Constr. Approx. (2004) 20: 325–352
DOI: 10.1007/s00365-004-0567-4

CONSTRUCTIVE APPROXIMATION

Pairs of Dual Wavelet Frames from Any Two Refinable Functions

Ingrid Daubechies and Bin Han

Abstract. Starting from any two compactly supported refinable functions in $L_2(\mathbf{R})$ with dilation factor d, we show that it is always possible to construct $2d$ wavelet functions with compact support such that they generate a pair of dual d-wavelet frames in $L_2(\mathbf{R})$. Moreover, the number of vanishing moments of each of these wavelet frames is equal to the approximation order of the dual MRA; this is the highest possible. In particular, when we consider symmetric refinable functions, the constructed dual wavelets are also symmetric or antisymmetric. As a consequence, for any compactly supported refinable function φ in $L_2(\mathbf{R})$, it is possible to construct, explicitly and easily, wavelets that are finite linear combinations of translates $\varphi(d\cdot -k)$, and that generate a wavelet frame with an arbitrarily preassigned number of vanishing moments. We illustrate the general theory by examples of such pairs of dual wavelet frames derived from B-spline functions.

1. Introduction

As a generalization of biorthogonal wavelets, pairs of dual wavelet frames have proved particularly useful in signal denoising and many other applications where translation invariance or redundancy is important. By allowing redundancy in a wavelet system, one has much more freedom in the choice of wavelets. It may also be easier to recognize patterns in a redundant transform. From the computational point of view, it is often easier to work with dual wavelet frames that are generated by MRAs. If one works with biorthogonal *bases*, then the two MRAs have to be linked in special ways [5]. In this paper, we are particularly interested in pairs of dual wavelet frames derived from refinable functions; we shall see that there are then no restrictions on the MRAs.

Before proceeding further, let us introduce some notation. Throughout this paper, by d we denote the dilation factor which is an integer with absolute value greater than one. For simplicity, throughout this paper, we further assume that d is a positive dilation factor since all the corresponding results in this paper for a negative dilation factor can be proved almost identically. The inner product $\langle \cdot, \cdot \rangle$ in $L_2(\mathbf{R})$ is defined to be

$$\langle f, g \rangle := \int_{\mathbf{R}} f(t)\overline{g(t)}\, dt, \qquad f, g \in L_2(\mathbf{R}).$$

Date received: July 31, 2000. Date revised: April 10, 2002. Date accepted: January 14, 2004. Communicated by Amos Ron.
AMS classification: 42C40, 42C15, 41A15.
Key words and phrases: Dual wavelet frames, Wavelet frames, Refinable functions, B-Spline functions.

Let $\{\psi^1, \dots, \psi^r\}$ be a finite set of functions in $L_2(\mathbf{R})$. We say that $\{\psi^1, \dots, \psi^r\}$ generates a d**-wavelet frame** in $L_2(\mathbf{R})$ if there exist positive constants C_1 and C_2 such that

$$(1.1)\qquad C_1\|f\|^2 \le \sum_{\ell=1}^{r}\sum_{j\in\mathbf{Z}}\sum_{k\in\mathbf{Z}} |\langle f, \psi^\ell_{j,k}\rangle|^2 \le C_2\|f\|^2 \qquad \forall\, f \in L_2(\mathbf{R}),$$

where $\|f\|^2 := \langle f, f\rangle$ and

$$\psi^\ell_{j,k} := |d|^{j/2}\psi^\ell(d^j \cdot -k), \qquad j \in \mathbf{Z}, \quad k \in \mathbf{Z}.$$

In particular, when $C_1 = C_2 = 1$ in (1.1), we say that $\{\psi^1, \dots, \psi^r\}$ generates a (normalized) **tight** d**-wavelet frame** in $L_2(\mathbf{R})$.

If both $\{\psi^1, \dots, \psi^r\}$ and $\{\tilde\psi^1, \dots, \tilde\psi^r\}$ generate d-wavelet frames in $L_2(\mathbf{R})$ and satisfy

$$\langle f, g\rangle = \sum_{\ell=1}^{r}\sum_{j\in\mathbf{Z}}\sum_{k\in\mathbf{Z}} \langle f, \psi^\ell_{j,k}\rangle\langle \tilde\psi^\ell_{j,k}, g\rangle \qquad \forall\, f, g \in L_2(\mathbf{R}),$$

then we say that $\{\psi^1, \dots, \psi^r\}$ and $\{\tilde\psi^1, \dots, \tilde\psi^r\}$ generate a **pair of dual** d**-wavelet frames** in $L_2(\mathbf{R})$. A pair of dual d-wavelet frames is also called a bi-frame in the literature [11]. Consequently, any function f in $L_2(\mathbf{R})$ has the following wavelet expansions:

$$f = \sum_{\ell=1}^{r}\sum_{j\in\mathbf{Z}}\sum_{k\in\mathbf{Z}} \langle f, \psi^\ell_{j,k}\rangle\tilde\psi^\ell_{j,k} = \sum_{\ell=1}^{r}\sum_{j\in\mathbf{Z}}\sum_{k\in\mathbf{Z}} \langle f, \tilde\psi^\ell_{j,k}\rangle\psi^\ell_{j,k}$$

with the series converging absolutely in the L_2 norm. Using the Fourier transform, one can give an explicit characterization for $\{\psi^1, \dots, \psi^r\}$ and $\{\tilde\psi^1, \dots, \tilde\psi^r\}$ to generate a pair of dual d-wavelet frames in $L_2(\mathbf{R})$; see [7], [11].

An important property of a wavelet system is its order of vanishing moments. We say that $\{\psi^1, \dots, \psi^r\}$ has **vanishing moments** of order n if

$$\int_{\mathbf{R}} t^k\psi^\ell(t)\, dt = 0 \qquad \forall\, \ell = 1, \dots, r \quad \text{and} \quad k = 0, \dots, n-1.$$

In this paper, we are particularly interested in obtaining pairs of dual wavelet frames that are derived from pairs of refinable functions with a general dilation factor. Let d be a dilation factor. A function φ is said to be d**-refinable** if

$$(1.2)\qquad \varphi = |d|\sum_{k\in\mathbf{Z}} a_k\varphi(d \cdot -k),$$

where a is a sequence on $\mathbf{Z}$, called the **mask** for φ. The **Fourier series** of a sequence a on $\mathbf{Z}$ is defined to be

$$(1.3)\qquad \hat a(\xi) := \sum_{k\in\mathbf{Z}} a_k e^{-ik\xi}, \qquad \xi \in \mathbf{R}.$$

Any mask a for a refinable function in this paper is assumed to be finitely supported with $\hat a(0) = \sum_{k\in\mathbf{Z}} a_k = 1$. We shall only consider L_2-solutions φ to (1.2) with a finitely supported mask a; because a is finitely supported, this solution φ is compactly supported, and (if it exists) uniquely defined up to normalization by $\hat{\varphi}(\xi) := \prod_{j=1}^{\infty} \hat{a}(d^{-j}\xi)$, $\xi \in \mathbf{R}$ (see [5]), where the Fourier transform is defined to be

$$\hat{f}(\xi) := \int_{\mathbf{R}} f(t)e^{-i\xi t}\,dt, \qquad f \in L_1(\mathbf{R}).$$

Since $\hat{a}(0) = 1$ and $\hat{\varphi}(\xi) := \prod_{j=1}^{\infty} \hat{a}(d^{-j}\xi)$, we always have $\hat{\varphi}(0) = 1$.

We say that a satisfies the **sum rules** of order n with respect to the lattice $d\mathbf{Z}$ if

$$\sum_{k\in d\mathbf{Z}} a_{k+j}(k+j)^{\ell} = \sum_{k\in d\mathbf{Z}} a_k k^{\ell} \qquad \forall\, \ell = 0,\ldots,n-1 \quad \text{and} \quad j \in \mathbf{Z}. \tag{1.4}$$

Equivalently, a finitely supported sequence a satisfies the sum rules of order n with respect to the lattice $d\mathbf{Z}$ if and only if

$$(1 + e^{-i\xi} + \cdots + e^{-i(d-1)\xi})^n \mid \hat{a}(\xi).$$

That is, $\hat{a}(\xi) = (1 + e^{-i\xi} + \cdots + e^{-i(d-1)\xi})^n p(\xi)$ for some 2π-periodic trigonometric *polynomial* p. Throughout this paper, we shall use the notation $q(\xi) \mid \hat{a}(\xi)$ to mean that $\hat{a}(\xi) = q(\xi)p(\xi)$ for some 2π-periodic trigonometric polynomial p.

Let φ and $\tilde{\varphi}$ be two d-refinable functions in $L_2(\mathbf{R})$ with finitely supported masks a and b, respectively. Suppose that a and b satisfy the sum rules of orders m and n with respect to the lattice $d\mathbf{Z}$ for some positive integers m and n, respectively. For any nonnegative integer N, we show in Section 3 that there exist finitely supported sequences $a^1,\ldots,a^d, b^1,\ldots,b^d$ such that by defining

$$\begin{aligned} \psi^{\ell} &= |d| \sum_{k\in\mathbf{Z}} a_k^{\ell}\varphi(d\cdot -k) \qquad \text{and} \\ \tilde{\psi}^{\ell} &= |d| \sum_{k\in\mathbf{Z}} b_k^{\ell}\tilde{\varphi}(d\cdot -k), \qquad \ell = 1,\ldots,d, \end{aligned} \tag{1.5}$$

(or equivalently, in the frequency domain, $\widehat{\psi^{\ell}}(d\xi) = \widehat{a^{\ell}}(\xi)\hat{\varphi}(\xi)$ and $\widehat{\tilde{\psi}^{\ell}}(d\xi) = \widehat{b^{\ell}}(\xi)\hat{\tilde{\varphi}}(\xi)$ for $\ell = 1,\ldots,d$). $\{\psi^1,\ldots,\psi^d\}$ and $\{\tilde{\psi}^1,\ldots,\tilde{\psi}^d\}$ generate a pair of dual d-wavelet frames in $L_2(\mathbf{R})$. Moreover, ψ^1, $\{\psi^2,\ldots,\psi^d\}$, $\tilde{\psi}^1$ and $\{\tilde{\psi}^2,\ldots,\tilde{\psi}^d\}$ have vanishing moments of orders n, $n+2N$, $m+2N$, and m, respectively. In addition, if both φ and $\tilde{\varphi}$ are real-valued and symmetric d-refinable functions, such that the symmetry centers of φ and $\tilde{\varphi}$ differ by a half-integer, then the wavelet functions $\psi^1,\ldots,\psi^d,\tilde{\psi}^1,\ldots,\tilde{\psi}^d$ can be chosen to be real-valued and be either symmetric or antisymmetric with the same symmetry center. See Sections 3 and 4 for more details.

The structure of this paper is as follows. In Section 2, we shall recall a general method for constructing pairs of dual wavelet frames derived from any two refinable functions; we also prove two auxiliary results that will be useful for our main theorems. In Section 3, we shall discuss how to obtain pairs of dual wavelet frames in a concrete and constructive way from any two refinable functions. An algorithm for constructing pairs of dual wavelet frames will be presented. Wavelet frames derived from refinable functions will be discussed in Section 3. In Section 4, we shall investigate how to derive pairs of real-valued and symmetric dual wavelet frames from any two real-valued and symmetric refinable functions. Finally, in Section 5 we give several examples of pairs of dual wavelet frames derived from B-spline functions.

A program consisting of a collection of MAPLE routines based on the algorithms and constructions of dual wavelet frames in this paper, which comes without warranty, can be downloaded at http://www.ualberta.ca/∼bhan. The examples in Section 5 are produced by this program.

2. Dual Wavelet Frames of High Vanishing Moments

In this section, we shall discuss how to construct dual wavelet frames with high vanishing moments from refinable functions.

The following lemma is a direct consequence of results from Cohen and Daubechies [3] and Villemoes [12]:

Lemma 2.1. *Let $\varphi \in L_2(\mathbf{R})$ be a d-refinable function with a dilation factor d and a finitely supported mask a. Let b be a finitely supported sequence on $\mathbf{Z}$ such that $\hat{b}(0) = 0$. Define a function ψ by $\hat{\psi}(d\xi) = \hat{b}(\xi)\hat{\varphi}(\xi)$. Then there exists a positive constant C such that*

$$\sum_{j\in\mathbf{Z}}\sum_{k\in\mathbf{Z}} |\langle f, \psi_{j,k}\rangle|^2 \le C\|f\|^2 \qquad \forall\ f \in L_2(\mathbf{R}), \tag{2.1}$$

where $\psi_{j,k} := |d|^{j/2}\psi(d^j \cdot -k)$.

