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Commutation for Irregular Subdivision

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Abstract. We present a generalization of the commutation formula to irregular subdivision schemes and wavelets. We show how, in the noninterpolating case, the divided differences need to be adapted to the subdivision scheme. As an example we include the construction of an entire family of biorthogonal compactly supported irregular knot B-spline wavelets starting from Lagrangian interpolation.

1. Introduction

This paper is a sequel to [7]. In that paper we used derived subdivision schemes and commutation formulas as a tool to compute the smoothness of the limit function for the 4-point scheme and for higher-order Lagrange interpolation schemes, generalized to irregular grids with irregular subdivision. The definition of derived subdivision schemes in [7] was geared to interpolating schemes only, and used heavily that such schemes produce polynomials up to a certain order. We recognized in [7] that for more general subdivision schemes a more careful analysis was necessary, but we did not elaborate on this. The present paper provides this more detailed analysis; in particular, we shall see that we have to define divided differences adapted to the subdivision scheme when we leave the interpolating framework and we shall discuss how this fits with a generalized commutation formula. We also show how to carry out inverse commutation in this framework, and how this analysis can be used to derive whole families of biorthogonal multiresolution analysis spaces. Finally, we also discuss how one can then define the wavelets associated to these schemes. At every step, it turns out that the geometry of the grid, and the sequences that converge to polynomials through subdivision, play an important role in many of the crucial definitions. We illustrate our constructions with the derived schemes of Lagrange interpolating subdivision and with B-spline schemes; as in the regular case, they turn out to be biorthogonal here as well.

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2. Commutation for Regular Subdivision

The commutation formalism in the regular setting allows one to go from one pair of biorthogonal multiresolution analyses and wavelets to an associated pair via differentiation and integration. Commutation for regular subdivision is described in [2], [9], [13], [12]. We give here a quick review of some of the main steps.

Let φ, $\widetilde{\varphi}$ be a dual pair of (compactly supported) scaling functions satisfying the refinement relations

$$\varphi(x) = 2\sum_k h_k \varphi(2x-k),$$
$$\widetilde{\varphi}(x) = 2\sum_k \widetilde{h}_k \widetilde{\varphi}(2x-k),$$

where we assume that the h_k, $\widetilde{h}_k$ are real. Biorthogonality of the $\varphi(x-l), \widetilde{\varphi}(x-l)$ implies that

(1) $$h(z)\widetilde{h}(1/z) + h(-z)\widetilde{h}(-1/z) = 1,$$

where $h(z) = \sum_k h_k z^k$, and $\widetilde{h}(z) = \sum_k \widetilde{h}_k z^k$. Typically, $h(z)$ and $\widetilde{h}(z)$ have a multiple zero at $z=-1$. Commutation now constructs a new pair $h^{[1]}$ and $\widetilde{h}^{[-1]}$, by means of

$$h^{[1]}(z) = 2h(z)(1+z^{-1})^{-1},$$
$$\widetilde{h}^{[-1]}(z) = \tfrac{1}{2}\widetilde{h}(z)(1+z).$$

It follows immediately that $h^{[1]}$, $\widetilde{h}^{[-1]}$ are again biorthogonal in the sense of 1. The corresponding scaling functions $\varphi^{[1]}, \widetilde{\varphi}^{[1]}$ are related to $\varphi, \widetilde{\varphi}$ by differentiation/integration, as shown by the following argument. We have

$$\hat{\varphi}(\xi) = \prod_{j=1}^{\infty} h(e^{-i2^{-j}\xi}) = \prod_{j=1}^{\infty} h^{[1]}(e^{-i2^{-j}\xi}) \prod_{j=1}^{\infty}\left(\frac{1+e^{i2^{-j}\xi}}{2}\right)$$
$$= \hat{\varphi}^{[1]}(\xi)\frac{-1+e^{i\xi}}{i\xi},$$

so that

(2) $$\frac{d}{dx}\varphi(x) = \varphi^{[1]}(x+1) - \varphi^{[1]}(x);$$

a similar derivation gives

(3) $$\frac{d}{dx}\widetilde{\varphi}^{[1]}(x) = \widetilde{\varphi}(x) - \widetilde{\varphi}(x-1).$$

Note that the primal side (2) uses forward differencing, while the dual side (3) uses backward differencing.

