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CONSTRUCTIVE APPROXIMATION

How Does Truncation of the Mask Affect a Refinable Function?

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Abstract. If the mask of a refinable function has infinitely many coefficients, or if the coefficients are irrational, then it is often replaced by a finite mask with coefficients with terminating decimal expansions when it comes to applications. This note studies how such truncation affects the refinable function.

1. Introduction

A *refinable function* ϕ is a function satisfying the following so-called *refinement equation*:

$$\phi(x) = \sum_{k \in \mathbf{Z}} b_k \phi(2x - k), \tag{1.1}$$

where $\{b_k\}$ is a fixed real scalar sequence called the *mask*. Using the Fourier transform of ϕ:

$$\widehat{\phi}(\xi) := \int_{-\infty}^{+\infty} \phi(x) e^{-i\xi x}\, dx,$$

we find that another form of (1.1) is

$$\widehat{\phi}(\xi) = \left[\frac{1}{2} \sum_{k} b_k e^{ik\xi/2}\right] \cdot \widehat{\phi}\left(\frac{\xi}{2}\right), \tag{1.2}$$

which implies, at least formally,

$$\widehat{\phi}(\xi) = \prod_{j=1}^{\infty} b(2^{-j}\xi)\widehat{\phi}(0), \tag{1.3}$$

where

$$b(\xi) := \frac{1}{2} \sum_{k \in \mathbf{Z}} b_k e^{ik\xi}$$

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is the filter. If $b(0) = 1$, and $|b(\xi) - b(\zeta)| \leq C|\xi - \zeta|^s$ for some $C, s > 0$, then it can easily be checked that the infinite product (1.3) is well defined for all $\xi \in \mathbf{R}$, and that $\widehat{\phi}$ is continuous, with the same Hölder exponent. We always suppose that b satisfies these conditions.

Functions of this kind are encountered in subdivision methods in computer graphics, where interpolation or other subdivision schemes are used to construct smooth curves or surfaces. For an excellent review, we refer to [CDM] and the papers cited there. Refinable functions also occur in wavelet theory as "scaling functions" which can be used as tools to construct orthonormal [D2] or biorthogonal wavelet bases [CDF] within a multiresolution analysis framework [M].

There have been many papers studying in detail the characterization and regularity of refinable functions, mostly with finite masks [MP], [DyLe], [DaLa], [R], [CR], [V], [E]; a few papers are also adapted to infinite masks [G], [He3], [CD]. This small note concentrates on the behavior of a refinable function when there is a small perturbation on the mask. It is motivated by the study of the iterates of a nonlinear operator related to wavelets in [Hu], which turn out to be linked with refinement equations associated with not necessarily finite masks. In order to plot the graphs of the corresponding refinable functions, we had to truncate the filters first. Then the problem naturally arises: how close is the graph thus obtained to the graph of the true function? In other words, how does truncation of the mask affect a refinable function?

A similar problem arises with the wavelets corresponding to infinite impulse response (IIR) filters in [HV]. In fact, the question is relevant even for compactly supported wavelets, corresponding to scaling functions with finite masks: in contrast with more traditional subdivision schemes, the coefficients in these masks are often irrational (because they have to satisfy many other nonlinear constraints; see [D1] or [D2]), computed numerically with reasonably high precision. Using terminating decimal (or binary) expansions corresponds again to a truncation of the mask, in the sense that the functions plotted in, e.g., [D1] or [D2] do not correspond exactly to $b(\xi)$ but to a very close $\tilde{b}(\xi)$. To see this, let us go back to formulas (1.1)–(1.3). It is clear from them that the function ϕ is completely determined by the mask b_k, up to normalization. If b satisfies some technical conditions, to which we return below, then $\phi(x)$ is approximated arbitrarily well by the functions $f_n(x)$ as $n \to \infty$, with f_n defined by

$$\widehat{f_n}(\xi) := \prod_{j=1}^{n} b(2^{-j}\xi)\widehat{f_0}(\xi), \tag{1.4}$$

where we take $f_0(x) = 1 - |x|$ if $|x| \leq 1$, $f_0(x) = 0$ otherwise. The f_n are then piecewise linear, with nodes at the $2^{-n}k$, $k \in \mathbf{Z}$. Since $\|\phi - f_n\|_{L^\infty(\mathbf{R})} \longrightarrow 0$, we can get a good idea of what ϕ looks like from plotting $f_n(x)$ with sufficiently large n. In practice the mask b_k or the filter $b(\xi)$ may have to be truncated (by cut-off if the $\{b_k\}$ are an infinite sequence or round-off if the b_k are irrational), which amounts to replacing $b(\xi)$ by $\tilde{b}(\xi)$. What we truly plot, therefore, is $\tilde{f}_n(x)$, defined by

$$\widehat{\tilde{f}}_n(\xi) := \prod_{j=1}^{n} \tilde{b}(2^{-j}\xi)\widehat{f_0}(\xi). \tag{1.5}$$

So we need also show that $\|f_n - \tilde{f}_n\|_{L^\infty}$ is small. This is the purpose of this note.

In all useful examples, $b(\xi)$ can be factored as

$$(1.6)\qquad b(\xi) = \left(\cos\frac{\xi}{2}\right)^r v(\xi),$$

with $r \geq 1$, $v \in C(0, 2\pi)$, $v(\pi) \neq 0$; a factorization of this type is in fact necessary in order to have L^∞-convergence of the f_n [DyLe]. We show that our truncation estimates depend on whether $\tilde{b}(\pi) \neq 0$ or $\tilde{b}(\pi) = 0$. In the latter case $\tilde{b}$ can be factored as well,

$$(1.7)\qquad \tilde{b}(\xi) = \left(\cos\frac{\xi}{2}\right)^{\tilde{r}} \tilde{v}(\xi),$$

with $\tilde{r} \geq 1$, $\tilde{v} \in C(0, 2\pi)$, $\tilde{v}(\pi) \neq 0$; modulo some technical conditions on b and $\tilde{b}$ we then show that $\sup_n \|f_n - \tilde{f}_n\|_{L^\infty(\mathbf{R})}$ can be made arbitrarily small by choosing $\|b - \tilde{b}\|_{L^\infty(0,2\pi)}$ sufficiently small. In the first case we show that, for fixed n, we can still make $\|f_n - \tilde{f}_n\|_{L^\infty(\mathbf{R})}$ small, but the norm of this difference may diverge as $n \to \infty$, however small $\|b - \tilde{b}\|_{L^\infty(0,2\pi)}$.