Proof. Since $\varphi \in L_2(\mathbf{R})$ is compactly supported, it is well-known that there exists a compactly supported function $\eta \in L_2(\mathbf{R})$ in the closure of $\operatorname{span}\{\varphi(\cdot - k) : k \in \mathbf{Z}\}$ such that the shifts of η are linearly independent and $\varphi = \sum_{k\in\mathbf{Z}} c_k\eta(\cdot - k)$ for some finitely supported sequence c (see [1], [10]). By [3, Theorem 5.1], for some $\alpha > 0$, $\eta \in W_2^\alpha(\mathbf{R}) := \{f \in L_2(\mathbf{R}) : \int_{\mathbf{R}}(1 + |\xi|^2)^\alpha|\hat{f}(\xi)|^2\,d\xi < \infty\}$. Therefore, as a finite linear combination of $\eta(\cdot - k)$, $\varphi \in W_2^\alpha(\mathbf{R})$. So the compactly supported function ψ lies in $W_2^\alpha(\mathbf{R})$ and $\int_{\mathbf{R}}\psi(t)\,dt = 0$. By [3, Theorem 5.1] or [12, Theorem 3.3], there exists a positive constant C such that (2.1) holds. ■

We point out to the reader that Lemma 2.1 has been generalized to the case of the multivariate multiwavelets in [8]. Let M be an $s \times s$ integer matrix such that all its eigenvalues are greater than one in modulus. Suppose that $\varphi = (\varphi^1, \dots, \varphi^r)^T \in (L_2(\mathbf{R}^s))^r$ is compactly supported and $\hat{\varphi}(M^T\xi) = \hat{a}(\xi)\hat{\varphi}(\xi)$ for some $r \times r$ matrix $\hat{a}(\xi)$ of 2π-periodic trigonometric polynomials. Then it was proved in [8] that there exists $\alpha > 0$ such that $\int_{\mathbf{R}^s}(1 + \|\xi\|^2)^\alpha|\widehat{\varphi^\ell}(\xi)|^2\,d\xi < \infty$ and $(1 + \|\cdot\|)^\alpha\widehat{\varphi^\ell} \in L_\infty$ for all $\ell = 1, \dots, r$. Moreover, for any $\psi = (\psi^1, \dots, \psi^r)^T$ which is defined by $\hat{\psi}(M^T\xi) = \hat{b}(\xi)\hat{\varphi}(\xi)$ for some $r \times r$ matrix $\hat{b}(\xi)$ of 2π-periodic trigonometric polynomials, if $\int_{\mathbf{R}^s}\psi^\ell(t)\,dt = 0$ for all $\ell = 1, \dots, r$, then there exists a positive constant C such that $\sum_{\ell=1}^r\sum_{j\in\mathbf{Z}}\sum_{k\in\mathbf{Z}^s}|\langle f, \psi_{j,k}^\ell\rangle|^2 \le C\|f\|^2$ for all $f \in L_2(\mathbf{R}^s)$, where $\psi_{j,k}^\ell := |\det M|^{j/2}\psi^\ell(M^j \cdot -k)$.

Pairs of dual wavelet frames can be obtained from refinable functions by the following result:

Theorem 2.2. *Let φ and $\tilde{\varphi}$ be two d-refinable functions in $L_2(\mathbf{R})$ with the dilation factor d and finitely supported masks a and b, respectively. Suppose that there are* *finitely supported sequences* $a^1, \dots, a^r, b^1, \dots, b^r$ *and a* 2π*-periodic trigonometric polynomial* Θ *such that*

$$\Theta(0) = 1, \qquad \widehat{a^\ell}(0) = \widehat{b^\ell}(0) = 0 \qquad \forall\, \ell = 1, \dots, r, \tag{2.2}$$

and

$$\overline{\begin{bmatrix} \hat{a}(\xi) & \widehat{a^1}(\xi) & \cdots & \widehat{a^r}(\xi) \\ \hat{a}\left(\xi + \frac{2\pi}{d}\right) & \widehat{a^1}\left(\xi + \frac{2\pi}{d}\right) & \cdots & \widehat{a^r}\left(\xi + \frac{2\pi}{d}\right) \\ \vdots & \vdots & \ddots & \vdots \\ \hat{a}\left(\xi + \frac{2\pi(d-1)}{d}\right) & \widehat{a^1}\left(\xi + \frac{2\pi(d-1)}{d}\right) & \cdots & \widehat{a^r}\left(\xi + \frac{2\pi(d-1)}{d}\right) \end{bmatrix}} \times \begin{bmatrix} \Theta(d\xi)\hat{b}(\xi) \\ \widehat{b^1}(\xi) \\ \vdots \\ \widehat{b^r}(\xi) \end{bmatrix} = \begin{bmatrix} \Theta(\xi) \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{2.3}$$

Define wavelet functions $\psi^1, \dots, \psi^r, \tilde{\psi}^1, \dots, \tilde{\psi}^r$ *as follows*:

$$\begin{aligned} \psi^\ell &= |d| \sum_{k \in \mathbf{Z}} a_k^\ell \varphi(d \cdot -k) \qquad \textit{and} \\ \tilde{\psi}^\ell &= |d| \sum_{k \in \mathbf{Z}} b_k^\ell \tilde{\varphi}(d \cdot -k), \qquad \ell = 1, \dots, r. \end{aligned} \tag{2.4}$$

Then $\{\psi^1, \dots, \psi^r\}$ *and* $\{\tilde{\psi}^1, \dots, \tilde{\psi}^r\}$ *generate a pair of dual* d*-wavelet frames in* $L_2(\mathbf{R})$.

Proof. By Lemma 2.1, there exists a positive constant C such that

$$\sum_{\ell=1}^{r} \sum_{j \in \mathbf{Z}} \sum_{k \in \mathbf{Z}} [|\langle f, \psi_{j,k}^\ell \rangle|^2 + |\langle f, \tilde{\psi}_{j,k}^\ell \rangle|^2] \le C \|f\|^2 \qquad \forall\, f \in L_2(\mathbf{R}),$$

where $\psi_{j,k}^\ell = |d|^{j/2} \psi^\ell(d^j \cdot -k)$ and $\tilde{\psi}_{j,k}^\ell = |d|^{j/2} \tilde{\psi}^\ell(d^j \cdot -k)$. Define η by $\hat{\eta}(\xi) := \Theta(\xi)\hat{\tilde{\varphi}}(\xi)$. By (2.3) and a simple calculation, for every $j \in \mathbf{Z}$, one has

$$\sum_{\ell=1}^{r} \sum_{k \in \mathbf{Z}} \langle f, \psi_{j,k}^\ell \rangle \langle \tilde{\psi}_{j,k}^\ell, g \rangle = \sum_{k \in \mathbf{Z}} \langle f, \varphi_{j+1,k} \rangle \langle \eta_{j+1,k}, g \rangle - \sum_{k \in \mathbf{Z}} \langle f, \varphi_{j,k} \rangle \langle \eta_{j,k}, g \rangle, \quad f, g \in L_2(\mathbf{R}).$$

The rest of the proof follows directly from Daubechies, Han, Ron, and Shen [6, Corollary 5.3]. ■

It is easy to see that (2.3) can be rewritten as follows:

$$\overline{\hat{a}(\xi + 2\pi j/d)}\hat{b}(\xi)\Theta(d\xi) + \sum_{\ell=1}^{r} \overline{\widehat{a^\ell}(\xi + 2\pi j/d)}\widehat{b^\ell}(\xi) = \delta_j \Theta(\xi), \quad j = 0, \dots, d-1, \tag{2.5}$$

where δ denotes the **Dirac sequence** such that $\delta_0 = 1$ and $\delta_j = 0$ for all $j \in \mathbf{Z}\backslash\{0\}$.

Tight wavelet frames and dual wavelet frames have been investigated in [2], [6] for the case $d = 2$. In this paper, we shall give a systematic study of dual wavelet frames with a general dilation factor. We mention that Theorem 2.2 can also be verified using the characterization of dual wavelet frames in [7], [11].

In order to prove the main results in this paper, the following result is crucial in our construction of dual wavelet frames from refinable functions:

Lemma 2.3. *Let d be a dilation factor. Let A and B be two finitely supported sequences on $\mathbf{Z}$ such that $\hat{A}(0) = \hat{B}(0) \neq 0$. Let $\Theta(\xi) := \sum_{k\in\mathbf{Z}} \Theta_k e^{-ik\xi}$ be a 2π-periodic trigonometric polynomial and we denote by $\Theta^{(j)}$ the jth derivative of the trigonometric polynomial Θ. Then, for any positive integer n,*

$$\Theta(0) = 1 \qquad \textit{and} \qquad (1 - e^{i\xi})^n \mid [\Theta(\xi)\hat{A}(\xi) - \Theta(d\xi)\hat{B}(\xi)] \tag{2.6}$$

if and only if

$$i^\ell \Theta^{(\ell)}(0) = \sum_{k\in\mathbf{Z}} \Theta_k k^\ell = \lambda_\ell, \qquad \ell = 0, \dots, n-1, \tag{2.7}$$

where $\lambda_0 = 1$ and λ_ℓ ($\ell \in \mathbf{N}$) are uniquely determined by the following recursive formula:

$$\lambda_\ell = \frac{1}{(d^\ell - 1)\hat{A}(0)} \sum_{j=0}^{\ell-1} \frac{\ell!}{j!\,(\ell-j)!} \left[\sum_{k\in\mathbf{Z}} A_k k^{\ell-j} - d^j \sum_{k\in\mathbf{Z}} B_k k^{\ell-j}\right] \lambda_j, \quad \ell \in \mathbf{N}. \tag{2.8}$$

Consequently, for any positive integer n there exists a 2π-periodic trigonometric polynomial Θ such that (2.6) *holds. In particular, if A and B are finitely supported real-valued sequences on $\mathbf{Z}$ such that $A_{s-k} = A_k$ and $B_{\tilde{s}-k} = B_k$ for all $k \in \mathbf{Z}$ for some integers s and $\tilde{s}$ such that $c = (s - \tilde{s})/(d-1)$ is an integer, then Θ can be chosen to be a 2π-periodic trigonometric polynomial with real coefficients such that $\Theta(\xi) = e^{-ic\xi}\overline{\Theta(\xi)}$; that is, $\Theta_{c-k} = \Theta_k$ for all $k \in \mathbf{Z}$.*

Proof. By the Leibniz differentiation formula and $\hat{A}(0) = \hat{B}(0)$, (2.6) is equivalent to $\Theta(0) = 1$ and

$$(d^\ell - 1)\hat{A}(0)\Theta^{(\ell)}(0) = \sum_{j=0}^{\ell-1} \frac{\ell!}{j!\,(\ell-j)!} \Theta^{(j)}(0)[\hat{A}^{(\ell-j)}(0) - d^j \hat{B}^{(\ell-j)}(0)], \quad \ell = 1, \dots, n-1.$$

It follows directly from the above equations that (2.6) is equivalent to (2.7).

When A and B are real-valued sequences, by (2.8), it is clear that we can choose Θ to be a 2π-periodic trigonometric polynomial with real coefficients. Note that $A_{s-k} = \overline{A_k}$ for all $k \in \mathbf{Z}$ if and only if $\hat{A}(\xi) = e^{-is\xi}\overline{\hat{A}(\xi)}$. Let θ be a 2π-periodic trigonometric polynomial with real coefficients such that $\theta(0) = 1$ and $(1 - e^{i\xi})^n \mid [\theta(\xi)\hat{A}(\xi) - \theta(d\xi)\hat{B}(\xi)]$. Set $\Theta(\xi) := [\theta(\xi) + e^{-ic\xi}\overline{\theta(\xi)}]/2$. Since

$$\begin{aligned} e^{-ic\xi}\overline{\theta(\xi)}\hat{A}(\xi) - e^{-idc\xi}\overline{\theta(d\xi)}\hat{B}(\xi) &= e^{-i(s+c)\xi}[\overline{\theta(\xi)}e^{is\xi}\hat{A}(\xi) - \overline{\theta(d\xi)}e^{i\tilde{s}\xi}\hat{B}(\xi)] \\ &= e^{-i(s+c)\xi}\overline{[\theta(\xi)\hat{A}(\xi) - \theta(d\xi)\hat{B}(\xi)]}, \end{aligned}$$

it is easy to check that (2.6) holds and $\Theta(\xi) = e^{-ic\xi}\overline{\Theta(\xi)}$. ■

Lemma 2.3 was also given in [9, Lemma 3.2] which generalized the special case $A(\xi) \equiv 1$ in an earlier version of this paper. The following result is important for us to construct pairs of symmetric dual wavelet frames from symmetric refinable functions:

Proposition 2.4. *Let d be a positive integer such that $d \geq 2$. For any positive integer N and any integer s_0, there exist 2π-periodic trigonometric polynomials $c^1, \ldots, c^d$ with real coefficients such that*:

(a) $\det C(0) \neq 0$, *where $C(\xi)$ is the matrix defined by*

$$C(\xi) := \begin{bmatrix} c^1(\xi) & c^2(\xi) & \cdots & c^d(\xi) \\ c^1\left(\xi + \frac{2\pi}{d}\right) & c^2\left(\xi + \frac{2\pi}{d}\right) & \cdots & c^d\left(\xi + \frac{2\pi}{d}\right) \\ \vdots & \vdots & \ddots & \vdots \\ c^1\left(\xi + \frac{2\pi(d-1)}{d}\right) & c^2\left(\xi + \frac{2\pi(d-1)}{d}\right) & \cdots & c^d\left(\xi + \frac{2\pi(d-1)}{d}\right) \end{bmatrix}. \tag{2.9}$$

(b) $(1 - e^{i\xi})^{2N} \mid c^j(\xi)$ *for all* $j = 2, \ldots, d$.
(c) *The 2π-periodic trigonometric polynomials $c^1, \ldots, c^d$ have real coefficients and*

$$c^j(\xi) = e^{-is_0\xi}\overline{c^j(\xi)}, \qquad j = 1, \ldots, N_{d,s_0},$$

and

$$c^j(\xi) = -e^{-is_0\xi}\overline{c^j(\xi)}, \qquad j = N_{d,s_0} + 1, \ldots, d,$$

where the integer N_{d,s_0} is defined to be

$$N_{d,s_0} := \begin{cases} \left\lfloor \dfrac{d+2}{2} \right\rfloor, & \text{when } s_0 \text{ is an even integer}, \\ \left\lfloor \dfrac{d+1}{2} \right\rfloor, & \text{when } s_0 \text{ is an odd integer}, \end{cases} \tag{2.10}$$

with $\lfloor x \rfloor$ denoting the greatest integer which is no greater than x.

In particular, when $d = 2$ and s_0 is even, one can choose $c^1(\xi) = e^{-is_0\xi/2}$ and $c^2(\xi) = e^{-is_0\xi/2}(1-\cos\xi)^N$. When $d = 2$ and s_0 is odd, one can choose $c^1(\xi) = e^{-i(s_0-1)\xi/2}(1+e^{-i\xi})$ and $c^2(\xi) = e^{-i(s_0-1-2N)\xi/2}(1-e^{-i\xi})^{2N+1}$.