The wavelets ψ, $\widetilde{\psi}$ associated with $\varphi, \widetilde{\varphi}$ are defined by

$$\psi(x) = 2\sum_k g_k \varphi(2x-k), \qquad \widetilde{\psi}(x) = 2\sum_k \widetilde{g}_k \widetilde{\varphi}(2x-k),$$

where $g(z) = \widetilde{h}(-1/z)z$ and $\widetilde{g}(z) = h(-1/z)z$. These wavelets are biorthogonal, i.e.,

$$\int \psi(x-k)\widetilde{\psi}(x-l)\,dx = \delta_{k,l};$$

moreover,

$$\int \psi(x-k)\widetilde{\varphi}(x-l)\,dx = 0 = \int \widetilde{\psi}(x-k)\varphi(x-l)\,dx.$$

If we then define $g^{[1]}$ and $\widetilde{g}^{[-1]}$ analogously,

$$g^{[1]}(z) = \widetilde{h}^{[-1]}(-1/z)z, \qquad \widetilde{g}^{[-1]}(z) = h^{[1]}(-1/z)z,$$

then $g^{[1]}(z) = g(z)(1-z^{-1})/2$, $\widetilde{g}^{[-1]}(z) = \widetilde{g}(z)2/(1-z)$, and

$$\begin{aligned}\hat{\psi}^{[1]}(\xi) &= \widetilde{g}(e^{-i\xi/2})\hat{\varphi}^{[1]}(\xi/2) = \frac{1-e^{i\xi}}{2}g(e^{-i\xi/2})\hat{\varphi}(\xi/2)\frac{i\xi}{e^{i\xi}-1}\\ &= -\frac{i\xi}{2}\hat{\psi}(\xi),\end{aligned}$$

so that

$$\psi^{[1]} = -\frac{1}{2}\frac{d\psi}{dx} \qquad \text{and} \qquad \widetilde{\psi} = \frac{1}{2}\frac{d\widetilde{\psi}^{[1]}}{dx}.$$

The new families of wavelets and scaling functions are again biorthogonal,

$$\begin{aligned}\int \varphi^{[1]}(x-k)\widetilde{\varphi}^{[1]}(x-l)\,dx &= \delta_{k,l},\\ \int \psi^{[1]}(x-k)\widetilde{\psi}^{[1]}(x-l)\,dx &= \delta_{k,l},\\ \int \psi^{[1]}(x-k)\widetilde{\varphi}^{[1]}(x-l)\,dx &= 0,\\ \int \widetilde{\psi}^{[1]}(x-k)\varphi^{[1]}(x-l)\,dx &= 0.\end{aligned}$$

This whole construction can be translated into a subdivision framework. Consider the following subdivision operator:

$$(Sy)_l = 2\sum_k h_{l-2k}y_k.$$

If we start with an initial sequence $a = (a_k)_{k\in\mathbf{Z}}$, then iterating the subdivision leads to the limit function $\sum_k a_k\varphi(x-k)$. We can similarly define $\widetilde{S}$, $S^{[1]}$, and $\widetilde{S}^{[-1]}$. Define now backwards and forwards differencing operators Δ and $\widetilde{\Delta}$ by

$$\begin{aligned}(\Delta a)_k &= a_{k+1} - a_k,\\ (\widetilde{\Delta} a)_k &= a_k - a_{k-1}.\end{aligned}$$

One then checks that

$$2\Delta S = S^{[1]}\Delta \qquad \text{and} \qquad 2\widetilde{\Delta}\widetilde{S}^{[-1]} = \widetilde{S}\widetilde{\Delta}.$$

Iterating this leads to

$$2^l \Delta S^l = (S^{[1]})^l \Delta;$$

in the limit $2^l \Delta$ "becomes" a derivative (since the grid distance at level l is 2^{-l}), and $S^l a$ tends to the function $\sum_k a_k \varphi(x-k)$, so that we obtain

$$\frac{d}{dx} \sum_k a_k \varphi(x-k) = \sum_k (\Delta a)_k \varphi^{[1]}(x-k),$$

which can also be derived directly from (2).