2. Convergence of the Approximation Functions f_n

We start by recalling a few results which can be found in many places in the literature, adapting them slightly to our purpose.

One of the main tools in the study of orthonormal wavelets and their regularity is the Perron–Frobenius or transfer operator associated with the mask (or powers of the mask) of the corresponding scaling function [CR], [G], [E], [V], [He1], [CD]. Given a continuous 2π-periodic function M, the transfer operator P_M, acting on 2π-periodic functions F, is defined by

$$(2.1)\qquad (P_M F)(\xi) := M\left(\frac{\xi}{2}\right) \cdot F\left(\frac{\xi}{2}\right) + M\left(\frac{\xi}{2} + \pi\right) \cdot F\left(\frac{\xi}{2} + \pi\right).$$

Its adjoint is also known as the "subdivision operator" [CDM] used in, e.g., [DyLe] and [R]. It can easily be checked that

$$(2.2)\qquad (P_M^* F)(\xi) = 2\overline{M(\xi)} F(2\xi),$$

where the adjoint is defined in the standard L^2-sense. A consequence of (2.2) is that

$$(2.3)\qquad \left[(P_{M_3}^*)^{n_3} (P_{M_2}^*)^{n_2} (P_{M_1}^*)^{n_1} F\right](\xi)$$
$$= 2^{n_1+n_2+n_3} \left[\prod_{j=0}^{n_3-1} \overline{M_3(2^j\xi)} \prod_{k=0}^{n_2-1} \overline{M_2(2^{k+n_3}\xi)} \prod_{\ell=0}^{n_1-1} \overline{M_1(2^{\ell+n_2+n_3}\xi)}\right]$$
$$\cdot F(2^{n_1+n_2+n_3}\xi).$$

Taking the inner product with a function G, we can rewrite (2.3) as

$$\int_{|\xi|\le\pi} \overline{F(\xi)}\left[P_{M_1}^{n_1}P_{M_2}^{n_2}P_{M_3}^{n_3}G\right](\xi)d\xi \tag{2.4}$$

$$= \int_{|\xi|\le 2^{n_1+n_2+n_3}\pi} \overline{F(\xi)}G(2^{-n_1-n_2-n_3}\xi)$$

$$\prod_{\ell=1}^{n_1} M_1(2^{-\ell}\xi) \prod_{k=n_1+1}^{n_1+n_2} M_2(2^{-k}\xi) \prod_{j=n_1+n_2+1}^{n_1+n_2+n_3} M_3(2^{-j}\xi)d\xi.$$

This is a generalization of a lemma found in, e.g., [CR], [C], [V], and [E]. If we put $F = G = 1$ in (2.4), then we see that

$$\left|\int_{|\xi|\le 2^{n_1+n_2+n_3}\pi} \prod_{\ell=1}^{n_1} M_1(2^{-\ell}\xi) \prod_{k=n_1+1}^{n_1+n_2} M_2(2^{-k}\xi) \prod_{j=n_1+n_2+1}^{n_1+n_2+n_3} M_3(2^{-j}\xi)\right| \tag{2.5}$$

$$\le 2\pi \|P_{M_1}^{n_1}\|\|P_{M_2}^{n_2}\|\|P_{M_3}^{n_3}\|;$$

here, and in what follows, $\|\|$ denotes the operator norm on the Banach space $C[0, 2\pi]$.

The spectral radii of transfer operators play an important role in estimating the regularity of refinable functions; many of the different approaches can be reduced to the computation of such a spectral radius in some appropriate space; see the discussion at the end of [CD]. We quote here one such result from the Ph.D. thesis of Hervé [He1]:

Proposition 2.1. *If $v(\xi)$ has only a finite number of zeros, then $\rho_{|v|}$, the spectral radius for the operator $P_{|v|}$ restricted to $C[0,2\pi]$, is given by*

$$\rho_{|v|} = \lim_{n\to\infty} \frac{S_n}{S_{n-1}}, \tag{2.6}$$

where

$$S_n = \sum_{k=0}^{2^n-1} \prod_{\ell=0}^{n-1} |v(2\pi 2^{\ell-n}k)|.$$

The following lemma gives easy lower and upper bounds on $\rho_{|v|}$:

Lemma 2.2. *If M is a positive, continuous 2π-periodic function, then ρ_M, the spectral radius of P_M on $C(0,2\pi)$, satisfies*

$$M(0) \le \rho_M \le \max_\xi[M(\xi) + M(\xi+\pi)].$$

Proof. 1. For any $\varepsilon > 0$, we can find $\alpha < 2\pi$ so that, for $0 \le \xi \le \alpha$,

$$M(\xi) \ge M(0) - \varepsilon.$$

2. Let $\varphi_{n,\alpha}$ be continuous 2π-periodic functions so that, for all $n \in \mathbf{N}$,

$$0 \le \varphi_{n,\alpha}(\xi) \le 1, \qquad \text{for all } \xi,$$

$$\varphi_{n,\alpha}(\xi) := 1, \qquad \frac{1}{3}2^{-n}\alpha \le \xi \le \frac{2}{3}2^{-n}\alpha,$$

$$\varphi_{n,\alpha}(\xi) := 0, \qquad 2^{-n}\alpha \le \xi \le 2\pi.$$

Then $\|\varphi_{n,\alpha}\|_{L^\infty} = 1$, hence

$$\begin{aligned}\sup_{\substack{f\in C[0,2\pi]\\ \|f\|_{L^\infty}=1}} \|(P_M)^n f\|_{L^\infty} &\ge \|P_M^n \varphi_{n,\alpha}\|_{L^\infty} \\ &\ge \frac{1}{2\pi}\int_{|\xi|\le\pi} (P_M^n\varphi_{n,\alpha})(\xi)\,d\xi \\ &= \frac{1}{2\pi}2^n \int_{|\zeta|\le\pi} \prod_{j=0}^{n-1} M(2^j\zeta)\varphi_{n,\alpha}(\zeta)\,d\zeta \qquad \text{(by (2.4))} \\ &= \frac{2^n}{2\pi}\int_{0\le\zeta\le 2^{-n}\alpha} \prod_{j=0}^{n-1} M(2^j\zeta)\varphi_{n,\alpha}(\zeta)\,d\zeta \\ &\ge \frac{2^n}{2\pi}(M(0)-\varepsilon)^n \int_{0\le\zeta\le 2^{-n}\alpha} \varphi_{n,\alpha}(\zeta)\,d\zeta \\ &\ge \frac{1}{2\pi}(M(0)-\varepsilon)^n \frac{\alpha}{3}.\end{aligned}$$

3. Consequently we have

$$\rho_M = \liminf_{n\to\infty} \|P_M^n\|^{1/n} \ge M(0) - \varepsilon,$$

which implies $\rho_M \ge M(0)$, since ε was arbitrarily small.