Proof. Observe that a 2π-periodic trigonometric polynomial c with real coefficients satisfies $c(\xi) = \overline{c(\xi)}$ if and only if $c(\xi) = p(\cos\xi)$ for some polynomial p with real coefficients. Let $m := \lfloor (d+2)/2 \rfloor$. The main idea in the following proof is that we divide the set $\{2\pi j/d \;:\; j = 0, \dots, d-1\}$ into three subsets I_1, I_2, and I_3, where $I_1 := \{j\pi \;:\; j = 0, \dots, 2m-d-1\}$, $I_2 := \{2\pi j/d \;:\; j = 1, \dots, d-m\}$, and $I_3 := \{2\pi - 2\pi j/d \;:\; j = 1, \dots, d-m\}$.

It is not difficult to see that there exist polynomials $p_0, \dots, p_{m-1}$ with real coefficients such that

$$p_j(\cos(2\pi k/d)) = \delta_{j-k}, \qquad j, k = 0, \dots, m-1,$$

and

$$p_j^{(\ell)}(1) = 0 \qquad \forall\, \ell = 0, \dots, N-1 \quad \text{and} \quad j = 1, \dots, m-1.$$

When s_0 is even, we define

$$c^j(\xi) = e^{-is_0\xi/2} p_{j-1}(\cos\xi), \qquad j = 1, \dots, m,$$

and

$$c^{j+m}(\xi) = e^{-is_0\xi/2}(e^{i\xi} - e^{-i\xi}) p_j(\cos\xi), \qquad j = 1, \dots, d-m.$$

In order to prove that $\det C(0) \neq 0$, it suffices to prove it for the case $s_0 = 0$. When $s_0 = 0$, by the choice of the polynomials $p_0, \dots, p_{m-1}$, it is easy to see that after performing suitable permutations on rows and columns of the matrix $C(0)$, the matrix $C(0)$ becomes

$$\begin{bmatrix} I_{2m-d} & 0 & 0 \\ 0 & I_{d-m} & \overline{E} - E \\ 0 & I_{d-m} & E - \overline{E} \end{bmatrix} \quad \text{with} \quad E := \operatorname{diag}(e^{-i2\pi/d}, e^{-i4\pi/d}, \dots, e^{-i2(d-m)\pi/d}).$$

Evidently, $|\det C(0)| = 4^{d-m} \prod_{j=1}^{d-m} \sin(2\pi j/d) \neq 0$.

When s_0 is odd and d is odd, we have $2m-1 = d$ and we define

$$c^j(\xi) = e^{-i(s_0-1)\xi/2}(1+e^{-i\xi}) p_{j-1}(\cos\xi), \qquad j = 1, \dots, m,$$

and

$$c^{j+m}(\xi) = e^{-i(s_0-1)\xi/2}(1-e^{-i\xi}) p_j(\cos\xi), \qquad j = 1, \dots, m-1.$$

When s_0 is odd and d is even, we have $2m-2 = d$ and we define

$$c^j(\xi) = e^{-i(s_0-1)\xi/2}(1+e^{-i\xi}) p_{j-1}(\cos\xi), \qquad j = 1, \dots, m-1,$$

and

$$c^{j+m-1}(\xi) = e^{-i(s_0-1)\xi/2}(1-e^{-i\xi}) p_j(\cos\xi), \qquad j = 1, \dots, m-1.$$

In order to prove that $\det C(0) \neq 0$, it is easy to see that it suffices to prove it for the case $s_0 = 1$. When $s_0 = 1$, by the choice of the polynomials $p_0, \dots, p_{m-1}$, it is easy to see that after performing suitable permutations on rows and columns of the matrix $C(0)$, the matrix $C(0)$ becomes

$$\begin{bmatrix} I_{2m-d} & 0 & 0 \\ 0 & I_{d-m} + E & I_{d-m} - E \\ 0 & I_{d-m} + \overline{E} & I_{d-m} - \overline{E} \end{bmatrix} \quad \text{with} \quad E := \operatorname{diag}(e^{-i2\pi/d}, e^{-i4\pi/d}, \dots, e^{-i2(d-m)\pi/d}).$$

Evidently, $|\det C(0)| = 4^{d-m} \prod_{j=1}^{d-m} \sin(2\pi j/d) \neq 0$. All other claims can be easily verified. ■

Proposition 2.4 still holds if we take $p_0 = 1$ in the above proof. We observe that the degrees of the 2π-periodic trigonometric polynomials $c^1, \dots, c^d$ constructed in the proof of Proposition 2.4 can be made even smaller.

3. Construction of Dual Wavelet Frames

In this section, we shall discuss how to construct pairs of dual d-wavelet frames from any two d-refinable functions.

Let φ and $\tilde{\varphi}$ be two d-refinable functions in $L_2(\mathbf{R})$ with finitely supported masks a and b, respectively. Suppose that a and b satisfy the sum rules of orders m and n with respect to the lattice $d\mathbf{Z}$ for some positive integers m and n, respectively; in other words, $(1 + e^{-i\xi} + \cdots + e^{-i(d-1)\xi})^m \mid \hat{a}(\xi)$ and $(1 + e^{-i\xi} + \cdots + e^{-i(d-1)\xi})^n \mid \hat{b}(\xi)$.

In order to construct a pair of dual d-wavelet frames by Theorem 2.2, we need to construct finitely supported sequences $a^1, \dots, a^r, b^1, \dots, b^r$ on $\mathbf{Z}$ and a 2π-periodic trigonometric polynomial Θ such that (2.2) and (2.3) are satisfied. In this section, let us consider the special case $r = d$. When $r = d$, the relation in (2.3) can be rewritten as follows:

$$\overline{\begin{bmatrix} \widehat{a^1}(\xi) & \widehat{a^2}(\xi) & \cdots & \widehat{a^d}(\xi) \\ \widehat{a^1}\left(\xi + \frac{2\pi}{d}\right) & \widehat{a^2}\left(\xi + \frac{2\pi}{d}\right) & \cdots & \widehat{a^d}\left(\xi + \frac{2\pi}{d}\right) \\ \vdots & \vdots & \ddots & \vdots \\ \widehat{a^1}\left(\xi + \frac{2\pi(d-1)}{d}\right) & \widehat{a^2}\left(\xi + \frac{2\pi(d-1)}{d}\right) & \cdots & \widehat{a^d}\left(\xi + \frac{2\pi(d-1)}{d}\right) \end{bmatrix}} \tag{3.1}$$
$$\times \begin{bmatrix} \widehat{b^1}(\xi) \\ \widehat{b^2}(\xi) \\ \vdots \\ \widehat{b^d}(\xi) \end{bmatrix} = \begin{bmatrix} \Theta(\xi) - \Theta(d\xi)\overline{\hat{a}(\xi)}\hat{b}(\xi) \\ -\Theta(d\xi)\overline{\hat{a}\left(\xi + \frac{2\pi}{d}\right)}\hat{b}(\xi) \\ \vdots \\ -\Theta(d\xi)\overline{\hat{a}\left(\xi + \frac{2(d-1)\pi}{d}\right)}\hat{b}(\xi) \end{bmatrix}.$$

Define the wavelet functions $\psi^1, \ldots, \psi^d, \tilde{\psi}^1, \ldots, \tilde{\psi}^d$ as in (1.5). Since $\hat{\varphi}(0) = 1$, it is easy to see that $\{\psi^1, \ldots, \psi^d\}$ has vanishing moments of order n if and only if $(1-e^{-i\xi})^n \mid \widehat{a^\ell}(\xi)$ for all $\ell = 1, \ldots, d$. So, in order to achieve high vanishing moments, it is necessary and natural to require that

$$\widehat{a^\ell}(\xi) = (1 - e^{-i\xi})^n g(\xi) c^\ell(\xi), \qquad \ell = 1, \ldots, d, \tag{3.2}$$

where $g, c^\ell, \ell = 1, \ldots, d$, are 2π-periodic trigonometric polynomials with g being a certain common divisor of all the 2π-periodic trigonometric polynomials $\widehat{a^\ell}, \ell = 1, \ldots, d$. Consequently, we have

$$\begin{bmatrix} \widehat{a^1}(\xi) & \widehat{a^2}(\xi) & \cdots & \widehat{a^d}(\xi) \\ \widehat{a^1}\left(\xi + \frac{2\pi}{d}\right) & \widehat{a^2}\left(\xi + \frac{2\pi}{d}\right) & \cdots & \widehat{a^d}\left(\xi + \frac{2\pi}{d}\right) \\ \vdots & \vdots & \ddots & \vdots \\ \widehat{a^1}\left(\xi + \frac{2\pi(d-1)}{d}\right) & \widehat{a^2}\left(\xi + \frac{2\pi(d-1)}{d}\right) & \cdots & \widehat{a^d}\left(\xi + \frac{2\pi(d-1)}{d}\right) \end{bmatrix}$$
$$= D(\xi)C(\xi)$$

with the matrix $C(\xi)$ being defined in (2.9) and $D(\xi)$ being the following diagonal matrix:

$$D(\xi) := \begin{bmatrix} (1 - e^{-i\xi})^n g(\xi) & & & \\ & (1 - e^{-i(\xi+2\pi/d)})^n g\left(\xi + \frac{2\pi}{d}\right) & & \\ & & \ddots & \\ & & & (1 - e^{-i(\xi+2\pi(d-1)/d)})^n g\left(\xi + \frac{2\pi(d-1)}{d}\right) \end{bmatrix}.$$

Denote

$$h(\xi) := \det C(\xi) \tag{3.3}$$

and

$$\begin{bmatrix} f^1(\xi) \\ f^2(\xi) \\ \vdots \\ f^d(\xi) \end{bmatrix} := \begin{bmatrix} \dfrac{\Theta(\xi) - \Theta(d\xi)\overline{\hat{a}(\xi)}\hat{b}(\xi)}{(1 - e^{i\xi})^n \overline{g(\xi)}\,\overline{h(\xi)}} \\ -\dfrac{\Theta(d\xi)\overline{\hat{a}(\xi + 2\pi/d)}\hat{b}(\xi)}{(1 - e^{i(\xi+2\pi/d)})^n \overline{g(\xi + 2\pi/d)}\,\overline{h(\xi)}} \\ \vdots \\ -\dfrac{\Theta(d\xi)\overline{\hat{a}(\xi + 2\pi(d-1)/d)}\hat{b}(\xi)}{(1 - e^{i(\xi+2\pi(d-1)/d)})^n \overline{g(\xi + 2\pi(d-1)/d)}\,\overline{h(\xi)}} \end{bmatrix}. \tag{3.4}$$

Now, the equation in (3.1) can be rewritten as follows:

$$
\begin{bmatrix} \widehat{b^1}(\xi) \\ \widehat{b^2}(\xi) \\ \vdots \\ \widehat{b^d}(\xi) \end{bmatrix} = [\overline{C(\xi)}]^{-1}[\overline{D(\xi)}]^{-1} \begin{bmatrix} \Theta(\xi) - \Theta(d\xi)\overline{\hat{a}(\xi)}\hat{b}(\xi) \\ -\Theta(d\xi)\overline{\hat{a}\left(\xi + \frac{2\pi}{d}\right)}\hat{b}(\xi) \\ \vdots \\ -\Theta(d\xi)\overline{\hat{a}\left(\xi + \frac{2(d-1)\pi}{d}\right)}\hat{b}(\xi) \end{bmatrix} = \overline{\operatorname{adj} C(\xi)} \begin{bmatrix} f^1(\xi) \\ f^2(\xi) \\ \vdots \\ f^d(\xi) \end{bmatrix}, \tag{3.5}
$$

where adj $C(\xi)$ denotes the adjacent matrix of $C(\xi)$, that is, $C(\xi)^{-1} = [\det C(\xi)]^{-1}$ adj $C(\xi)$. Clearly, all the entries in adj $C(\xi)$ are 2π-periodic trigonometric polynomials since all the entries of $C(\xi)$ are 2π-periodic trigonometric polynomials.

Now, according to Theorem 2.2, the challenging question that remains is to choose an appropriate 2π-periodic trigonometric polynomial Θ with $\Theta(0) = 1$ such that $f^1, \dots, f^d$ are 2π-periodic trigonometric polynomials and $\widehat{b^\ell}(0) = 0$ for all $\ell = 1, \dots, d$.

We have the following result on pairs of dual wavelet frames:

Theorem 3.1. *Let $\varphi \in L_2(\mathbf{R})$ and $\tilde{\varphi} \in L_2(\mathbf{R})$ be two arbitrary d-refinable functions with dilation factor d and finitely supported masks a and b, respectively. Suppose that a and b satisfy the sum rules of orders m and n with respect to the lattice $d\mathbf{Z}$ for some positive integers m and n, respectively. Construct a 2π-periodic trigonometric polynomial Θ by Lemma* 2.3 *such that*

$$
\Theta(0) = 1, \qquad (1 - e^{i\xi})^{n+m} \mid [\Theta(\xi) - \Theta(d\xi)\overline{\hat{a}(\xi)}\hat{b}(\xi)]. \tag{3.6}
$$

Define the finitely supported sequences $a^1, \dots, a^d, b^1, \dots, b^d$ on $\mathbf{Z}$ as in (3.2) *and* (3.5) *by taking $g(\xi) = 1$ and $c^\ell(\xi) = e^{-i(\ell-1)\xi}$, $\ell = 1, \dots, d$. Then $\{\psi^1, \dots, \psi^d\}$ and $\{\tilde{\psi}^1, \dots, \tilde{\psi}^d\}$, which are defined in* (1.5), *generate a pair of dual d-wavelet frames in $L_2(\mathbf{R})$. Moreover, $\{\psi^1, \dots, \psi^d\}$ and $\{\tilde{\psi}^1, \dots, \tilde{\psi}^d\}$ have vanishing moments of orders n and m, respectively. Note that $\psi^\ell = \psi^1(\cdot - (\ell-1)/d)$ for $\ell = 1, \dots, d$.*

Proof. Since the mask b satisfies the sum rules of order n with respect to the lattice $d\mathbf{Z}$, we have $(1+e^{-i\xi}+\cdots+e^{-i(d-1)\xi})^n \mid \hat{b}(\xi)$. Consequently, $(1-e^{i(\xi+2\pi j/d)})^n \mid \hat{b}(\xi)$ for all $j = 1, \dots, d-1$. By $c^\ell(\xi) = e^{-i(\ell-1)\xi}$, $\ell = 1, \dots, d$, we observe that $h(\xi) := \det C(\xi)$ is a monomial since $C(\xi)\overline{C(\xi)}^T = dI_d$, where the matrix $C(\xi)$ is defined in (2.9). Now, by the fact that $g(\xi) = 1$ and $h(\xi)$ is a monomial, it is straightforward to see that $f^2, \dots, f^d$ are 2π-periodic trigonometric polynomials. Since $(1+e^{-i\xi}+\cdots+e^{-i(d-1)\xi})^m \mid \hat{a}(\xi)$, we have $(1-e^{i\xi})^m \mid \hat{a}(\xi+2\pi j/d)$ for all $j = 1, \dots, d-1$ and, therefore, $(1-e^{i\xi})^m \mid f^j(\xi)$ for all $j = 2, \dots, d$.