We shall recognize many of these formulas in the more general (and notationally heavier) irregular setting of the remainder of this paper, except that we shall not be able to use the Fourier transform in our derivations.

3. Multilevel Grids

We start with a sequence of grids on the real line $X = \{x_j \mid j \in \mathbf{N}\}$. Each grid x_j is a strictly increasing sequence of points $(x_{j,k} \in \mathbf{R})_{k \in \mathbf{Z}}$. We refer to j as the level of the grid point $x_{j,k}$, where $j = 0$ is the initial, coarsest level. The length of the interval between $x_{j,k}$ and $x_{j,k+1}$ is given by $d_{j,k}$:

$$d_{j,k} = x_{j,k+1} - x_{j,k}.$$

We say that a grid is *regular* in case $d_{j,k}$ does not depend on k. We shall also use the term *grid size* on level j, for the quantity $d_j = \sup_k d_{j,k}$. As $j \to \infty$ we want the grids to become dense, "with no holes left." This leads to the following definition:

Definition 1. We say that a sequence of grids $X = \{x_j \mid j \in \mathbf{N}\}$ form a *multilevel grid*, in case

$$\sum_{j=0}^{\infty} d_j < \infty,$$

where $d_j = \sup_k x_{j,k+1} - x_{j,k}$.

Let, for all $y \in \mathbf{R}$, $x_{j,k_j(y)}$ be the closest grid point on level j to the left of y, i.e., $k_j(y) = \max\{l : x_{j,l} \leq y\}$. Then the definition implies that for all $y \in \mathbf{R}$:

$$\lim_{j \to \infty} x_{j,k_j(y)} = y. \tag{4}$$

Often multilevel grids are built through binary refinement: in every refinement step we insert one odd point $x_{j+1,2k+1}$ between each adjacent pair of even points $x_{j,k} = x_{j+1,2k}$ and $x_{j,k+1} = x_{j+1,2k+2}$. This leads to the following definition:

Definition 2. We say that a multilevel grid is *two-nested* in case the x_j grids are consecutive binary refinements of the initial grid X_0, i.e., $x_{j+1,2k} = x_{j,k}$ for all $j \in \mathbf{N}$ and $k \in \mathbf{Z}$.

Note that in a two-nested grid, all points $x_{j+j',2^{j'}k}$ coincide for $j,\ j' \in \mathbf{N}$ and $k \in \mathbf{Z}$. In case these points do not exactly coincide, but the sequence in j' has a limit for all j and k we call the grid two-threadable:

Definition 3. We say that a multilevel grid is *two-threadable* in case the limit

$$\mathbf{x}_{j,k} = \lim_{j' \to \infty} x_{j+j',2^{j'}k} \tag{5}$$

exists for all j and k.

The reason for the name "threadable" is that you can think of $x_{j,k}$ being connected to $x_{j+1,2k}$ with a "thread." The $\mathbf{x}_{j,k}$ themselves form a multilevel grid which we call the *limit grid*. It is clear that if the multilevel grid X is two-threadable, then the limit grid $\mathbf{X} = \{\mathbf{x}_{j,k};\ j \in \mathbf{Z}_+, k \in \mathbf{Z}\}$ is two-nested.

Examples. 1. The standard regular dyadic grid is given by $x_{j,k} = 2^{-j}k$ and clearly is two-nested.

2. An example of a regular, but nonnested grid is given by $x_{j,k} = 2^{-j}(k + \frac{1}{2})$. This grid is two-threadable, where the limit grid is the standard regular dyadic grid.