4. The upper bound is even easier: we have

$$\begin{aligned}\|P_M f\|_{L^\infty} &= \max_\xi \left| M\left(\frac{\xi}{2}\right) f\left(\frac{\xi}{2}\right) + M\left(\frac{\xi}{2}+\pi\right) f\left(\frac{\xi}{2}+\pi\right)\right| \\ &\le \|f\|_{L^\infty} \max_\xi \left| M\left(\frac{\xi}{2}\right) + M\left(\frac{\xi}{2}+\pi\right)\right|,\end{aligned}$$

implying $\rho_M \le \|P_M\| \le \max_\xi |M(\xi) + M(\xi+\pi)|$. ■

These bounds are rather coarse, but they suffice for our purpose.

Spectral radii of transfer operators are also important in the discussion of the convergence of the f_n to ϕ; see, e.g., [R] or [DyLe]. The following proposition is similar to the results proved there, although we take the spectral radius on a slightly different space.

Proposition 2.3. *Let $b(\xi)$ be a 2π-periodic C^∞ function with $b(0) = 1$ and*

$$|b(\xi)| = \left|\cos\frac{\xi}{2}\right|^r \cdot |v(\xi)|, \qquad r \in \mathbf{N}, \quad \mathbf{v} \in \mathbf{C}^1, \quad \mathbf{v}(\pi) \ne \mathbf{0}.$$

Let $\beta = \rho_{|v|}$ be the spectral radius of $P_{|v|}$ restricted to $C[0,2\pi]$. If $\beta < 2^r$, then the functions $f_n(x)$ defined by

(2.7)
$$\widehat{f_n}(\xi) := \prod_{j=1}^{n} b(2^{-j}\xi)\widehat{f_0}(2^{-n}\xi),$$

with

(2.8)
$$f_0(x) := (1 - |x|)\chi_{[-1,1]}(x),$$

converge uniformly to the function ϕ defined by

$$\widehat{\phi}(\xi) := \prod_{j=1}^{\infty} b(2^{-j}\xi).$$

Moreover, for all $s < r - \log_2 \beta$, ϕ belongs to the Hölder space $C^s(\mathbf{R})$, and the convergence of the f_n is exponentially fast with a rate at least $\tilde{s} = \min(1,s)$,

$$\|f_n - \phi\|_{L^\infty} \leq C(\tilde{s})\, 2^{-\tilde{s}n}.$$

Remark. 1. f_0 can be replaced by any function satisfying $xf_0 \in L^1(\mathbf{R})$, $\widehat{f_0}(2\pi\ell) = \delta_{\ell,0}$, and $|\widehat{f_0}(\xi)| < C(1 + |\xi|)^{-1-\varepsilon}$ for some $C, \varepsilon > 0$.

2. The regularity conditions on b can be weakened significantly, but it does not really matter: in all practical applications b is in fact analytic in a strip $|\text{Im}\xi| \leq \alpha$.

Proof. 1. We first show that $\phi \in C^s(\mathbf{R})$ for all $s < r - \log_2 \beta$. Estimates of this type can be found in, e.g., [C], [V], [He1], [He3]; this particular one, using the spectral radius on $C[0, 2\pi]$, is borrowed from [He3]. We include the argument because we need some of the estimates again later. We want to prove that, for any $\varepsilon > 0$,

$$I_n := \int_{2^{n-1}\pi \leq |\xi| \leq 2^n\pi} \prod_{j=1}^{\infty} |b(2^{-j}\xi)| d\xi \leq C\left(\frac{\beta + \varepsilon}{2^r}\right)^n.$$

Now using the classical formulas

(2.9)
$$\prod_{j=1}^{n} \cos\frac{\xi}{2^{j+1}} = \frac{\sin(\xi/2)}{2^n \sin(2^{-n-1}\xi)}$$

and (taking $n \to \infty$ in (2.9))

(2.10)
$$\prod_{j=1}^{\infty} \cos\frac{\xi}{2^{j+1}} = \frac{\sin(\xi/2)}{(\xi/2)},$$

we find

$$\begin{aligned} I_n &= \int_{2^{n-1}\pi \le |\xi| \le 2^n\pi} \prod_{j=1}^{\infty} \left|\cos \frac{\xi}{2^{j+1}}\right|^r \prod_{j=1}^{\infty} |v(2^{-j}\xi)| d\xi \\ &= \int_{2^{n-1}\pi \le |\xi| \le 2^n\pi} \left|\frac{\sin(\xi/2)}{\xi/2}\right|^r \prod_{j=1}^{\infty} |v(2^{-j}\xi)| d\xi \\ &\le C(2^{n-1}\pi)^{-r} \int_{2^{n-1}\pi \le |\xi| \le 2^n\pi} \prod_{j=1}^{\infty} |v(2^{-j}\xi)| d\xi. \end{aligned}$$

Note that if we let

$$h(\xi) := \prod_{j=1}^{\infty} |v(2^{-j}\xi)| = \prod_{j=1}^{n} |v(2^{-j}\xi)| h(2^{-n}\xi),$$

then, since $v \in C^1$ and $v(0) = 1$, we know that h is continuous. It follows that, for $2^{n-1}\pi \le |\xi| \le 2^n\pi$, $|h(2^{-n}\xi)| \le \sup_{\pi/2 \le |\zeta| \le \pi} |h(\zeta)| = C$, implying $|h(\xi)| \le C\prod_{j=1}^{n} |v(2^{-j}\xi)|$. Hence

$$\begin{aligned} I_n &\le C\,2^{-rn} \int_{2^{n-1}\pi \le |\xi| \le 2^n\pi} \prod_{j=1}^{n} |v(2^{-j}\xi)|\, d\xi \\ &\le C\,2^{-rn} \int_{|\xi| \le 2^n\pi} \prod_{j=1}^{n} |v(2^{-j}\xi)|\, d\xi \\ &\le C' 2^{-rn}(\beta+\varepsilon)^n = C'\left(\frac{\beta+\varepsilon}{2^r}\right)^n \qquad \text{(by (2.5)).} \end{aligned}$$

Then

$$\int_{-\infty}^{+\infty} (1+|\xi|)^s |\widehat{\phi}(\xi)|\, d\xi \le C\left[1 + \sum_n 2^{ns}\left((\beta+\varepsilon)/2^r\right)^n\right] < \infty$$

if $s < r - \log_2(\beta+\varepsilon)$, implying $\phi \in C^s(\mathbf{R})$.