On the other hand, since $g(\xi) = 1$ and $h(\xi)$ is a monomial, it follows directly from (3.6) that f^1 is a 2π-periodic trigonometric polynomial and $(1 - e^{i\xi})^m \mid f^1(\xi)$. The proof is completed by our discussion before this theorem. ∎

We point out that Theorem 3.1 holds for general 2π-periodic trigonometric polynomials $c^1, \dots, c^d$ provided that $\det C(\xi)$ is a monomial, where $C(\xi)$ is defined in (2.9).

For a 2π-periodic trigonometric polynomial p, we define

$$Z(p, \xi_0) := \sup\{\ell \in \mathbf{N} \cup \{0\} : (e^{i\xi} - e^{i\xi_0})^\ell \mid p(\xi)\} = \inf\{\ell \in \mathbf{N} \cup \{0\} : p^{(\ell)}(\xi_0) \neq 0\}. \tag{3.7}$$

That is, $Z(p, \xi_0)$ denotes the multiplicity of the zeros of $p(\xi)$ at the point $\xi = \xi_0$.

Now we can generalize Theorem 3.1 and the following is the main result in this section:

Theorem 3.2. *Let φ and $\tilde{\varphi}$ be two d-refinable functions in $L_2(\mathbf{R})$ with the dilation factor d and finitely supported masks a and b, respectively. Suppose that a and b satisfy the sum rules of orders m and n with respect to the lattice $d\mathbf{Z}$ for some positive integers m and n, respectively; that is, $(1 + e^{-i\xi} + \cdots + e^{-i(d-1)\xi})^m \mid \hat{a}(\xi)$ and $(1 + e^{-i\xi} + \cdots + e^{-i(d-1)\xi})^n \mid \hat{b}(\xi)$. Let $g, c^1, \dots, c^d$ be 2π-periodic trigonometric polynomials. Define $h(\xi) := \det C(\xi)$, where the matrix $C(\xi)$ is defined in* (2.9). *Then there exists a 2π-periodic trigonometric polynomial Θ such that*:

(a) $\Theta(0) = 1$.
(b) *All $f^\ell, \ell = 1, \dots, d$, which are defined in* (3.4), *are 2π-periodic trigonometric polynomials*.
(c) $(1 - e^{i\xi})^m \mid f^\ell(\xi)$ *for all $\ell = 1, \dots, d$*,

if and only if the following two conditions hold:

$$(1 + e^{-i\xi} + \cdots + e^{-i(d-1)\xi})^{n+Z(g,0)+Z(h,0)} \mid \hat{b}(\xi) \tag{3.8}$$

and

$$Z(\hat{a}, 2\pi j/d) - Z(g, 2\pi j/d) - Z(h, 0) \geq m \qquad \forall\, j = 1, \dots, d-1. \tag{3.9}$$

(*For example, when $g(\xi) \equiv 1$ and $h(0) \neq 0$, then* (3.8) *and* (3.9) *are automatically satisfied.*) *For any 2π-periodic trigonometric polynomial Θ such that* (a), (b), *and* (c) *are satisfied* (*such Θ can be easily obtained by solving a system of linear equations which are induced by the three conditions in* (a), (b), *and* (c) *by long division*), *let $a^1, \dots, a^d, b^1, \dots, b^d$ be defined in* (3.2) *and* (3.5). *Define the wavelet functions $\psi^1, \dots, \psi^d, \tilde{\psi}^1, \dots, \tilde{\psi}^d$ as in* (1.5). *Then $\{\psi^1, \dots, \psi^d\}$ and $\{\tilde{\psi}^1, \dots, \tilde{\psi}^d\}$ generate a pair of dual d-wavelet frames with compact support in $L_2(\mathbf{R})$. Moreover, $\{\psi^1, \dots, \psi^d\}$ and $\{\tilde{\psi}^1, \dots, \tilde{\psi}^d\}$ have vanishing moments of orders n and m, respectively.*

Proof. *Sufficiency*. For simplicity of the presentation, let us assume here that $g(\xi) \equiv 1$ and $h(0) \neq 0$; the complete proof for the general case can be found in the Appendix.

Since $h(0) \neq 0$, we define $\theta_1(\xi) := e^{i\xi(d-1)/2}\overline{h(\xi/d)}/[\overline{h(0)}]^2$. By the fact $h(\xi + 2\pi/d) = (-1)^{d-1}h(\xi)$, we have $\theta_1(\xi + 2\pi) = e^{i(\xi+2\pi)(d-1)/2}\overline{h(\xi/d + 2\pi/d)}/[\overline{h(0)}]^2 =$

$\theta_1(\xi)$ and, therefore, θ_1 is a 2π-periodic trigonometric polynomial. Since $\overline{h(0)}\theta_1(0) = \overline{h(0)}\theta_1(0)\overline{\hat{a}(0)}\hat{b}(0) = 1$, by Lemma 2.3, there exists a 2π-periodic trigonometric polynomial θ_2 such that $\theta_2(0) = 1$ and

$$(3.10)\qquad (1 - e^{i\xi})^{n+m} \mid \theta_2(\xi)[\overline{h(\xi)}\theta_1(\xi)] - \theta_2(d\xi)[\overline{h(d\xi)}\theta_1(d\xi)\overline{\hat{a}(\xi)}\hat{b}(\xi)].$$

Now we take $\Theta(\xi) := \theta_2(\xi)\overline{h(\xi)}\theta_1(\xi)$. Obviously, $\Theta(0) = 1$. Noting that $(1 + e^{-i\xi} + \cdots + e^{-i(d-1)\xi})^m \mid \hat{a}(\xi)$ and $(1 + e^{-i\xi} + \cdots + e^{-i(d-1)\xi})^n \mid \hat{b}(\xi)$, by a simple calculation (see the Appendix for more detail), we can verify that indeed all f^ℓ are 2π-periodic trigonometric polynomials and $(1 - e^{i\xi})^m \mid f^\ell(\xi)$ for all $\ell = 1, \ldots, d$.

Necessity. Since all f^ℓ are 2π-periodic trigonometric polynomials and $(1 - e^{i\xi})^m \mid f^\ell(\xi)$ for all $\ell = 2, \ldots, d$, we have $Z(f^{j+1}, 0) \geq m$ and $Z(f^{j+1}, -2\pi j/d) \geq 0$ for all $j = 1, \ldots, d-1$. By the definition of f^ℓ in (3.4) and $\Theta(0) = \hat{b}(0) = 1$, we deduce

$$Z(\hat{a}, 2\pi j/d) - Z(g, 2\pi j/d) - Z(h, 0) = Z(f^{j+1}, 0) \geq m \qquad \forall\, j = 1, \ldots, d-1.$$

So, (3.9) must hold. Similarly, by the definition of f^ℓ in (3.4) and $\Theta(0) = \hat{a}(0) = 1$, we have

$$Z(\hat{b}, -2\pi j/d) - n - Z(g, 0) - Z(h, 0) = Z(f^{j+1}, -2\pi j/d) \geq 0 \qquad \forall\, j = 1, \ldots, d-1,$$

which is equivalent to (3.8). ■

B-Spline functions are of great interest in many applications. The B-spline function of order m ($m \in \mathbf{N}$), denoted by B_m throughout this paper, can be obtained via the following recursive formula: $B_1 = \chi_{[0,1]}$, the characteristic function of the interval [0, 1], and

$$(3.11)\qquad B_m(x) := \int_0^1 B_{m-1}(x - t)\, dt, \qquad x \in \mathbf{R}, \quad m = 2, 3, \ldots.$$

The B-spline function $B_m \in C^{m-2}(\mathbf{R})$ is a function of piecewise polynomials of degree less than m, vanishes outside the interval $[0, m]$, and is symmetric about the point $x = m/2$ (i.e., $B_m(m-x) = B_m(x)$ for all $x \in \mathbf{R}$). It is well-known that the B-spline function B_m is a d-refinable function satisfying

$$\widehat{B_m}(d\xi) = \left(\frac{1 + e^{-i\xi} + \cdots + e^{-i(d-1)\xi}}{d}\right)^m \widehat{B_m}(\xi).$$

Now we have the following result on wavelet frames which is a direct consequence of Theorem 3.2:

Corollary 3.3. *Let $\varphi \in L_2(\mathbf{R})$ be a d-refinable function with the dilation factor d and a finitely supported mask a. Choose 2π-periodic trigonometric polynomials $g, c^1, \ldots, c^d$ such that*

$$(3.12)\qquad Z(\hat{a}, 2\pi j/d) > Z(g, 2\pi j/d) + Z(h, 0) \qquad \forall\, j = 1, \ldots, d-1,$$

where $h(\xi) := \det C(\xi)$ *and the matrix* $C(\xi)$ *is defined in* (2.9). *For any positive integer* n, *define the wavelet functions* $\psi^1, \dots, \psi^d$ *by*

$$\widehat{\psi^\ell}(d\xi) = (1 - e^{-i\xi})^n g(\xi) c^\ell(\xi)\hat{\varphi}(\xi), \qquad \ell = 1, \dots, d. \tag{3.13}$$

Then $\{\psi^1, \dots, \psi^d\}$ *generates a* d*-wavelet frame in* $L_2(\mathbf{R})$ *and has vanishing moments of order* n. *Moreover, there exist compactly supported functions* $\tilde{\psi}^1, \dots, \tilde{\psi}^d$, *which can be derived explicitly from any* d*-refinable function in* $L_2(\mathbf{R})$ *whose mask is finitely supported and satisfies the sum rules of order* $n + Z(g, 0) + Z(h, 0)$ *with respect to the lattice* $d\mathbf{Z}$, *such that* $\{\psi^1, \dots, \psi^d\}$ *and* $\{\tilde{\psi}^1, \dots, \tilde{\psi}^d\}$ *generate a pair of dual* d*-wavelet frames in* $L_2(\mathbf{R})$.

Proof. Take $\tilde{\varphi}$ to be any d-refinable function in $L_2(\mathbf{R})$ whose mask is finitely supported and satisfies the sum rules of order $n + Z(g, 0) + Z(h, 0)$ with respect to the lattice $d\mathbf{Z}$. For example, we can take $\tilde{\varphi} = B_{n+Z(g,0)+Z(h,0)}$ to be the B-spline function of order $n + Z(g, 0) + Z(h, 0)$ which is defined in (3.11). Observe that (3.12) is equivalent to the condition in (3.9) with $m = 1$. Now Corollary 3.3 follows directly from Theorem 3.2. ■

If in Corollary 3.3 we choose $c^\ell(\xi) = e^{-i(\ell-1)\xi}$, $\ell = 1, \dots, d$, then we have the following result:

Corollary 3.4. *Let* φ *be a* d*-refinable function in* $L_2(\mathbf{R})$ *with the dilation factor* d *and a finitely supported mask* a. *For any* 2π*-periodic trigonometric polynomial* g *such that*

$$g(0) = 0 \qquad \text{and} \qquad Z(\hat{a}, 2\pi j/d) > Z(g, 2\pi j/d) \qquad \forall\, j = 1, \dots, d-1, \tag{3.14}$$

define a wavelet function ψ *by* $\hat{\psi}(\xi) = g(\xi)\hat{\varphi}(\xi)$. *Then* ψ *generates a* d*-wavelet frame in* $L_2(\mathbf{R})$; *that is,* $\{\psi_{j,k} : j, k \in \mathbf{Z}\}$ *is a frame in* $L_2(\mathbf{R})$, *where* $\psi_{j,k} := |d|^{j/2}\psi(d^j \cdot -k)$. *Moreover, there exist compactly supported functions* $\tilde{\psi}^1, \dots, \tilde{\psi}^d$ *with arbitrary smoothness such that* $\{\psi(d\cdot), \psi(d \cdot -1), \dots, \psi(d \cdot -d + 1)\}$ *and* $\{\tilde{\psi}^1, \tilde{\psi}^2, \dots, \tilde{\psi}^d\}$ *generate a pair of dual* d*-wavelet frames in* $L_2(\mathbf{R})$.

Proof. Let $c^\ell(\xi) = e^{-i(\ell-1)\xi}$, $\ell = 1, \dots, d$. Then it is easy to see that $h(0) = \det C(0) \neq 0$ since $C(\xi)\overline{C(\xi)}^T = dI_d$, where the matrix $C(\xi)$ is defined in (2.9). Clearly, by (3.14), (3.12) holds and $g(0)c^\ell(0) = 0$ for all $\ell = 1, \dots, d$. Therefore, by Corollary 3.3, $\{\psi^1, \dots, \psi^d\}$, which is defined in (3.13) with $n = 0$, generates a d-wavelet frame. It is easy to check that $\psi^\ell = |d|\psi(d \cdot -\ell + 1)$ for $\ell = 1, \dots, d$. Now the claim follows from the fact that $\{\psi^1, \dots, \psi^d\}$ and $\{|d|^{1/2}\psi\}$ generate the same d-wavelet frame in $L_2(\mathbf{R})$. ■

Let us make some remarks here for the above results. Through the proof of Theorem 3.2 in the Appendix and the following argument, we shall see that the quantities $Z(\widehat{a^\ell}, 2\pi j/d)$, $j = 0, \dots, d-1$, which denote the multiplicity of zeros of the trigonometric polynomials $\widehat{a^\ell}(\xi)$ at $\xi = 2\pi j/d$, play a critical role in the construction of pairs of dual wavelet frames via Theorem 2.2. In the following, we shall see that (3.14) in Corollary 3.4 is not only a sufficient condition for having an MRA wavelet frame, but also a necessary condition. More precisely, (3.14) must hold if via Theorem 2.2 there exist compactly supported functions $\tilde{\psi}^1, \dots, \tilde{\psi}^d$ such that $\{\psi(d\cdot), \psi(d\cdot-1), \dots, \psi(d\cdot-d+1)\}$, which is given in Corollary 3.4, and $\{\tilde{\psi}^1, \tilde{\psi}^2, \dots, \tilde{\psi}^d\}$ generate a pair of dual d-wavelet frames in $L_2(\mathbf{R})$.