3. An irregular two-nested grid is given by $x_{0,k} = k$, $x_{j+1,2k} = x_{j,k}$, and $x_{j+1,2k+1} = \beta x_{j,k} + (1-\beta)x_{j,k+1}$ for $0 < \beta < 1$.

4. One can take any two-nested grid X and build a multilevel grid Y which is two-threadable, but not two-nested by letting $y_{j,k} = \frac{1}{2}(x_{j,k} + x_{j,k+1})$. The limit grid then again is the grid X.

5. A not two-threadable grid is $x_j = 3^{-j}\mathbf{Z}$. Clearly the definitions can be generalized to q-threadable and q-nested, but this is not our focus right now.

6. In the first four examples the grid size decays exponentially with j. A regular, nonnested, nonthreadable grid, with nonexponential decay of the grid step is given by $x_{j,k} = k/j^q$ with $q > 0$.

From now on we consider only two-threadable or two-nested grids.

On the other hand, we consider sequences $f_j = (f_{j,k} \in \mathbf{R})_{k \in \mathbf{Z}}$, $j \geq 0$. With each sequence f_j and grid x_j we associate a piecewise constant function $\theta_{\{x_j,\, f_j\}}$ by means of

$$\theta_{\{x_j,\, f_j\}}(y) = f_{j,k_j(y)} \qquad \text{or} \qquad \theta_{\{x_j,\, f_j\}} = \sum_k f_{j,k}\,\chi_{[x_{j,k},x_{j,k+1})}.$$

We shall be interested in choices for x_j and f_j so that the $\theta_{\{x_j,\, f_j\}}$ converge as $j \to \infty$, either pointwise, or in L^p.

Remarks. 1. There is nothing special about using piecewise *constant* functions; for instance, if the limit function is continuous, then using piecewise *linear* functions that interpolate the $(x_{j,k},\ f_{j,k})$ points would not change the definition of convergence nor the limit function.

2. Typically when we are given a two-threadable grid, we will replace it with its limit grid which is nested. We can do this because if $\theta_{\{x_j,\, f_j\}}$ converges pointwise a.e. to a continuous limit function, then $\theta_{\{\mathbf{x}_j,\, f_j\}}$ converges to the same function. Thus we can often focus on two-nested grids.

4. Irregular Subdivision

We shall consider particular sequences f_j that can be computed iteratively using a sequence S of linear operators S_j, $j \geq 0$:

$$f_{j+1} = S_j f_j.$$

We call f_0 the initial sequence.

A *subdivision scheme* now is a pair (S, X), where X is a two-nested grid or a two-threadable grid; for an initial sequence f_0 the ambition of subdivision is to synthesize a continuous limit function $f(y)$ defined for all $y \in \mathbf{R}$ by following the "threads." This leads to the following definition:

Definition 4. A subdivision scheme (S, X) *converges* if for any initial sequence f_0, the limit

$$F(y) = \lim_{j\to\infty} \theta_{\{\mathbf{x}_j, f_j\}}(y),$$

exists for all $y \in \mathbf{R}$.

Remarks. 1. Note that convergence depends both on the S_j operators and on the grid points $x_{j,k}$.

2. The definition assumes that subdivision is started from initial data f_0 on level 0. We can just as well start from any other level J and compute f_j for $j > J$ starting from an initial sequence f_J. If the subdivision scheme converges according to Definition (4) then it also converges when started on level J.

We denote the elements of the infinite subdivision matrix S_j by $S_{j,l,k}$ where l denotes rows and k denotes columns; then

$$f_{j+1,l} = \sum_k S_{j,l,k}\, f_{j,k}.$$

We want to enforce some locality on S_j in the sense that a new value $f_{j+1,l}$ only depends on a finite number of old values $f_{j,k}$ around $k = l/2$.