2. To prove the rest of the proposition, we first note that, using the identity (2.9) to replace (2.10) in the estimate above, the same technique of proof can be used to prove that, for any $0 < \alpha \le \pi$,

$$\int_{2^n\alpha \le |\xi| \le 2^n\pi} \prod_{j=1}^{n} |b(2^{-j}\xi)|\, d\xi \le C(\alpha)\, 2^{-ns}. \tag{2.11}$$

3. We now prove the L^1-convergence of the sequence $\widehat{f_n}$ to $\widehat{\phi}$, which will imply the uniform convergence of $f_n(x)$ to $\phi(x)$. Since b is smooth, so is $\widehat{\phi}$; for any R we can therefore find C_R so that $|\widehat{\phi}(\xi) - 1| = |\widehat{\phi}(\xi) - \widehat{\phi}(0)| \le C_R|\xi|$ if $|\xi| \le R$. On the other hand, $\widehat{f_0}(\xi) = 4\xi^{-2}(\sin\frac{\xi}{2})^2$, hence $|\widehat{f_0}(\xi) - 1| \le C|\xi|^2$. It follows that, for $|\xi| \le \pi$,

$$|\widehat{\phi}(\xi) - \widehat{f_0}(\xi)| \le C|\xi|^{\tilde{s}}. \tag{2.12}$$

4. Divide $\|\widehat{f_n} - \widehat{\phi}\|_{L^1}$ into three parts:

$$\|\widehat{f_n} - \widehat{\phi}\|_{L^1} \le \int_{|\xi|\le 2^n\pi} |\widehat{f_n}(\xi) - \widehat{\phi}(\xi)|\, d\xi + \int_{|\xi|\ge 2^n\pi} |\widehat{f_n}(\xi)|\, d\xi$$
$$+ \int_{|\xi|\ge 2^n\pi} |\widehat{\phi}(\xi)|\, d\xi =: I_1 + I_2 + I_3.$$

Then

$$I_3 := \int_{|\xi|\ge 2^n\pi} (1+|\xi|)^{-\tilde{s}}(1+|\xi|)^{\tilde{s}} |\widehat{\phi}(\xi)|\, d\xi \tag{2.13}$$
$$\le 2^{-n\tilde{s}} \int_{-\infty}^{\infty} (1+|\xi|)^{s} |\widehat{\phi}(\xi)|\, d\xi \le C\, 2^{-n\tilde{s}}.$$

5. For I_2 we have

$$I_2 := \int_{|\xi|\ge 2^n\pi} \prod_{j=1}^{n} |b(2^{-j}\xi)| |\widehat{f_0}(2^{-n}\xi)|\, d\xi$$
$$= \int_{|\xi|\le 2^n\pi} \prod_{j=1}^{n} |b(2^{-j}\xi)| \sum_{\ell\ne 0} |\widehat{f_0}(2^{-n}\xi + 2\ell\pi)|\, d\xi.$$

However,

$$\sum_{\ell\ne 0} |\widehat{f_0}(2^{-n}\xi + 2\ell\pi)| = |\sin(2^{-n-1}\xi)|^2 \sum_{\ell\ne 0} |\ell\pi + 2^{-n-1}\xi|^{-2},$$

which, for $|\xi| \le 2^n\pi$, is bounded by $C(2^{-n}|\xi|)^2 \le C'(2^{-n}|\xi|)^{\tilde{s}}$. Hence

$$I_2 \le C' 2^{-n\tilde{s}} \int_{|\xi|\le 2^n\pi} |\xi|^{\tilde{s}} \prod_{j=1}^{n} |b(2^{-j}\xi)|\, d\xi. \tag{2.14}$$

6. On the other hand, (2.12) implies

$$I_1 := \int_{|\xi|\le 2^n\pi} \left[\prod_{j=1}^{n} |b(2^{-j}\xi)|\right] |\widehat{f_0}(2^{-n}\xi) - \widehat{\phi}(2^{-n}\xi)|\, d\xi \tag{2.15}$$
$$\le C\, 2^{-n\tilde{s}} \int_{|\xi|\le 2^n\pi} |\xi|^{\tilde{s}} \prod_{j=1}^{n} |b(2^{-j}\xi)|\, d\xi.$$

This is of the same type as (2.14), and we can therefore treat the two terms together.

7. Since $\widehat{\phi}$ is continuous in 0, and since $\widehat{\phi}(0) = 1$, an $\alpha \in (0, \pi]$ exists such that

$$|\xi| \le \alpha \quad \Rightarrow \quad |\widehat{\phi}(\xi)| \ge C > 0. \tag{2.16}$$

Now divide the integral in (2.14) or (2.15) into two parts, on the domains $|\xi| \le 2^n\alpha$, and $2^n\alpha \le |\xi| \le 2^n\pi$. Then

$$\int_{|\xi|\le 2^n\alpha} |\xi|^{\tilde{s}} \prod_{j=1}^{n} |b(2^{-j}\xi)|\, d\xi$$

$$\le C^{-1} \int_{|\xi|\le 2^n\alpha} (1+|\xi|)^s \left[\prod_{j=1}^{n} |b(2^{-j}\xi)|\right] |\widehat{\phi}(2^{-n}\xi)|\, d\xi$$

$$\le C^{-1} \int_{-\infty}^{\infty} (1+|\xi|)^s |\widehat{\phi}(\xi)|\, d\xi < \infty.$$

On the other hand, using (2.11) we find that

$$\int_{2^n\alpha\le|\xi|\le 2^n\pi} |\xi|^{\tilde{s}} \prod_{j=1}^{n} |b(2^{-j}\xi)|\, d\xi \le \pi^s 2^{n\tilde{s}} C(\alpha)\, 2^{-ns} < \infty.$$

Combining this with (2.14), (2.15), and (2.16), we find that

$$\|f_n - \phi\|_{L^\infty} \le \|\widehat{f_n} - \widehat{\phi}\|_{L^1} \le C 2^{-n\tilde{s}},$$

as announced. ■

3. Controlling the Truncation Effects

We start by estimating, for fixed n, the difference $\|f_n - \tilde{f}_n\|_{L^\infty(\mathbf{R})}$ as a function of $\|b - \tilde{b}\|_{L^\infty(0,2\pi)}$.