To see this point, by Theorem 2.2, (2.5) must hold with $\Theta, \hat{a}, \hat{b}, \widehat{a^\ell}, \widehat{b^\ell}, \ell = 1, \dots, r$, being some 2π-periodic trigonometric polynomials such that (2.2) holds. In particular, it follows from (2.5) that

$$\overline{\hat{a}(\xi + 2\pi j/d)}\hat{b}(\xi)\Theta(d\xi) = -\sum_{\ell=1}^{r} \overline{\widehat{a^\ell}(\xi + 2\pi j/d)}\widehat{b^\ell}(\xi) \qquad \forall\ j = 1, \dots, d-1. \tag{3.15}$$

Therefore, by $\Theta(0) = \hat{b}(0) = 1$, it follows from (3.15) that we must have

$$Z(\hat{a}, 2\pi j/d) \geq \min\{Z(\widehat{a^\ell}, 2\pi j/d) + Z(\widehat{b^\ell}, 0)\ :\ \ell = 1, \dots, r\} \qquad \forall\ j = 1, \dots, d-1. \tag{3.16}$$

Similarly, by $\Theta(0) = \hat{a}(0) = 1$, it follows from (3.15) that

$$Z(\hat{b}, -2\pi j/d) \geq \min\{Z(\widehat{a^\ell}, 0) + Z(\widehat{b^\ell}, -2\pi j/d)\ :\ \ell = 1, \dots, r\} \qquad \forall\ j = 1, \dots, d-1. \tag{3.17}$$

By our choice of $\widehat{a^\ell}(\xi) = (1 - e^{-i\xi})^n g(\xi) c^\ell(\xi)$ in (3.2) and $r = d$, we have $Z(\widehat{a^\ell}, 2\pi j/d) = Z(g, 2\pi j/d) + Z(c^\ell, 2\pi j/d)$ for all $j = 1, \dots, d-1$, and $Z(\widehat{a^\ell}, 0) = n + Z(g, 0) + Z(c^\ell, 0)$ for all $\ell = 1, \dots, d$. Consequently, we conclude from (3.16) that

$$\begin{aligned} Z(\hat{a}, 2\pi j/d) - Z(g, 2\pi j/d) - \min\{Z(c^\ell, 2\pi j/d)\ :\ \ell = 1, \dots, d\} \\ \geq \min\{Z(\widehat{b^\ell}, 0)\ :\ \ell = 1, \dots, d\} \qquad \forall\ j = 1, \dots, d-1, \end{aligned} \tag{3.18}$$

which is quite similar to (3.9). Similarly, it follows from (3.17) that

$$Z(\hat{b}, 2\pi j/d) \geq n + Z(g, 0) + \min\{Z(c^\ell, 0)\ :\ \ell = 1, \dots, d\} \qquad \forall\ j = 1, \dots, d-1.$$

That is,

$$(1 + e^{-i\xi} + \cdots + e^{-i(d-1)\xi})^{n+Z(g,0)+\min\{Z(c^\ell,0)\,:\,\ell=1,\dots,d\}} \mid \hat{b}(\xi), \tag{3.19}$$

which is quite similar to (3.8).

By Theorem 2.2, it is necessary that $\widehat{b^\ell}(0) = 0$ and, therefore, $\min\{Z(\widehat{b^\ell}, 0)\ :\ \ell = 1, \dots, d\} > 0$. Since in Corollary 3.4 we set $c^\ell(\xi) = e^{-i(\ell-1)\xi}$, $\ell = 1, \dots, d$, we deduce from (3.18) that $Z(\hat{a}, 2\pi j/d) > Z(g, 2\pi j/d)$ for all $j = 1, \dots, d-1$. So, (3.14) must be a necessary condition in Corollary 3.4.

Let us consider the following simple example:

Example 3.5. Let B_m be the B-spline function of order m defined in (3.11). Then B_m is d-refinable with mask $\hat{a}(\xi) = d^{-m}(1 + e^{-i\xi} + \cdots + e^{-i(d-1)\xi})^m$ for any dilation factor $d \geq 2$. For any positive integer n, define

$$\psi = \sum_{k=0}^{n} (-1)^k \frac{n!}{k!\,(n-k)!} B_m(\cdot - k).$$

(That is, $\hat{\psi}(\xi) = (1 - e^{-i\xi})^n \widehat{B_m}(\xi)$.) Then ψ has vanishing moments of order n. We apply Corollary 3.4 with the special choice $g(\xi) = (1 - e^{-i\xi})^n$; note that g does not depend on d and $g(\xi) \neq 0$ for all $\xi \in (0, 2\pi)$. We then conclude that ψ generates a d-wavelet frame in $L_2(\mathbf{R})$ for any dilation factor $d \geq 2$.

Finally, we demonstrate that by appropriately choosing the 2π-periodic trigonometric polynomials c^ℓ, $\ell = 1, \dots, d$, the set $\{\psi^2, \dots, \psi^d\}$ of wavelet functions can have vanishing moments of arbitrary order.

Corollary 3.6. *Let φ and $\tilde{\varphi}$ be two d-refinable functions in $L_2(\mathbf{R})$ with the dilation factor d and finitely supported masks a and b, respectively. Suppose that a and b satisfy the sum rules of orders m and n with respect to the lattice $d\mathbf{Z}$ for some positive integers m and n, respectively. Let N be an arbitrary nonnegative integer. Then one can construct finitely supported sequences $a^1, \dots, a^d, b^1, \dots, b^d$ such that by defining the functions $\psi^1, \dots, \psi^d, \tilde{\psi}^1, \dots, \psi^d$ as in* (1.5):

(a) $\{\psi^1, \dots, \psi^d\}$ *and* $\{\tilde{\psi}^1, \dots, \tilde{\psi}^d\}$ *generate a pair of dual d-wavelet frames in* $L_2(\mathbf{R})$.
(b) $\psi^1, \{\psi^2, \dots, \psi^d\}, \tilde{\psi}^1$ *and* $\{\tilde{\psi}^2, \dots, \tilde{\psi}^d\}$ *have vanishing moments of orders n, $n + 2N$, $m + 2N$, and m, respectively.*

Proof. Let $c^1, \dots, c^d$ be the 2π-periodic trigonometric polynomials obtained in Proposition 2.4 with $s_0 = 0$. By Proposition 2.4, $h(0) = \det C(0) \neq 0$. Take $g(\xi) = 1$. The rest of the claim can be verified similarly as in the proof of Theorem 3.1 with the modification: replace the factor $(1 - e^{i\xi})^{n+m}$ in (3.10) by $(1 - e^{i\xi})^{n+m+2N}$. ■

When φ and $\tilde{\varphi}$ are real-valued and symmetric d-refinable functions, in this section, we didn't discuss whether one can obtain a pair of real-valued and symmetric dual d-wavelet frames from φ and $\tilde{\varphi}$. In the next section, we shall address such an issue in detail.

4. Real-Valued and Symmetric Dual Wavelet Frames

Given two real-valued and symmetric d-refinable functions in $L_2(\mathbf{R})$, it is of interest to construct from two such d-refinable functions pairs of dual d-wavelet frames which are also real-valued and symmetric. In this section, we shall discuss in detail how to obtain pairs of real-valued and symmetric dual wavelet frames from two real-valued and symmetric refinable functions.

Proposition 4.1. *Let $a, a^1, \dots, a^r, b, b^1, \dots, b^r$ be finitely supported sequences on $\mathbf{Z}$ and let Θ be a 2π-periodic trigonometric polynomial such that* (2.2) *and* (2.5) *are satisfied. Suppose that a, b and $a^1, \dots, a^r$ are sequences of real numbers. Define new sequences $\tilde{b}^1, \dots, \tilde{b}^r$ and a new 2π-periodic trigonometric polynomial $\tilde{\Theta}$ as follows*:

$$\tilde{\Theta}(\xi) := [\Theta(\xi) + \overline{\Theta(-\xi)}]/2$$

and

$$\widehat{\tilde{b}^\ell}(\xi) := [\widehat{b^\ell}(\xi) + \overline{\widehat{b^\ell}(-\xi)}]/2, \qquad \ell = 1, \dots, r.$$

Then $\tilde{\Theta}$ *is a* 2π*-periodic trigonometric polynomial with real coefficients and all* $\tilde{b}^1, \dots, \tilde{b}^r$ *are sequences of real numbers satisfying*

$$\tilde{\Theta}(0) = 1, \qquad \widehat{a^\ell}(0) = \widehat{\tilde{b}^\ell}(0) = 0, \qquad \ell = 1, \dots, r, \tag{4.1}$$

and

$$\overline{\hat{a}(\xi + 2\pi j/d)}\hat{b}(\xi)\tilde{\Theta}(d\xi) + \sum_{\ell=1}^{r} \overline{\widehat{a^\ell}(\xi + 2\pi j/d)}\,\widehat{\tilde{b}^\ell}(\xi) = \delta_j \tilde{\Theta}(\xi), \qquad j = 0, \dots, d-1. \tag{4.2}$$

Proof. Note that a is a sequence of real numbers if and only if $\overline{\hat{a}(-\xi)} = \hat{a}(\xi)$. Since $a, a^1, \dots, a^r$ are sequences of real numbers, taking the complex conjugate on both sides of (2.5) and replacing ξ by $-\xi$, we deduce from (2.5) that

$$\overline{\hat{a}(\xi - 2\pi j/d)}\,\overline{\hat{b}(-\xi)}\,\overline{\Theta(-d\xi)} + \sum_{\ell=1}^{r} \overline{\widehat{a^\ell}(\xi - 2\pi j/d)}\,\overline{\widehat{b^\ell}(-\xi)} = \delta_j \overline{\Theta(-\xi)}, \qquad j = 0, \dots, d-1.$$

Since b is a sequence of real numbers, we have $\overline{\hat{b}(-\xi)} = \hat{b}(\xi)$. Therefore,

$$\overline{\hat{a}(\xi + 2\pi j/d)}\hat{b}(\xi)\overline{\Theta(-d\xi)} + \sum_{\ell=1}^{r} \overline{\widehat{a^\ell}(\xi + 2\pi j/d)}\,\overline{\widehat{b^\ell}(-\xi)} = \delta_j \overline{\Theta(-\xi)}, \qquad j = 0, \dots, d-1.$$

Equation (4.2) can be easily verified by adding the above identity to (2.5). ■

When φ and $\tilde{\varphi}$ are real-valued refinable functions in $L_2(\mathbf{R})$, by Theorem 3.2 and Proposition 4.1, we can always obtain pairs of real-valued dual wavelet frames.

Proposition 4.2. *Let* $a, a^1, \dots, a^r, b, b^1, \dots, b^r$ *be finitely supported sequences on* $\mathbf{Z}$ *such that*

$$a_{s-k} = \overline{a_k} \quad \text{and} \quad b_{\tilde{s}-k} = \overline{b_k} \qquad \forall\, k \in \mathbf{Z}, \tag{4.3}$$

for some integers s *and* $\tilde{s}$ *such that* $c = (s - \tilde{s})/(d-1)$ *is an integer, and*

$$a^\ell_{s_\ell - k} = \varepsilon_\ell \overline{a^\ell_k} \qquad \forall\, k \in \mathbf{Z} \quad \text{and} \quad \ell = 1, \dots, r, \tag{4.4}$$

for some $\varepsilon_\ell \in \{-1, 1\}$ *and some integers* s_ℓ, $\ell = 1, \dots, r$, *such that* $(s - s_\ell)/d$ *is an integer for all* $\ell = 1, \dots, r$. *Suppose that* (2.2) *and* (2.5) *are satisfied with a* 2π*-periodic trigonometric polynomial* Θ. *Define new sequences* $\tilde{b}^1, \dots, \tilde{b}^r$ *and a new* 2π*-periodic trigonometric polynomial* $\tilde{\Theta}$ *as follows*:

$$\tilde{\Theta}(\xi) := [\Theta(\xi) + \overline{\Theta(\xi)} e^{-ic\xi}]/2$$

and

$$\widehat{\tilde{b}^\ell}(\xi) := [\widehat{b^\ell}(\xi) + \varepsilon_\ell \overline{\widehat{b^\ell}(\xi)} e^{-i(s_\ell + c)\xi}]/2, \qquad \ell = 1, \dots, r.$$

Then both (4.1) *and* (4.2) *are satisfied. Moreover,* $\Theta(\xi) = e^{-ic\xi}\overline{\Theta(\xi)}$ *and*

$$\tilde{b}^\ell_{s_\ell+c-k} = \varepsilon_\ell \overline{\tilde{b}^\ell_k} \qquad \forall\, k \in \mathbf{Z} \quad \textit{and} \quad \ell = 1, \ldots, r. \tag{4.5}$$

Proof. Note that $a^\ell_{s_\ell-k} = \varepsilon_\ell \overline{a^\ell_k}$ for all $k \in \mathbf{Z}$ if and only if $\widehat{a^\ell}(\xi) = \varepsilon_\ell e^{-is_\ell\xi}\overline{\widehat{a^\ell}(\xi)}$. Take the complex conjugate on (2.5), we have

$$\hat{a}(\xi + 2\pi j/d)\overline{\hat{b}(\xi)}\,\overline{\Theta(d\xi)} + \sum_{\ell=1}^{r} \widehat{a^\ell}(\xi + 2\pi j/d)\overline{\widehat{b^\ell}(\xi)} = \delta_j \overline{\Theta(\xi)}, \qquad j = 0, \ldots, d-1.$$