Definition 5. We say that a subdivision scheme is *local* in case, for some $L \in \mathbf{N}$:

$$S_{j,l,k} = 0 \qquad \text{for} \quad |k - \lfloor l/2 \rfloor| > L.$$

A new value at the point $x_{j+1,l}$ can then be found by the *finite* sum

$$f_{j+1,l} = \sum_{k=\lfloor l/2\rfloor - L}^{\lfloor l/2\rfloor + L} S_{j,l,k}\, f_{j,k}.$$

In this paper we consider only local subdivision schemes.

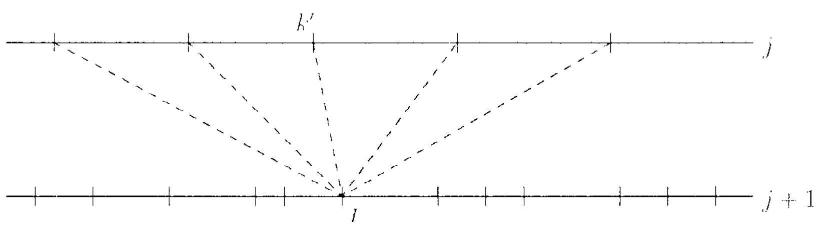

Fig. 1. The computation of $f_{j+1,l}$ only involves $2L + 1$ values $f_{j,k}$ centered around $k' = \lfloor l/2 \rfloor$.

It is natural to require that in a subdivision scheme the operators S_j reproduce constant sequences; we shall call such schemes *affine*.This property guarantees that if the scheme is used for building curves, the resulting curve is independent of the coordinate system. Schemes that are not affine have little practical relevance.

Definition 6. We say that a subdivision scheme (S, X) is *affine* in case

(6)
$$\sum_k S_{j,l,k} = 1,$$

for all $j > 0$ and $l \in \mathbf{Z}$.

Affine schemes can produce constants. We shall also consider, in addition, schemes which can produce linear functions.

Definition 7. We say that an affine subdivision scheme (S, X) *produces linear functions* if there exists a sequence $a_0 = (a_{0,k})_{k\in\mathbf{Z}}$ so that the sequences a_j defined iteratively by $a_{j+1} = S_j a_j$ constitute a two-threadable multilevel grid with limit grid $\mathbf{X}$, i.e.,

(7)
$$\lim_{j'\to\infty} a_{j+j',k2^{j'}} = \mathbf{x}_{j,k}.$$

This implies that the limit function corresponding to the a_j sequences is $F(y) = y$. Note that (7) is similar to (5); the crucial difference though is that the a_j sequences are related through subdivision while the x_j typically are not.

In the regular case it is easy to write down a condition for a subdivision scheme to produce linears or higher-order polynomials. This is due to the fact that the subdivision coefficients are constant from level to level. However, in the irregular case the coefficients change from level to level and are only given to use as a listing of the $S_{j,k,l}$. Thus one has no handle on what happens on finer and finer levels and therefore cannot write down a simple condition on polynomial reproduction. If the $S_{j,k,l}$ were given to us as a formula depending on the geometry, such a condition may be possible. However, this would hold only for very particular schemes and we did not consider this.

Definition 8. We say that a subdivision scheme (S, X) is *interpolating* if $f_{j,k} = f_{j+1,2k}$ for all $j \in \mathbf{N}$ and $k \in \mathbf{Z}$.

An equivalent condition is that $S_{j,2k,k+m} = \delta_{m,0}$ for all j. If an interpolating scheme produces linear functions, then the a_j have to be equal to the $\mathbf{x}_j$.

Definition 9. A subdivision scheme (S, X) is *bounded* if its coefficients are uniformly bounded, i.e., a constant C_S exists so that

$$\left|S_{j,l,k}\right| < C_S,$$

for all j, l, and k.

A local and bounded scheme automatically defines an operator from ℓ^∞ to ℓ^∞.