Proposition 3.1. *Let $b, \tilde{b}$ be continuous 2π-periodic functions; we denote the spectral radii of the associated transfer operators $P_{|b|}$ and $P_{|\tilde{b}|}$ on $C[0,2\pi]$ by ρ and $\tilde{\rho}$. Define f_n, $\tilde{f}_n$ by* (1.4) *and* (1.5). *Then, for any $\varepsilon > 0$, we can find $C > 0$ (C depending on ε) so that*

(3.1) $$\|f_n - \tilde{f}_n\|_{L^\infty(\mathbf{R})} \le C(\gamma + \varepsilon)^n$$

where $\gamma = \max(\rho, \tilde{\rho})$.

Remark. 1. Note that we did not even assume $\tilde{b}(0) = 1$ here, and we can in fact apply (3.1) even when $\tilde{b}(0) \ne 1$. (True to what we announced in the Introduction, we always assume $b(0) = 1$.)

2. In practice we always apply this to situations where the $\widehat{f_n}$ converge to an L^1-function ϕ, and where b has some Hölder continuity. In that case $\rho = 1$ (see, e.g., [He3]).

Proof. 1. We have

$$\begin{aligned}\|f_n - \tilde{f}_n\|_{L^\infty(\mathbf{R})} &\le \|\widehat{f_n} - \widehat{\tilde{f}}_n\|_{L^1}\\ &= \left\| \left[\prod_{j=1}^{n} b(2^{-j}\xi) - \prod_{j=1}^{n} \tilde{b}(2^{-j}\xi) \right] \widehat{f_0}(2^{-n}\xi) \right\|_{L^1}\\ &= \int_{|\xi|\le 2^n\pi} \left| \prod_{j=1}^{n} b(2^{-j}\xi) - \prod_{j=1}^{n} \tilde{b}(2^{-j}\xi) \right| \sum_{\ell\in\mathbf{Z}} \left| \widehat{f_0}(2^{-n}\xi + 2\ell\pi) \right| d\xi\\ &= 2^n \int_{|\xi|\le\pi} \left| \prod_{j=0}^{n-1} b(2^{j}\xi) - \prod_{j=0}^{n-1} \tilde{b}(2^{j}\xi) \right| \sum_{\ell\in\mathbf{Z}} \left| \widehat{f_0}(\xi + 2\ell\pi) \right| d\xi.\end{aligned}$$

Because of the decay of $\widehat{f_0}$, we find that $\sum_{\ell\in\mathbf{Z}} |\widehat{f_0}(\xi + 2\ell\pi)|$ is bounded. Therefore we only need to control

$$B_n = 2^n \int_{|\xi|\le\pi} \left| \prod_{j=0}^{n-1} b(2^{j}\xi) - \prod_{j=0}^{n-1} \tilde{b}(2^{j}\xi) \right| d\xi. \tag{3.2}$$

We can rewrite the integrand in (3.2) as

$$\begin{aligned}&\prod_{j=0}^{n-1} b(2^{j}\xi) - \prod_{j=0}^{n-1} \tilde{b}(2^{j}\xi)\\ &\qquad = \sum_{k=0}^{n-1} \left[\prod_{j=0}^{k-1} \tilde{b}(2^{j}\xi) \right] \left[b(2^k\xi) - \tilde{b}(2^k\xi) \right] \left[\prod_{j=k+1}^{n-1} b(2^{j}\xi) \right],\end{aligned}$$

with the standard convention that $\prod_{j=n_1}^{n_2} \alpha_j = 1$ if $n_2 < n_1$. It follows that

$$B_n \le \|b - \tilde{b}\|_{L^\infty(0,2\pi)} \int_{|\xi|\le 2^n\pi} \sum_{\ell=0}^{n-1} \prod_{m=1}^{\ell} |b(2^{-m}\xi)| \prod_{m=\ell+2}^{n} |\tilde{b}(2^{-m}\xi)|\, d\xi.$$

By (2.5), this leads to the bound

$$B_n \le C\|b - \tilde{b}\|_{L^\infty(0,2\pi)} \sum_{\ell=0}^{n-1} \|P_{|b|}^{\ell}\| \|P_1\| \|P_{|\tilde{b}|}^{n-\ell-1}\|, \tag{3.3}$$

where P_1 is the transfer operator corresponding to $M(\xi) = 1$.

2. Given $\varepsilon > 0$, we can find $C' > 0$ so that, for all k,

$$\|P_{|b|}^{k}\| \le C'\left(\rho + \frac{\varepsilon}{2}\right)^k, \qquad \|P_{|\tilde{b}|}^{k}\| \le C'\left(\tilde{\rho} + \frac{\varepsilon}{2}\right)^k. \tag{3.4}$$

Consequently

$$B_n \leq C''\|b - \tilde{b}\|_{L^\infty(0,2\pi)} \sum_{\ell=0}^{n-1} \left(\rho + \frac{\varepsilon}{2}\right)^\ell \left(\tilde{\rho} + \frac{\varepsilon}{2}\right)^{n-1-\ell} \tag{3.5}$$

$$\leq C''\|b - \tilde{b}\|_{L^\infty(0,2\pi)} n \left(\gamma + \frac{\varepsilon}{2}\right)^{n-1}$$

$$\leq C'''\|b - \tilde{b}\|_{L^\infty(0,2\pi)} (\gamma + \varepsilon)^n.$$

■

This is what made the graphs in, e.g., [D1] or [D2] work. In most of the figures of scaling functions there, $n = 7$ or 8 was used; for blowups such as in Figure 7.1 in [D2], larger n were needed, but the largest value for n ever used was $n = 21$. The mask coefficients were rounded off at the eighth decimal, leading to $\|b - \tilde{b}\|_{L^\infty} \simeq 10^{-8}$. For the particular case of Figure 7.1 in [D2] it can be checked moreover that $\rho = 1$, $|\tilde{\rho} - 1| \lesssim 2 \cdot 10^{-8}$, while, for $\varepsilon = 10^{-3}$, $C' \simeq 3$ can be taken in (3.4); together with $C = 2\pi$ in (3.3) this leads to