Note that $\hat{a}(\xi) = e^{-is\xi}\overline{\hat{a}(\xi)}$ and $\hat{b}(\xi) = e^{-i\tilde{s}\xi}\overline{\hat{b}(\xi)}$. Since $\widehat{a^\ell}(\xi) = \varepsilon_\ell e^{-is_\ell\xi}\overline{\widehat{a^\ell}(\xi)}$ for $\ell = 1, \ldots, r$, the above equation becomes

$$\begin{aligned} &e^{-is(\xi+2\pi j/d)}\overline{\hat{a}(\xi + 2\pi j/d)}e^{i\tilde{s}\xi}\hat{b}(\xi)\overline{\Theta(d\xi)} \\ &\qquad + \sum_{\ell=1}^{r} \varepsilon_\ell e^{-is_\ell(\xi+2\pi j/d)}\overline{\hat{a}^\ell(\xi + 2\pi j/d)}\,\overline{\widehat{b^\ell}(\xi)} = \delta_j \overline{\Theta(\xi)}, \end{aligned} \tag{4.6}$$

for $j = 0, \ldots, d-1$. By assumption, $s - \tilde{s} = (d-1)c$ and $s - s_\ell \in d\mathbf{Z}$ for all $\ell = 1, \ldots, r$. Multiplying the factor $e^{i(s2\pi j/d - c\xi)}$ with both sides of equation (4.6), we have that for $j = 0, \ldots, d-1$,

$$\begin{aligned} \delta_j e^{-ic\xi}\overline{\Theta(\xi)} &= \delta_j e^{i(s2\pi j/d - c\xi)}\overline{\Theta(\xi)} \\ &= \overline{\hat{a}(\xi + 2\pi j/d)}\hat{b}(\xi)e^{i(\tilde{s}-s-c)\xi}\overline{\Theta(d\xi)} \\ &\quad + \sum_{\ell=1}^{r} \varepsilon_\ell e^{i(s-s_\ell)2\pi j/d}\overline{\widehat{a^\ell}(\xi + 2\pi j/d)}e^{-i(s_\ell+c)\xi}\overline{\widehat{b^\ell}(\xi)} \\ &= \overline{\hat{a}(\xi + 2\pi j/d)}\hat{b}(\xi)e^{-idc\xi}\overline{\Theta(d\xi)} + \sum_{\ell=1}^{r} \overline{\widehat{a^\ell}(\xi + 2\pi j/d)}\varepsilon_\ell e^{-i(s_\ell+c)\xi}\overline{\hat{b}^\ell(\xi)}. \end{aligned}$$

Equation (4.2) can be verified by adding the above identity to (2.5). All other claims can be easily checked by computation. ■

Let φ and $\tilde{\varphi}$ be two d-refinable functions with finitely supported masks a and b, respectively. Then (4.3) implies that $\overline{\varphi} = \varphi(s/(d-1) - \cdot)$ and $\overline{\tilde{\varphi}} = \tilde{\varphi}(\tilde{s}/(d-1) - \cdot)$. Define $\widehat{\psi^\ell}(d\xi) := \widehat{a^\ell}(\xi)\hat{\varphi}(\xi)$ and $\widehat{\tilde{\psi}^\ell}(d\xi) := \widehat{\tilde{b}^\ell}(\xi)\hat{\tilde{\varphi}}(\xi)$ for $\ell = 1, \ldots, r$. Then (4.4) and (4.5) in Proposition 4.2 imply that

$$\overline{\psi^\ell} \;=\; \varepsilon_\ell \psi^\ell\left(\frac{(d-1)s_\ell + s}{d(d-1)} - \cdot\right)$$

and

$$\overline{\tilde{\psi}^\ell} \;=\; \varepsilon_\ell \tilde{\psi}^\ell\left(\frac{(d-1)s_\ell + s}{d(d-1)} - \cdot\right), \qquad \ell = 1, \ldots, r.$$

Now we have the following result on constructing pairs of real-valued and symmetric dual wavelet frames from two real-valued and symmetric refinable functions:

Theorem 4.3. *Let φ and $\tilde{\varphi}$ be two real-valued d-refinable functions in $L_2(\mathbf{R})$ with the dilation factor d and finitely supported masks a and b, respectively. Let N be a nonnegative integer and let J be an integer. Suppose that a and b satisfy the sum rules of orders m and n with respect to the lattice $d\mathbf{Z}$ for some positive integers m and n, respectively. Further, assume that*

$$a_{s-k} = \overline{a_k} \qquad \text{and} \qquad b_{\tilde{s}-k} = \overline{b_k} \qquad \forall\, k \in \mathbf{Z}, \tag{4.7}$$

for some integers s and $\tilde{s}$ such that $(s-\tilde{s})/(d-1)$ is an integer. Then one can construct finitely supported sequences $a^1, \dots, a^d, b^1, \dots, b^d$ of real numbers and a 2π-periodic trigonometric polynomial Θ satisfying (2.2) *and* (2.3) *such that the wavelet functions $\psi^1, \dots, \psi^d, \tilde{\psi}^1, \dots, \tilde{\psi}^d$, which are defined in* (1.5), *satisfy*:

(a) *$\{\psi^1, \dots, \psi^d\}$ and $\{\tilde{\psi}^1, \dots, \tilde{\psi}^d\}$ generate a pair of dual d-wavelet frames in $L_2(\mathbf{R})$.*
(b) *$\psi^1, \{\psi^2, \dots, \psi^d\}, \tilde{\psi}^1$ and $\{\tilde{\psi}^2, \dots, \tilde{\psi}^d\}$ have vanishing moments of orders $n, n+2N, m+2N$, and m, respectively.*
(c) *All the functions $\psi^1, \dots, \psi^d, \tilde{\psi}^1, \dots, \tilde{\psi}^d$ are real-valued and are either symmetric or antisymmetric about the point $J/2 + s/(2d-2)$. More precisely, for all $\ell = 1, \dots, d$,*

$$\psi^\ell(x) \;=\; \varepsilon_\ell \psi^\ell\left(J + \frac{s}{d-1} - x\right)$$

and

$$\tilde{\psi}^\ell(x) \;=\; \varepsilon_\ell \tilde{\psi}^\ell\left(J + \frac{s}{d-1} - x\right) \qquad \forall\, x \in \mathbf{R},$$

where $\varepsilon_\ell = (-1)^n$, $\ell = 1, \dots, N_{d,s-n}$, and $\varepsilon_\ell = (-1)^{n+1}$, $\ell = N_{d,s-n}+1, \dots, d$, with the integer $N_{d,s-n}$ being defined in (2.10).

Proof. Let $c^1, \dots, c^d$ be 2π-periodic trigonometric polynomials satisfying all the conditions in Proposition 2.4 with $s_0 = s + dJ - n$. Let $g(\xi) \equiv 1$ and define the sequences $a^1, \dots, a^d$ as in (3.2). It is evident that $a^\ell_{s+dJ-k} = \varepsilon_\ell a^\ell_k$ for all $k \in \mathbf{Z}$ and $\ell = 1, \dots, d$. Let $h(\xi) := \det C(\xi)$, where the matrix $C(\xi)$ is defined in (2.9). Observe that $h(\xi + 2\pi/d) = (-1)^{d-1} h(\xi)$. In the proof of Theorem 3.2, we can take $\theta_1(\xi) = h(\xi/d)$ when d is odd and $\theta_1(\xi) = e^{-i\xi/2} h(\xi/d)$ when d is even. The claim follows directly from Theorem 3.2, Corollary 3.6, and Propositions 4.1 and 4.2. ■

When a and b are symmetric sequences satisfying (4.7), in order to have a symmetric 2π-periodic trigonometric polynomial Θ, such that $\Theta(\xi) - \Theta(d\xi)\overline{\hat{a}(\xi)}\hat{b}(\xi)$ is also either symmetric or antisymmetric, it is not difficult to see that one naturally requires $(s-\tilde{s})/(d-1)$ to be an integer. The restriction that $(s-\tilde{s})/(d-1)$ should be an integer automatically disappears when $d = 2$.

Finally, we mention that the earlier version of this paper inspired the authors of [9] to generalize the results in this paper to the case of d-refinable function vectors. For details on constructing pairs of dual d-wavelet frames from d-refinable function vectors, see [9].

5. Examples of Dual Wavelet Frames

In this section, we shall give several examples to illustrate the main results in this paper on the construction of pairs of dual wavelet frames from pairs of refinable functions.

The following examples follow easily from the results in Sections 3 and 4 and are produced by the program which consists of a collection of MAPLE routines and is available from `http://www.ualberta.ca/~bhan`.

Example 5.1. Let the dilation factor $d = 2$. Let $\varphi = \tilde{\varphi} = B_2$ be the B-spline function of order 2 given in (3.11). Taking $N = 1$ and $J = 0$, by Theorem 4.3, we have

$$\begin{aligned}
\Theta &= \frac{1}{960}[71(z^{-4}+z^4) - 252(z^{-3}+z^3) + 120(z^{-2}+z^2) + 492(z^{-1}+z) + 98],\\
\widehat{a^1} &= -(1-z)^2, \qquad \widehat{a^2} = -z^{-1}(1-z)^4,\\
\widehat{b^1} &= \frac{(1-z)^4}{15360}[71(z^{-8}+z^6) + 426(z^{-7}+z^5) + 2233(z^{-6}+z^4) + 8428(z^{-5}+z^3)\\
&\quad + 21191(z^{-4}+z^2) + 39422(z^{-3}+z) + 57569(z^{-2}+1)\\
&\quad + 65336\, z^{-1}],\\
\widehat{b^2} &= \frac{(1-z)^2}{3840}[71(z^{-6}+z^6) + 284(z^{-5}+z^5) + 458(z^{-4}+z^4) + 412(z^{-3}+z^3)\\
&\quad + 85(z^{-2}+z^2) - 726(z^{-1}+z) - 1228],
\end{aligned}$$

where $z = e^{-i\xi}$. Then $\{\psi^1, \psi^2\}$ and $\{\tilde{\psi}^1, \tilde{\psi}^2\}$ generate a pair of dual 2-wavelet frames. The functions $\psi^1, \psi^2, \tilde{\psi}^1, \tilde{\psi}^2$ have vanishing moments of orders 2, 4, 4, 2, respectively, and they are either symmetric or antisymmetric about the point 1. See Figure 1 for their graphs.

Example 5.2. Let the dilation factor $d = 2$ and $\varphi = \tilde{\varphi} = B_4$. By Theorem 3.1, we have

$$\begin{aligned}
\Theta &= \frac{1}{15120}[-311(z^{-3}+z^3) + 3168(z^{-2}+z^2) - 14913(z^{-1}+z) + 39232],\\
\widehat{a^1} &= (1-z)^4, \qquad \widehat{a^2} := z(1-z)^4,\\
\widehat{b^1} &= \frac{(1-z)^4}{3870720}[311(z^{-6}+z^6) + 2488(z^{-5}+z^5) + 16736(z^{-4}+z^4)\\
&\quad + 81640(z^{-3}+z^3) + 250049(z^{-2}+z^2) + 494944(z^{-1}+z)\\
&\quad + 654400],\\
\widehat{b^2} &= \frac{(1-z)^4}{483840}[311(z^{-4}+z^6) + 2488(z^{-3}+z^5) + 10205(z^{-2}+z^4)\\
&\quad + 29392(z^{-1}+z^3) + 61868(1+z^2) + 86704\, z],
\end{aligned}$$

where $z = e^{-i\xi}$. Then $\{\psi^1, \psi^2\}$ and $\{\tilde{\psi}^1, \tilde{\psi}^2\}$ generate a pair of dual 2-wavelet frames. Both $\{\psi^1, \psi^2\}$ and $\{\tilde{\psi}^1, \tilde{\psi}^2\}$ have vanishing moments of order 4. See Figure 2 for their graphs.

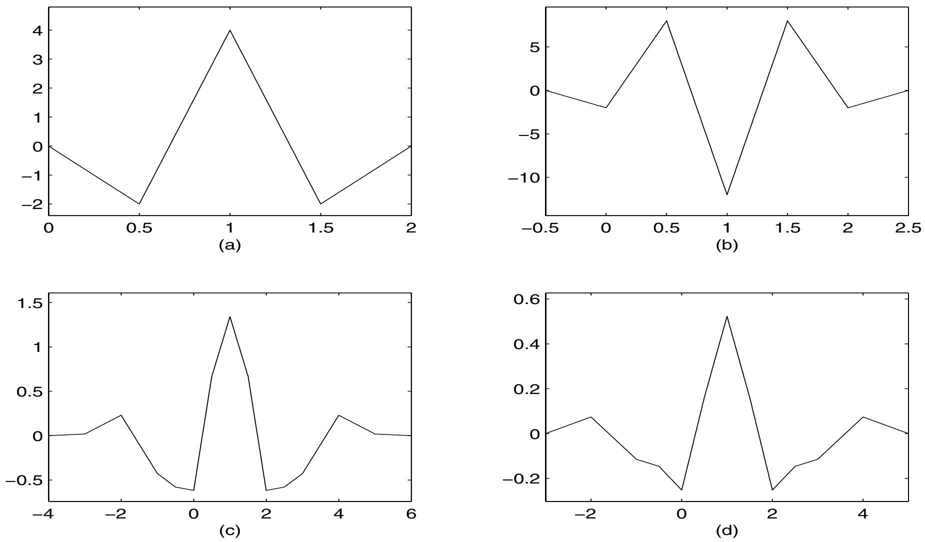

Fig. 1. Parts (a) and (b) are the graphs of the wavelet functions ψ^1 and ψ^2 in Example 5.1. Parts (c) and (d) are the graphs of their dual wavelet functions $\tilde{\psi}^1$ and $\tilde{\psi}^2$. The functions $\psi^1, \psi^2, \tilde{\psi}^1, \tilde{\psi}^2$ have vanishing moments of orders 2, 4, 4, 2, respectively, and they are either symmetric or antisymmetric about the point 1. $\{\psi^1, \psi^2\}$ and $\{\tilde{\psi}^1, \tilde{\psi}^2\}$ generate a pair of dual 2-wavelet frames.