5. Commutation for Differences

Our purpose is to investigate how subdivision interacts with finite differences. We therefore define the $\Delta f_j = (\Delta f_{j,k})_{k\in\mathbf{Z}}$ sequences by

$$\Delta f_{j,k} = f_{j,k+1} - f_{j,k}.$$

For a given local subdivision scheme (S, X) for the f_j we would like to build another local subdivision scheme (T, X) for the Δf_j sequences. The following proposition shows that this can always be done if S is affine:

Proposition 10. *If a local subdivision scheme (S, X) for f_j sequences is affine, then we can find a local subdivision scheme (T, X) for the difference sequences Δf_j. More precisely,*

$$T_{j,l,m} = \sum_{k=m+1}^{\lfloor (l+1)/2\rfloor + L} (S_{j,l+1,k} - S_{j,l,k}),$$

with the convention that $\sum_{k=k_1}^{k_2}$ is zero if $k_2 < k_1$.

Proof.

$$\begin{aligned}
(8)\qquad \Delta f_{j+1,l} &= f_{j+1,l+1} - f_{j+1,l} = \sum_{k=\lfloor l/2\rfloor - L}^{\lfloor (l+1)/2\rfloor + L} (S_{j,l+1,k} - S_{j,l,k})\, f_{j,k} \\
&= \sum_{k=\lfloor l/2\rfloor - L+1}^{\lfloor (l+1)/2\rfloor + L} (S_{j,l+1,k} - S_{j,l,k})\,(f_{j,k} - f_{j,\lfloor l/2\rfloor - L}) \quad \text{(affine)} \\
&= \sum_{k=\lfloor l/2\rfloor - L+1}^{\lfloor (l+1)/2\rfloor + L} (S_{j,l+1,k} - S_{j,l,k}) \sum_{m=\lfloor l/2\rfloor - L}^{k-1} \Delta f_{j,m} \\
&= \sum_{m=\lfloor l/2\rfloor - L}^{\lfloor (l+1)/2\rfloor + L-1} T_{j,l,m}\, \Delta f_{j,m},
\end{aligned}$$

where $T_{j,l,m}$ is given as above. ■

If l is even then $\lfloor l/2\rfloor = \lfloor (l+1)/2\rfloor$. Thus only $2L$ terms are needed to compute $\Delta f_{j+1,l}$, so that the subdivision scheme for the differences becomes shorter. In other words, the number of nonzero entries in every column of the matrix representation decreases by one in the transition from S to T. We thus expect the subdivision scheme for the differences to be easier to analyze than the original one. (This is exactly what happened for the interpolating schemes considered in [7].)

The above construction leads to a commutation formula for the differences

$$\Delta S_j f_j = T_j \Delta f_j.$$

We shall use the differences Δf_j to investigate the smoothness of the limit function F (if it exists). It therefore makes sense to consider *divided* differences. We shall see that these have an advantage from the point of view of commutation formulas. The scheme T is typically not affine, but once we make the transition to appropriately defined divided differences, the corresponding subdivision scheme can be affine again, and we can iterate our procedure.

6. Divided Differences

6.1. *Definition of Standard Divided Differences*

We want to consider the divided differences of the sequence f_j defined on the grid x_j assuming $f_{j,k} = F(x_{j,k})$. In case of standard divided differences, the zeroth-order divided difference is simply the function value [15]:

$$[x_{j,k}]F = f_{j,k},$$

and the higher-order divided differences are defined recursively

$$[x_{j,k}, \ldots, x_{j,k+p}]F = \frac{[x_{j,k}, \ldots, x_{j,k+p-1}]F - [x_{j,k+1}, \ldots, x_{j,k+p}]F}{(x_{j,k+p} - x_{j,k})/p}. \tag{9}$$

Note that (9) differs from the more common definition by the normalization factor p, introduced here for convenience. It is well known that if F is sufficiently smooth, then its finite divided differences of order p converge to the pth derivative of F.

If the subdivision scheme is interpolating, then $f_{j,k}$ are in fact equal to the function values $F(x_{j,k})$ of the limit function F and the above-described divided differences can be applied, see [7]. However, for noninterpolating subdivision we cannot view the $f_{j,k}$ as function values of F, and standard divided differences are not the appropriate tool to approximate the derivatives or to study the regularity of F; hence we need to adapt the definition of finite differences to the subdivision scheme.