$$\|f_{21} - \tilde{f}_{21}\|_{L^\infty(\mathbf{R})} \lesssim 2\pi \cdot 9 \cdot 21(1.0005)^{21} \cdot 10^{-8}$$

$$\simeq 1.2 \cdot 10^{-5},$$

which is still acceptably small for plotting purposes. For larger blowups more care might have to be taken however: since $\rho = 1$, we have $\gamma \geq 1$, leading to an exploding upper bound (3.5) for $n \to \infty$. The rate of explosion can be mitigated by choosing ε very small, but note that C' in (3.4), hence C'' in (3.5), typically increase to ∞ as ε shrinks to 0. Moreover, because of the round-off errors, we may well end up with $|\tilde{b}(0)| > 1$, even though the difference will be of the order of the round-off error itself; because of Lemma 2.2 this then implies however $\tilde{\rho} > 1$, hence $\gamma > 1$ in (3.1), so that our upper bound explodes even faster when $n \to \infty$. (Note that $\tilde{\rho} > 1$ is possible even when $|\tilde{b}(0)| = 1$.)

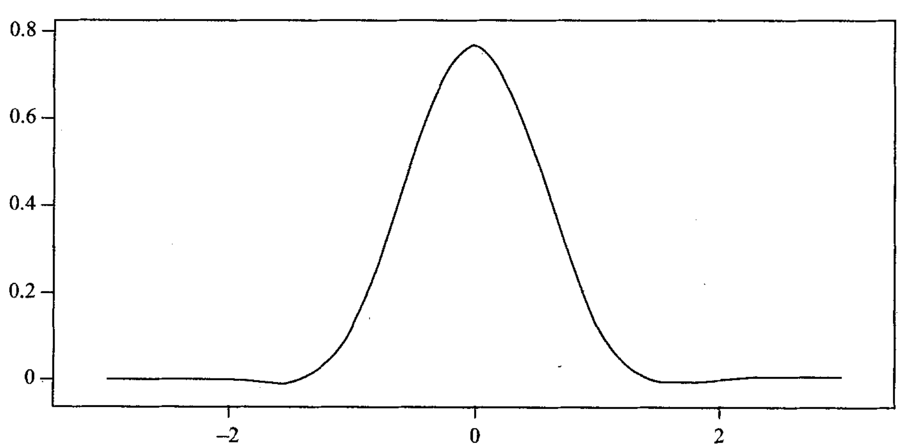

Figure 1. Graph of $\tilde{f}_7$. One has $\|\tilde{f}_7 - \phi\|_{L^\infty} \lesssim 0.005$.

The following example, albeit purely academic, shows that this effect is real, and not just an artifact of our technique of proof. Take

$$b(\xi) = \left(\frac{1+\cos\xi}{2}\right)^2 \left(\frac{4}{3} - \frac{1}{3}\cos\xi\right)$$
$$= \frac{5}{12} + \frac{25}{48}\cos\xi + \frac{1}{12}\cos 2\xi - \frac{1}{48}\cos 3\xi.$$

Rounding off all the mask coefficients after the third decimal we obtain

(3.6) $\tilde{b}(\xi) = 0.417 + 0.260(e^{i\xi} + e^{-i\xi}) + 0.042(e^{2i\xi} + e^{-2i\xi}) - 0.010(e^{3i\xi} + e^{-3i\xi})$.

Note that $\tilde{b}(0) = 1.001 > 1$, hence $\tilde{\rho} \geq 1.001 > 1$. Figure 1 shows $\tilde{f}_7$; estimating $\|\phi - \tilde{f}_7\| \leq \|\phi - f_7\| + \|f_7 - \tilde{f}_7\|$, it is found that the difference ($\simeq 0.005$) is not quite negligible at the scale of the figure (we have taken a rather coarse cut-off), but it is small. To show the divergence for large values of n, we track the value of $\tilde{f}_n$ at one single point, $\tilde{f}_n(0)$. This value can be computed very easily. It follows from viewing (1.5) as a subdivision scheme, or as a "cascade algorithm" in the language of Section 6.5 in [D2], that the vector

$$u_n = \left(\tilde{f}_n(-2^{-n+1}), \tilde{f}_n(-2^{-n}), \tilde{f}_n(0), \tilde{f}_n(2^{-n}), \tilde{f}_n(2^{-n+1})\right)$$

can be obtained via a recursive equation,

$$u_n = Tu_{n-1}, \tag{3.7}$$

where $u_0 = (0, 0, 1, 0, 0)$, and T is a fixed matrix,

$$T = \begin{pmatrix} \tilde{b}_{-2} & \tilde{b}_0 & \tilde{b}_2 & 0 & 0 \\ \tilde{b}_{-3} & \tilde{b}_{-1} & \tilde{b}_1 & \tilde{b}_3 & 0 \\ 0 & \tilde{b}_{-2} & \tilde{b}_0 & \tilde{b}_2 & 0 \\ 0 & \tilde{b}_{-3} & \tilde{b}_{-1} & \tilde{b}_1 & \tilde{b}_3 \\ 0 & 0 & \tilde{b}_{-2} & \tilde{b}_0 & \tilde{b}_2 \end{pmatrix},$$

where $\tilde{b}_k$ is the coefficient of $e^{-ik\xi}$ in $2 \times$ (3.6). Figure 2 shows a graph of $\tilde{f}_n(0) - \phi(0)$ as a function of n. The divergence of $\tilde{f}_n(0)$ for $n \to \infty$ follows of course directly from (3.7) and the fact that $(0, 0, 1, 0, 0)$ contains a component along the eigenvector of T with the largest eigenvalue, which is > 1 (this eigenvalue is in fact equal to $\tilde{\rho}$). Note that the divergence could also be predicted from the results in [DyLe]: convergence of the $\tilde{f}_n$ would have implied $\tilde{b}(0) = 1$, $\tilde{b}(\pi) = 0$, in contradiction with $\tilde{b}(0) = 1.001$ (see above).