When the dilation factor $d = 2$ and $\varphi = \tilde{\varphi} = B_m$, pairs of dual 2-wavelet frames derived from φ and $\tilde{\varphi}$ have also been constructed in [6]. It turns out that up to some integer shifts the construction in [6] for B-spline functions coincides with the construction in Theorem 3.1 for B-spline functions with the particular choice Θ as given by

$$\begin{aligned} \Theta(\xi) &= P_m(\sin^2 \xi/2) \qquad \text{with} \\ \left(1 + \sum_{j=1}^{\infty} \frac{(2j-1)!!}{(2j)!!\,(2j+1)} x^j\right)^{2m} &= P_m(x) + O(|x|^{2m}), \qquad x \to 0. \end{aligned}$$

So Example 5.2 was also obtained in [6].

Example 5.3. Let the dilation factor $d = 2$. Let $\varphi = B_4$ and $\tilde{\varphi} = B_2$. By Theorem 3.1, we have

$$\begin{aligned} \Theta &= \frac{1}{240}[13(z^{-1} + z^3) - 112(1 + z^2) + 438z], \\ \widehat{a^1} &= -(1 - z)^2, \qquad \widehat{a^2} = -z(1 - z)^2, \\ \widehat{b^1} &= \frac{(1 - z)^4}{15360}[13(z^{-4} + z^4) + 78(z^{-3} + z^3) + 356(z^{-2} + z^2) \\ &\qquad + 1226(z^{-1} + z) + 2334], \\ \widehat{b^2} &= \frac{(1 - z)^4}{7680}[39(z^{-2} + z^4) + 234(z^{-1} + z^3) + 613(1 + z^2) + 948z], \end{aligned}$$

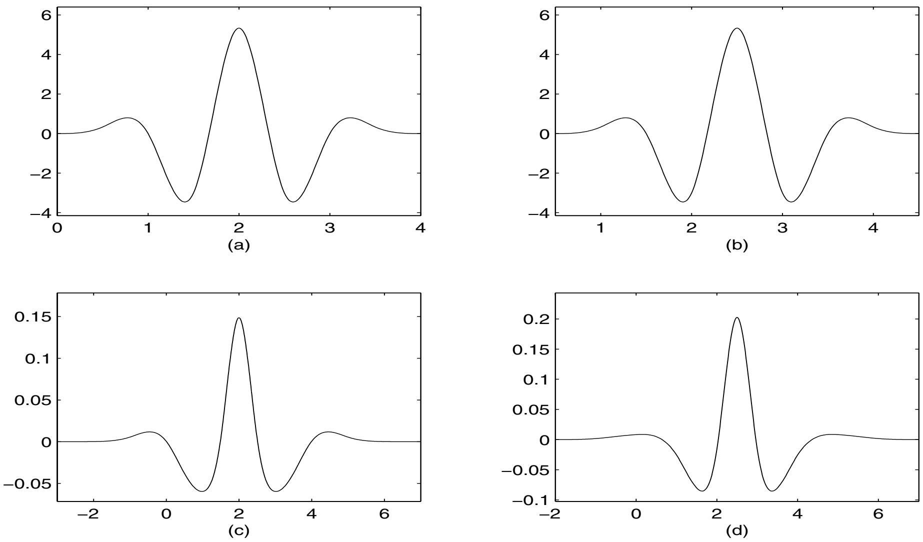

Fig. 2. Parts (a) and (b) are the graphs of the wavelet functions ψ^1 and ψ^2 in Example 5.2. Parts (c) and (d) are the graphs of their dual wavelet functions $\tilde{\psi}^1$ and $\tilde{\psi}^2$. Both $\{\psi^1, \psi^2\}$ and $\{\tilde{\psi}^1, \tilde{\psi}^2\}$ have vanishing moments of order 4 and generate a pair of dual 2-wavelet frames.

where $z = e^{-i\xi}$. Then $\{\psi^1, \psi^2\}$ and $\{\tilde{\psi}^1, \tilde{\psi}^2\}$ generate a pair of dual 2-wavelet frames and have vanishing moments of orders 2 and 4, respectively. See Figure 3 for their graphs.

Example 5.4. Let the dilation factor $d = 3$ and $\varphi = \tilde{\varphi} = B_3$. By Theorem 3.1 and Proposition 4.1, we have

$$
\begin{aligned}
\Theta &= \frac{1}{240}[13(z^{-2}+z^2) - 112(z^{-1}+z) + 438],\\
\widehat{a^1} &= (1-z)^3, \qquad \widehat{a^2} = z(1-z)^3, \qquad \widehat{a^3} = z^2(1-z)^3,\\
\widehat{b^1} &= \frac{(1-z)^3}{174960}[13(z^{-9}+z^9) + 78(z^{-8}+z^8) + 273(z^{-7}+z^7) + 1266(z^{-6}+z^6)\\
&\qquad + 4866(z^{-5}+z^5) + 14574(z^{-4}+z^4) + 32805(z^{-3}+z^3)\\
&\qquad + 58110(z^{-2}+z^2) + 82398(z^{-1}+z) + 94804],\\
\widehat{b^2} &= \frac{(1-z)^3}{58320}[91z^{-6} + 546z^{-5} + 1911z^{-4} + 4858z^{-3} + 10038z^{-2} + 17934z^{-1}\\
&\qquad + 27466 + 34596z + 29880z^2 + 19370z^3 + 10038z^4 + 4272z^5\\
&\qquad + 1622z^6 + 546z^7 + 156z^8 + 26z^9],\\
\widehat{b^3} &= -z^6\widehat{b^2}(1/z),
\end{aligned}
$$

where $z = e^{-i\xi}$. Then $\{\psi^1, \psi^2, \psi^3\}$ and $\{\tilde{\psi}^1, \tilde{\psi}^2, \tilde{\psi}^3\}$ generate a pair of dual 3-wavelet frames. Both $\{\psi^1, \psi^2, \psi^3\}$ and $\{\tilde{\psi}^1, \tilde{\psi}^2, \tilde{\psi}^3\}$ have vanishing moments of order 3. See Figure 4 for their graphs.

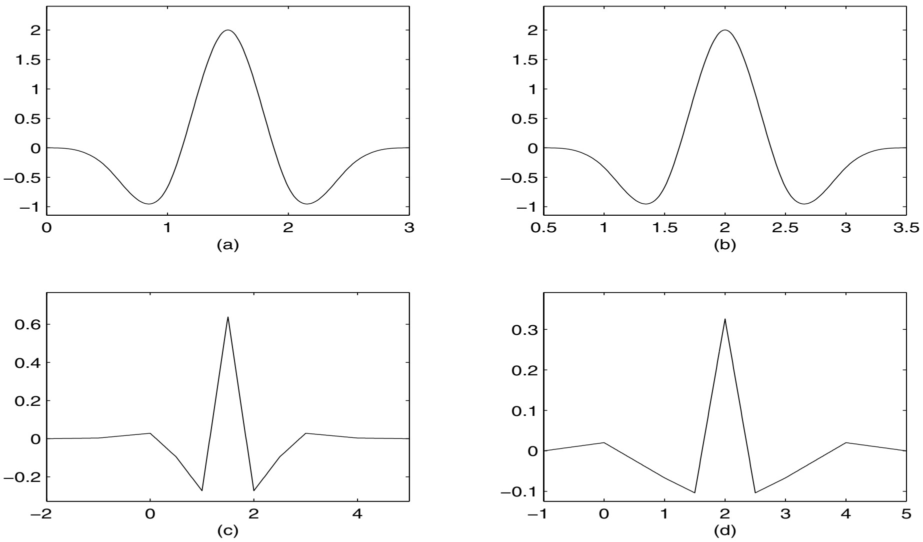

Fig. 3. Parts (a) and (b) are the graphs of the wavelet functions ψ^1 and ψ^2 in Example 5.3. Parts (c) and (d) are the graphs of their dual wavelet functions $\tilde{\psi}^1$ and $\tilde{\psi}^2$. $\{\psi^1, \psi^2\}$ and $\{\tilde{\psi}^1, \tilde{\psi}^2\}$ have vanishing moments of orders 2 and 4, respectively, and they generate a pair of dual 2-wavelet frames.

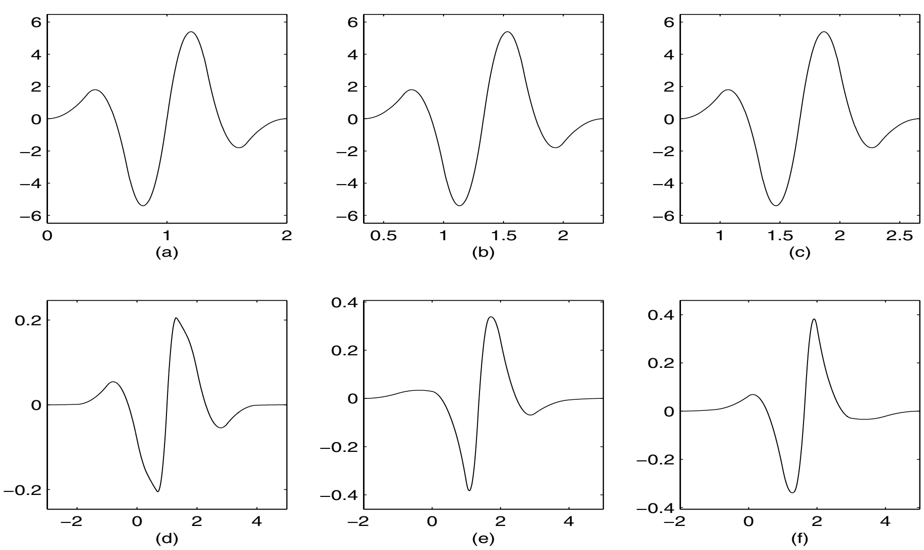

Fig. 4. Parts (a), (b), and (c) are the graphs of the wavelet functions ψ^1, ψ^2, ψ^3 in Example 5.4. Parts (d), (e), and (f) are the graphs of their dual wavelet functions $\tilde{\psi}^1$, $\tilde{\psi}^2$, $\tilde{\psi}^3$. $\{\psi^1, \psi^2, \psi^3\}$ and $\{\tilde{\psi}^1, \tilde{\psi}^2, \tilde{\psi}^3\}$ have vanishing moments of order 3 and generate a pair of dual 3-wavelet frames.

Finally, let us present an example using Theorem 3.2.

Example 5.5. Let the dilation factor $d = 2$ and $\varphi = \tilde{\varphi} = B_3$. Take $n = 2$, $g = -1 - e^{-i\xi}$, $c^1 = 1$, and $c^2 = e^{-i\xi} - e^{i\xi}$ in (3.2). Therefore, we have $h = \det C(\xi) = 2(e^{i\xi} - e^{-i\xi})$ with the matrix $C(\xi)$ being defined in (2.9). Clearly, $g(\pi) = h(0) = 0$. By Theorem 3.2, we set

$$\Theta = z^{-2}(1+z)^2(-1+4z-z^2)/8, \qquad \widehat{a^1} = -(z-1)^2(1+z),$$
$$\widehat{a^2} := z^{-1}(1-z)^3(1+z)^2,$$

where $z := e^{-i\xi}$. By the definition of f^1 and f^2 in (3.4), we have

$$f^1 = \frac{1}{1024}(1-z)[(z^{-3}+z^5)+8(z^{-2}+z^4)+30(z^{-1}+z^3)+72(1+z^2)+122z],$$
$$f^2 = \frac{1}{1024}z^{-3}(1-z)(1+z^2)^2(1-4z^2+z^4).$$

Consequently, it follows from (3.5) that

$$\widehat{b^1} = -\frac{1}{512}(1-z)^2(1+z)[(z^{-4}+z^4)+4(z^{-3}+z^3)+14(z^{-2}+z^2) + 36(z^{-1}+z)+58],$$
$$\widehat{b^2} = \frac{1}{128}(z-1)[(z^{-2}+z^4)+4(z^{-1}+z^3)+9(1+z^2)+16z].$$

Then $\{\psi^1, \psi^2\}$ and $\{\tilde{\psi}^1, \tilde{\psi}^2\}$ generate a pair of dual 2-wavelet frames. See Figure 5 for their graphs.

