \mathcal{F}$ *and all* $v \in \mathcal{C}$, *the sequence* $(\|\mathbf{A}^n v - h\|)_{n\in\mathbb{N}}$ *is nonincreasing and thus has a limit.*

PROOF: Since $\mathbf{A}$ is nonexpansive, we have indeed

$$\|\mathbf{A}^{n+1} v - h\| = \|\mathbf{A}\mathbf{A}^n v - \mathbf{A}h\| \le \|\mathbf{A}^n v - h\| .$$

□

We can now proceed to prove Theorem B.1.

PROOF OF THEOREM B.1: Let v be any element in $\mathcal{C}$. Take an arbitrary $h \in \mathcal{F}$. By Lemma B.5, we then have

$$\begin{aligned}\limsup_{n\to\infty} \|\mathbf{A}^n v\| &\le \limsup_{n\to\infty} \|\mathbf{A}^n v - h\| + \|h\| \\ &= \|h\| + \lim_{n\to\infty} \|\mathbf{A}^n v - h\| < \infty .\end{aligned}$$

Since the $\mathbf{A}^n v$ are thus uniformly bounded, it follows from the Banach-Alaoglu theorem that they must have at least one weak accumulation point.

The following argument shows that this accumulation point is unique. Suppose we have two different accumulation points:

$$\text{w-}\lim_{k\to\infty} \mathbf{A}^{n_k} v = u \quad \text{and} \quad \text{w-}\lim_{\ell\to\infty} \mathbf{A}^{\tilde{n}_\ell} v = \tilde{u} \quad \text{with } u \ne \tilde{u} .$$

By Lemma B.4, u and $\tilde{u}$ must both lie in $\mathcal{F}$, and by Lemma B.5, the limits $\lim_{n\to\infty} \|\mathbf{A}^n v - u\|$ and $\lim_{n\to\infty} \|\mathbf{A}^n v - \tilde{u}\|$ both exist. Since $\tilde{u} \ne u$, we obtain from Lemma B.2 that $\liminf_{k\to\infty} \|\mathbf{A}^{n_k} v - \tilde{u}\| > \liminf_{k\to\infty} \|\mathbf{A}^{n_k} v - u\|$. On the other hand, because $(\|\mathbf{A}^{n_k} v - \tilde{u}\|)_{k\in\mathbb{N}}$ and $(\|\mathbf{A}^{n_k} v - u\|)_{k\in\mathbb{N}}$ are each a subsequence of a convergent sequence, $\liminf_{k\to\infty} \|\mathbf{A}^{n_k} v - \tilde{u}\| = \lim_{n\to\infty} \|\mathbf{A}^n v - \tilde{u}\|$ and $\liminf_{k\to\infty} \|\mathbf{A}^{n_k} v - u\| = \lim_{k\to\infty} \|\mathbf{A}^{n_k} v - u\|$. It follows that $\lim_{n\to\infty} \|\mathbf{A}^n v - \tilde{u}\| > \lim_{n\to\infty} \|\mathbf{A}^n v - u\|$. In a completely analogous way (working with the subsequence $\mathbf{A}^{\tilde{n}_l} v$ instead of $\mathbf{A}^{n_k} v$) one derives the opposite strict inequality. Since both cannot be valid simultaneously, the assumption of the existence of two different weak accumulation points for $(\mathbf{A}^n v)_{n\in\mathbb{N}}$ is false. It thus follows that $\mathbf{A}^n v$ converges weakly to this unique weak accumulation point. □

Remark B.6. It is essential to require that the set $\mathcal{F}$ is not empty since there are asymptotically regular, nonexpansive maps that possess no fixed point. However, the only place where we used this assumption was in showing that the $\|\mathbf{A}^n v\|$ were bounded. If one can prove this boundedness by some other means (e.g., by a variational principle as we did in the iterative algorithm), then we automatically have a weakly convergent subsequence $(\mathbf{A}^{n_k} v)_{k\in\mathbb{N}}$, and thus, by Lemma B.4, an element of $\mathcal{F}$.

Remark B.7. The simplification of the original argument of [36] (obtained through deriving the contradiction in the proof of Theorem B.1) avoids having to appeal to the convexity of $\mathcal{F}$ (which is true but not immediately obvious) and having to introduce the auxiliary sets $\mathcal{F}_\delta$ used in [36].

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