Remark. 1. It may seem that the divergence in the example above is due to $\tilde{b}(0) > 1$. It can be shown, however, that $\|\tilde{f}_n - f_n\|_{L^\infty}$ may still diverge for $n \to \infty$ even when $\tilde{b}(0) = 1$. An example is

$$b(\xi) = \left(\frac{1+\cos\xi}{2}\right)^2 \left(\frac{4}{3} - \frac{6{,}251}{25{,}000}e^{i\xi} - \frac{6{,}247}{75{,}000}e^{-i\xi}\right);$$

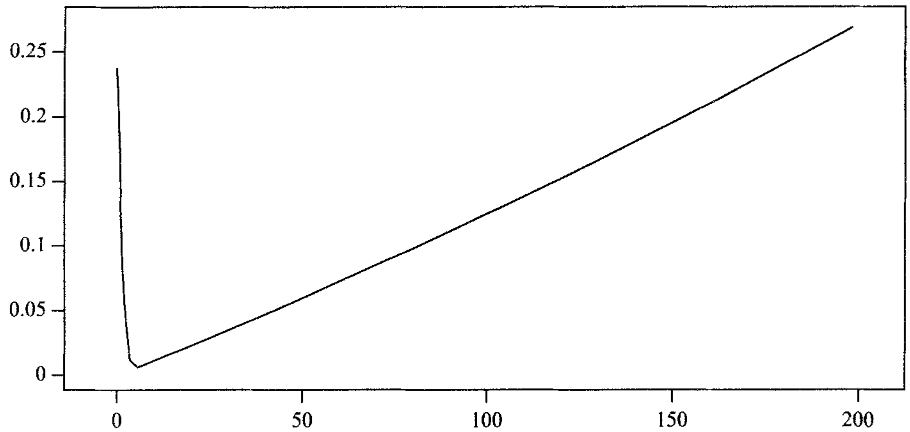

Figure 2. Graph of $\tilde{f}_n(0) - \phi(0)$ as a function of n.

rounding off the coefficients to the third decimal leads to

$$\begin{aligned}\tilde{b}(\xi) = &- 0.016e^{3i\xi} + 0.021e^{2i\xi} + 0.234e^{i\xi} + 0.417 \\ &+ 0.286e^{-i\xi} + 0.063e^{-2i\xi} - 0.005e^{-3i\xi},\end{aligned}$$

with $\tilde{b}(0) = 1$. Nevertheless, the graph of $\tilde{f}_n(0) - \phi(0)$, as a function of n, given in Figure 3, shows the same divergence. Of course, we have again $\tilde{b}(\pi) \neq 0$; examples with $\tilde{b}(\pi) = 0$ can be handled by Theorem 3.2 below, and lead to much better results.

2. It has been noted by various authors that to obtain approximate graphs of ϕ, it is more efficient to start by computing the $\phi(n)$ from

$$\phi(n) = \sum_k b_k \phi(2n - k)$$

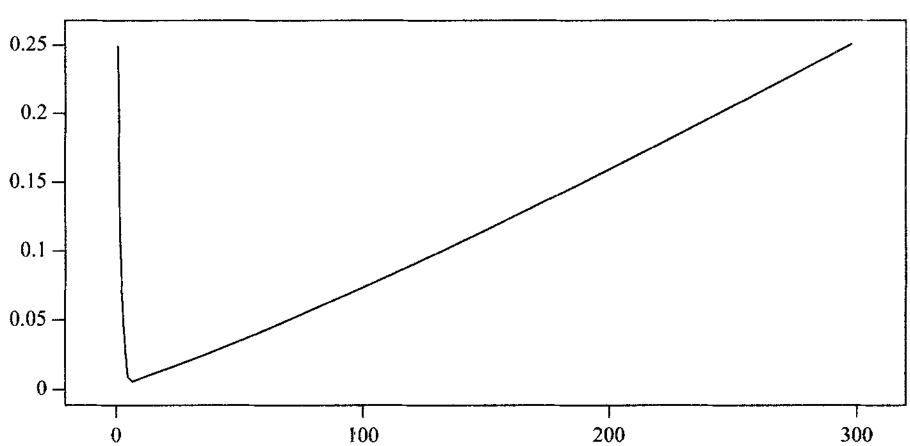

Figure 3. Graph of $\tilde{f}_n(0) - \phi(0)$ for another example (see the text).

(which reduces to finding an eigenvector of an $(N-1)\times(N-1)$ matrix if ϕ is continuous and $b_k = 0$ for $k < 0$ or $k > N$), and then computing successively the missing $\phi(\frac{k}{2})$, $\phi(\frac{\ell}{4}), \ldots$ by repeatedly using (1.1). In fact, W. Sweldens has drawn our attention to at least one case where this direct algorithm converges while the subdivision algorithm diverges; ϕ is necessarily noncontinuous in this case (see pp. 212 and 275 in the second edition of [D2]). A similar analysis of truncation can be performed there, and bounds similar to (3.5), with the same type of divergence for $n \to \infty$ in general, are found.

If the truncation of $b(\xi)$ is done a little more carefully, preserving (at least in part) the factorizability (1.6), then $\|f_n - \tilde{f}_n\|_{L^\infty(\mathbf{R})}$ need not diverge, as is shown by:

Theorem 3.2. *Let $b, \tilde{b}$, be 2π-periodic Hölder continuous functions with $b(0) = \tilde{b}(0) = 1$. Define f_n and $\tilde{f}_n$ as above. Assume, furthermore, that*

$$|b(\xi)| = \left|\cos\frac{\xi}{2}\right|^r \cdot |v(\xi)|, \qquad r \in \mathbf{N}, \quad \mathbf{v} \in \mathbf{C}[\mathbf{0}, \mathbf{2}\pi], \quad \mathbf{v}(\pi) \neq \mathbf{0}. \tag{3.8}$$

$$|\tilde{b}(\xi)| = \left|\cos\frac{\xi}{2}\right|^{\tilde{r}} \cdot |\tilde{v}(\xi)|, \qquad \tilde{r} \in \mathbf{N}, \quad \tilde{\mathbf{v}} \in \mathbf{C}[\mathbf{0}, \mathbf{2}\pi], \quad \tilde{\mathbf{v}}(\pi) \neq \mathbf{0}. \tag{3.9}$$