6. Appendix

Proof of Theorem 3.2. We only need to prove the sufficiency part for the general case. Note that $h(\xi + 2\pi/d) = (-1)^{d-1}h(\xi)$. By the definition of $Z(p, \xi_0)$ in (3.7), rewrite $\overline{g(\xi)} = (1 - e^{i\xi})^{Z(g,0)} g_1(\xi)$ and $\overline{h(\xi)} = (1 - e^{i\xi})^{Z(h,0)} h_1(\xi)$ for some 2π-periodic trigonometric polynomials g_1 and h_1 such that $g_1(0) \neq 0$ and $h_1(0) \neq 0$. Define 2π-periodic trigonometric polynomials F_k and G_k by

$$\text{(6.1)} \qquad \frac{F_0(\xi)}{G_0(\xi)} := \frac{\overline{\hat{a}(\xi)}\hat{b}(\xi)}{g_1(\xi)h_1(\xi)} \quad \text{and}$$
$$\frac{F_k(\xi)}{G_k(\xi)} := \frac{\overline{\hat{a}(\xi)}\hat{b}(\xi - 2\pi k/d)}{(1-e^{i\xi})^n\overline{g(\xi)}\,\overline{h(\xi)}}, \qquad k = 1, \ldots, d-1,$$

where F_k and G_k have no common zeros on the set $(2\pi/d)\mathbf{Z}$ for $k = 0, \ldots, d-1$. Now we claim that

$$\text{(6.2)} \qquad G_k(2\pi j/d) \neq 0 \qquad \forall\, j = 0, \ldots, d-1 \quad \text{and} \quad k = 0, \ldots, d-1.$$

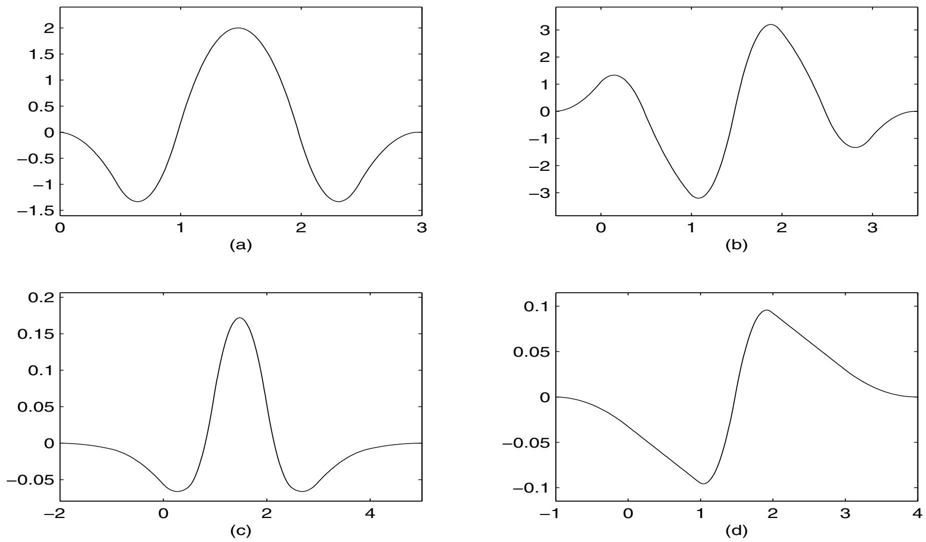

Fig. 5. Parts (a) and (b) are the graphs of the wavelet functions ψ^1 and ψ^2 in Example 5.5. Parts (c) and (d) are the graphs of their dual wavelet functions $\tilde{\psi}^1$ and $\tilde{\psi}^2$. $\{\psi^1, \psi^2\}$ and $\{\tilde{\psi}^1, \tilde{\psi}^2\}$ have vanishing moments of orders 2 and 1, and they generate a pair of dual 2-wavelet frames.

Since F_k and G_k have no common zeros on $(2\pi/d)\mathbf{Z}$, in order to show (6.2), it suffices to show that

$$(6.3)\qquad Z(F_k, 2\pi j/d) - Z(G_k, 2\pi j/d) \geq 0 \qquad \forall\ j = 0, \ldots, d-1 \quad \text{and} \quad k = 0, \ldots, d-1,$$

which implies that $Z(G_k, 2\pi j/d) = 0$ and therefore, (6.2) holds.

Note that (3.8) implies that $Z(\hat{b}, -2\pi k/d) \geq n + Z(g, 0) + Z(h, 0)$ for all $k = 1, \ldots, d-1$. By our assumptions in (3.8), it follows from (6.1) that for every $k = 1, \ldots, d-1$,

$$\begin{aligned} Z(F_0, 0) - Z(G_0, 0) &= 0 \qquad \text{and} \\ Z(F_k, 0) - Z(G_k, 0) &= Z(\hat{b}, -2\pi k/d) - n - Z(g, 0) - Z(h, 0) \geq 0. \end{aligned}$$

Similarly, for every $j = 1, \ldots, d-1$, by (3.9), we have

$$\begin{aligned} Z(F_0, 2\pi j/d) - Z(G_0, 2\pi j/d) &= Z(\hat{b}, 2\pi j/d) + Z(\hat{a}, 2\pi j/d) - Z(g_1, 2\pi j/d) \\ &\quad - Z(h_1, 2\pi j/d) \\ &= Z(\hat{b}, 2\pi j/d) + Z(\hat{a}, 2\pi j/d) - Z(g, 2\pi j/d) \\ &\quad - Z(h, 0) \geq 0 \end{aligned}$$

and for every $k = 1, \ldots, d-1$,

$$\begin{aligned} &Z(F_k, 2\pi j/d) - Z(G_k, 2\pi j/d) \\ &\qquad = Z(\hat{b}, 2\pi(j-k)/d) + Z(\hat{a}, 2\pi j/d) - Z(g, 2\pi j/d) - Z(h, 0) \geq 0. \end{aligned}$$

So, (6.3) holds and, therefore, (6.2) must be true. Define

$$\theta_1(\xi) := \prod_{k=0}^{d-1}\prod_{j=0}^{d-1} G_k(\xi/d + 2\pi j/d), \qquad \xi \in \mathbf{R}. \tag{6.4}$$

By (6.2), we have $\theta_1(0) \neq 0$ and θ_1 is a 2π-periodic trigonometric polynomial since $\theta_1(\xi+2\pi) = \theta_1(\xi)$. By Lemma 2.3, there exists a 2π-periodic trigonometric polynomial θ_2 such that $\theta_2(0) = 1$ and

$$\begin{aligned}(1 - e^{i\xi})^{n+m+Z(g,0)+Z(h,0)} \mid \theta_2(\xi)[\theta_1(\xi) g_1(\xi) h_1(\xi)] \\ - \theta_2(d\xi)[\theta_1(d\xi) g_1(d\xi) h_1(d\xi)\overline{\hat{a}(\xi)}\hat{b}(\xi)].\end{aligned} \tag{6.5}$$

Since $\theta_1(0)$, $g_1(0)$, and $h_1(0)$ are nonzero numbers, we now define

$$\Theta(\xi) := \frac{\theta_2(\xi)\theta_1(\xi) g_1(\xi) h_1(\xi)}{\theta_1(0) g_1(0) h_1(0)}. \tag{6.6}$$

Clearly, $\Theta(0) = 1$ and by (6.5), it is easy to see that

$$(1 - e^{i\xi})^{n+m+Z(g,0)+Z(h,0)} \mid [\Theta(\xi) - \Theta(d\xi)\overline{a(\xi)} b(\xi)]. \tag{6.7}$$

In the following, we show that with the choice of Θ in (6.6), all f^ℓ must be 2π-periodic trigonometric polynomials satisfying $(1 - e^{i\xi})^m \mid f^\ell(\xi)$ for all $\ell = 1, \ldots, d$. By computation, we have

$$\begin{aligned}
f^1(\xi) &= \frac{\Theta(\xi) - \Theta(d\xi)\overline{\hat{a}(\xi)}\hat{b}(\xi)}{(1 - e^{i\xi})^{n+Z(g,0)+Z(h,0)} g_1(\xi) h_1(\xi)} \\
&= \frac{(1 - e^{i\xi})^m}{(1 - e^{i\xi})^{n+m+Z(g,0)+Z(h,0)}} \left[\frac{\Theta(\xi)}{g_1(\xi) h_1(\xi)} - \frac{\Theta(d\xi)\overline{\hat{a}(\xi)}\hat{b}(\xi)}{g_1(\xi) h_1(\xi)} \right] \\
&= \frac{(1 - e^{i\xi})^m}{(1 - e^{i\xi})^{n+m+Z(g,0)+Z(h,0)}} \left[\frac{\theta_1(\xi)\theta_2(\xi)}{\theta_1(0) g_1(0) h_1(0)} - \frac{\Theta(d\xi)}{G_0(\xi)} F_0(\xi) \right] \\
&= \frac{(1 - e^{i\xi})^m}{(1 - e^{i\xi})^{n+m+Z(g,0)+Z(h,0)}} \\
&\quad \times \left[\frac{\theta_1(\xi)\theta_2(\xi)}{\theta_1(0) g_1(0) h_1(0)} - \frac{\theta_1(d\xi)}{G_0(\xi)} \frac{g_1(d\xi) h_1(d\xi)\theta_2(d\xi)}{\theta_1(0) g_1(0) h_1(0)} F_0(\xi) \right].
\end{aligned}$$

By the definition of θ_1 in (6.4), we see that $\theta_1(d\xi)/G_0(\xi)$ is a 2π-periodic trigonometric polynomial. Consequently, it follows from (6.7) and the above identity that f^1 is a 2π-periodic trigonometric polynomial satisfying $(1 - e^{i\xi})^m \mid f^1(\xi)$.

For $j = 1, \ldots, d-1$, by computation and the fact $h(\xi + 2\pi/d) = (-1)^{d-1} h(\xi)$, we have

$$\begin{aligned}
f^{j+1}(\xi) &= -\frac{\Theta(d\xi)\overline{\hat{a}(\xi + 2\pi j/d)}\hat{b}(\xi)}{(1 - e^{i(\xi+2\pi j/d)})^n \overline{g(\xi + 2\pi j/d)}\,\overline{h(\xi)}} \\
&= (-1)^{(d-1)j+1} \frac{\Theta(d\xi)}{G_j(\xi + 2\pi j/d)} F_j(\xi + 2\pi j/d) \\
&= (-1)^{(d-1)j+1} \frac{\theta_1(d\xi)}{G_j(\xi + 2\pi j/d)} \frac{g_1(d\xi) h_1(d\xi)\theta_2(d\xi)}{\theta_1(0) g_1(0) h_1(0)} F_j(\xi + 2\pi j/d).
\end{aligned}$$

By the definition of θ_1 in (6.4), we see that $\theta_1(d\xi)/G_j(\xi + 2\pi j/d)$ is a 2π-periodic trigonometric polynomial. By (3.4) and (3.9), we have

$$Z(f^{j+1}, 0) = Z(\hat{a}, 2\pi j/d) - Z(g, 2\pi j/d) - Z(h, 0) \geq m, \qquad j = 1, \dots, d-1.$$

Consequently, f^j, $j = 2, \dots, d$, are 2π-periodic trigonometric polynomials satisfying $(1 - e^{i\xi})^m \mid f^j(\xi)$ for all $j = 2, \dots, d$. So, by Theorem 2.2, $\{\psi^1, \dots, \psi^d\}$ and $\{\tilde{\psi}^1, \dots, \tilde{\psi}^d\}$ generate a pair of dual d-wavelet frames in $L_2(\mathbf{R})$. Moreover, $\{\psi^1, \dots, \psi^d\}$ has vanishing moments of order n and $\{\tilde{\psi}^1, \dots, \tilde{\psi}^d\}$ has vanishing moments of order m. ■

Let $\hat{a}$, $\hat{b}$, g, and h be given. Using long division, we observe that the conditions in (a), (b), (c) of Theorem 3.2 are equivalent to a set of linear equations on the coefficients of the 2π-periodic trigonometric polynomial Θ. Therefore, one can obtain a desirable 2π-periodic trigonometric polynomial Θ with smallest degree by solving a set of linear equations; the existence of such desirable Θ is guaranteed by the above proof of Theorem 3.2.

Acknowledgments. We would like to thank the referees for their helpful comments and suggestions which improved the presentation of this paper. In particular, we thank one of the referees for suggesting the first part of the proof of Lemma 2.3 which shortens our original proof. The authors also thank Dr. Qun Mo at the University of Alberta for several discussions with the second author which led to the remarks after Corollary 3.4. The second author also thanks the Program in Applied and Computational Mathematics at Princeton University for their hospitality during his visit at the Program in Applied and Computational Mathematics in the year 1999–2000. Research of Ingrid Daubechies was partially supported by grants NSF (DMS-9872890, DMS-9706753) and AFOSR (F49620-98-1-0044). Research of Bin Han was partially supported by the Natural Sciences and Engineering Research Council of Canada (NSERC Canada) under Grant G121210654, and by Alberta Innovation and Science REE under Grant G227120136.

References

1. A. Ben-Artzi, A. Ron (1990): *On the integer translates of a compactly supported function*: *Dual bases and linear projectors*. SIAM J. Math. Anal., **21**:1550–1562.
2. C. K. Chui, W. He, J. Stöckler (2002): *Compactly supported tight and sibling frames with maximum vanishing moments*. Appl. Comput. Harmon. Anal., **13**:224–262.
3. A. Cohen, I. Daubechies (1992): *A stability criterion for biorthogonal wavelet bases and their related subband coding scheme*. Duke Math. J., **68**:313–335.
4. I. Daubechies (1990): *The wavelet transform*, *time-frequency localization and signal analysis*. IEEE Trans. Inform. Theory, **36**:961–1005.
5. I. Daubechies (1992): *Ten lectures on wavelets*. In: CBMS-NSF Regional Conference Series in Applied Mathematics, Vol. 61. Philadelphia, PA: SIAM.
6. I. Daubechies, B. Han, A. Ron, Z. W. Shen (2003): *Framelets*: *MRA-based constructions of wavelet frames*. Appl. Comput. Harmon. Anal., **14**:1–46.
7. B. Han (1997): *On dual wavelet tight frames*. Appl. Comput. Harmon. Anal., **4**:380–413.
8. B. Han (2003): *Compactly supported tight wavelet frames and orthonormal wavelets of exponential decay with a general dilation matrix*. J. Comput. Appl. Math., **155**:43–67.

9. B. HAN, Q. MO (2003): *Multiwavelet frames from refinable function vectors*. Adv. in Comput. Math., **18**:211–245.
10. A. RON (1990): *Factorization theorems for univariate splines on regular grids*. Israel J. Math., **70**:48–68.
11. A. RON, Z. W. SHEN (1997): *Affine systems in $L_2(\mathbf{R}^d)$ II: Dual systems*. J. Fourier Anal. Appl., **3**:617–637.
12. L. VILLEMOES (1993): *Sobolev regularity of wavelets and stability of iterated filter banks*. In: Progress in Wavelet Analysis and Applications (Y. Meyer, S. Roques, eds.), pp. 243–251.

I. Daubechies
Program in Applied and Computational Mathematics
Princeton University
Princeton, NJ 08544
USA
ingrid@math.princeton.edu
www: http://www.princeton.edu/∼icd

B. Han
Department of Mathematical and Statistical Sciences
University of Alberta
Edmonton, Alberta
Canada T6G 2G1
bhan@math.ualberta.ca
www: http://www.ualberta.ca/∼bhan