Let $\beta, \tilde{\beta}$ be the spectral radii of $P_{|v|}, P_{|\tilde{v}|}$ restricted to $C[0,2\pi]$. If

$$\beta < 2^r \quad \text{and} \quad \tilde{\beta} < 2^{\tilde{r}},$$

then, for any $\lambda < 1$, we can find $C > 0$ (C depending on λ) such that

$$\sup_{n\in\mathbf{N}} \|f_n - \tilde{f}_n\|_{L^\infty} \le C\|b - \tilde{b}\|^\lambda_{L^\infty}. \tag{3.10}$$

Proof. We know from Proposition 2.3 that $C > 0$ exists such that

$$\|f_n - f_m\|_{L^\infty} \le C2^{-s\min(m,n)},$$
$$\|\tilde{f}_n - \tilde{f}_m\|_{L^\infty} \le C2^{-\tilde{s}\min(m,n)},$$

where $s, \tilde{s} \le 1$ and $s < r - \log_2\beta$, $\tilde{s} < \tilde{r} - \log_2\tilde{\beta}$. On the other hand, we know from Proposition 3.1 that for any $\varepsilon > 0$, we can find $C(\varepsilon)$ so that

$$\|\tilde{f}_m - f_m\|_{L^\infty} \le C(\varepsilon)(1+\varepsilon)^m\|b - \tilde{b}\|_{L^\infty},$$

where we have used the spectral radii of $P_{|b|}$, $P_{|\tilde{b}|}$ that are both equal to 1 (see the remarks at the start of Section 3). Consequently, for given $m \in \mathbf{N}$, and $t > 0$, we can find $C > 0$ so that, for all $n \ge m$,

$$\begin{aligned}\|\tilde{f}_n - f_n\|_{L^\infty} &\le \|\tilde{f}_n - \tilde{f}_m\|_{L^\infty} + \|\tilde{f}_m - f_m\|_{L^\infty} + \|f_m - f_n\|_{L^\infty}\\ &\le C\left[2^{-Sm} + \|b - \tilde{b}\|_{L^\infty}2^{tm}\right],\end{aligned}$$

where $S = \min(s, \tilde{s})$. The quantity between the square brackets is minimal for $2^m = (St^{-1}\,\|b - \tilde{b}\|^{-1})^{(S+t)^{-1}}$, and it is then bounded by $K\|b - \tilde{b}\|^{S(S+t)^{-1}}$, where K depends only on S and t. By choosing t sufficiently small, (3.10) follows. ∎

One can easily adapt this argument to prove the following corollaries:

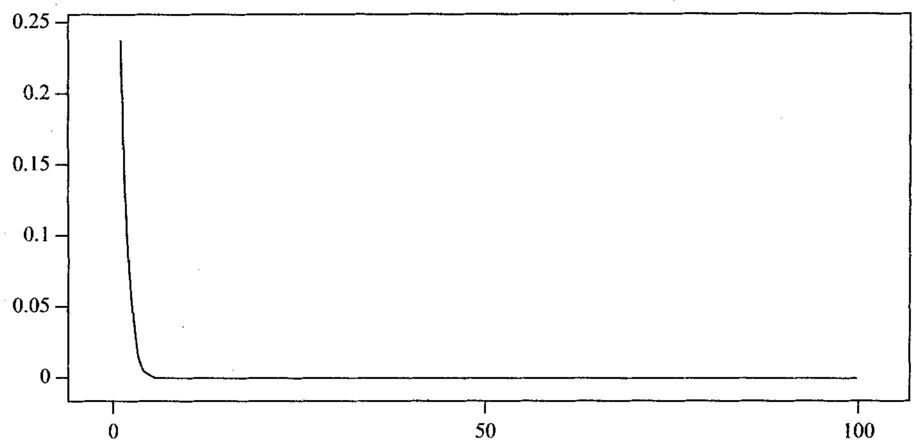

Figure 4. Graph of $\tilde{\tilde{f}}_n(0) - \phi(0)$, where ϕ is the same function as in Figs. 1 and 2, but the truncated mask $\tilde{\tilde{b}}(\xi)$ respects the factorization of $b(\xi)$.

Corollary 3.3. *Under the same assumptions as in Theorem* 3.2, *we also have, for all* $s < \min(r - \log_2 \beta, \tilde{r} - \log_2 \tilde{\beta})$, $\sup_{n \in \mathbf{N}} \|f_n - \tilde{f}_n\|_{C^s(\mathbf{R})} \to 0$ *if* $\|b - \tilde{b}\|_{L^\infty} \to 0$.

Corollary 3.4. *If* $b, \tilde{b}$ *are* 2π*-periodic and Hölder continuous, satisfying* (3.8) *and* (3.9), *and if* β_2, $\tilde{\beta}_2$, *the spectral radii of* $P_{|v|^2}$, $P_{|\tilde{v}|^2}$ *on* $C[0,2\pi]$, *satisfy* $\beta_2 < 2^{2r}$, *resp.* $\tilde{\beta}_2 < 2^{2\tilde{r}}$, *then* $\|f_n - \tilde{f}_n\|_{W^s(\mathbf{R})} = \int |\widehat{f_n}(\xi) - \widehat{\tilde{f}}_n(\xi)|^2 (1 + |\xi|^2)^s \, d\xi$ *can be made arbitrarily small by choosing* $\|b - \tilde{b}\|_{L^\infty(\mathbf{R})}$ *sufficiently small, if* $s < \min(r - \frac{1}{2}\log \beta_2, \tilde{r} - \frac{1}{2}\log \tilde{\beta}_2)$.

In practice, one can usually choose (as, e.g., in [Hu]) $\tilde{r} = r$, and it suffices then to control $\|v - \tilde{v}\|_{L^\infty(\mathbf{R})}$. Note that a subdivision scheme with mask $\tilde{b}$ can also be implemented much more efficiently if $\tilde{b}$ is factorizable (basically because adding, and multiplication or division by powers of 2 take much less time than multiplication by arbitrary numbers), which is another reason for preserving the factorization (1.6) when truncating. We conclude by revisiting the example given above. If instead of (3.6) we choose the truncation

$$\tilde{\tilde{b}}(\xi) = \left(\frac{1 + \cos\xi}{2}\right)^2 (1.33 - .33\cos\xi),$$

where we use an even more brutal roundoff on $v(\xi)$ (after the second decimal of the coefficients), but preserving the factorization, then the corresponding graph of $\tilde{\tilde{f}}_n(0) - \phi(0)$ is given by Figure 4. The difference between $\tilde{\tilde{f}}_n(0)$ and $\phi(0)$ converges rapidly to its limit, -0.001.

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