We even find some trace of this phenomenon in the case of standard divided differences discussed above, as soon as we consider higher-order differences: although it is entirely natural to use the denominator $(x_{j,k+1} - x_{j,k})$ in the first-order divided difference, it is not clear immediately why $(x_{j,k+p} - x_{j,k})/p$ is the right choice for the higher-order differences. We will show that this is related to the fact that the subdivision scheme for the divided differences of an interpolating scheme is no longer interpolating. Whereas, in an interpolating case, the $f_{j,k}$ live at the $x_{j,k}$, we will see that the pth-order divided differences do not live at $x_{j,k}$ but rather at $(x_{j,k}+x_{j,k+1}+\cdots+x_{j,k+p-1})/p$. For the limiting behavior this distinction is irrelevant given that the two locations become arbitrarily close.

However, it explains in a natural way why $x_{j,k+p} - x_{j,k}$ shows up in (9). Indeed, we can think of (9) as the ratio between the difference of two samples, divided by the difference of their locations:

$$[x_{j,k}, \dots, x_{j,k+p}]F = \frac{[x_{j,k}, \dots, x_{j,k+p-1}]F - [x_{j,k+1}, \dots, x_{j,k+p}]F}{(x_{j,k} + \cdots + x_{j,k+p-1})/p - (x_{j,k+1} + \cdots + x_{j,k+p})/p}.$$

In the next subsection we show how the standard divided differences can be generalized correctly.

6.2. *Generalized Divided Differences*

We start out by introducing new notation for divided differences. Consider a subdivision scheme $(S^{[0]}, X)$ which generates values $f_{j,k}^{[0]}$ and a limit function $F^{[0]}$. Note that for noninterpolating schemes the $f_{j,k}^{[0]}$ are not necessarily equal to $F^{[0]}(x_{j,k})$. The superscript [0] here merely indicates that this is the *first* subdivision scheme we consider. Later we will compute divided differences of the $f^{[0]}$ sequences which we will denote by $f^{[1]}$, but we may also find sequences whose divided differences are $f^{[1]}$ and which we will denote by $f^{[-1]}$. Thus the superscript does not necessarily indicate the order of the divided differences and needs to be seen as a relative number. We also sometimes omit the superscript [0] if it is clear from the context.

Our generalized finite difference operator will be of the form

$$f_{j,k}^{[p]} = \frac{f_{j,k+1}^{[p-1]} - f_{j,k}^{[p-1]}}{\xi_{j,k+1}^{[p]} - \xi_{j,k}^{[p]}}, \tag{10}$$

where $p \in \mathbf{N}$ and the $\xi_{j,k}^{[p]}$ form a monotone sequence that will be defined below and for which

$$\lim_{j' \to \infty} \xi_{j+j',k2^{j'}}^{[p]} = \mathbf{x}_{j,k} \qquad \text{for all} \quad j \in \mathbf{Z}_+, \quad k \in \mathbf{Z}. \tag{11}$$

From the definition (10) it is clear that we can think of the $f_{j,k}^{[p]}$ as living at $\xi_{j,k}^{[p]}$. The $\xi_{j,k}^{[p]}$ will be chosen such that the $f_{j,k}^{[p]}$ do converge to the pth derivative of $F^{[0]}$ if function $F^{[0]}$ is sufficiently smooth.

Now let $f_j^{[p]}$ be the sequence $(f_{j,k}^{[p]})_{k \in \mathbf{Z}}$. Then the sequences $f_j^{[p]}$ are related through the difference operator $D_j^{[p]}$:

$$f_j^{[p]} = D_j^{[p]} f_j^{[p-1]}.$$

The following are two basic convergence results concerning divided difference sequences:

Proposition 11. *Suppose that $\Xi = \{\xi_j\}$ is a multilevel grid and that the subdivision scheme (S, X) is local, affine, and bounded.*

If the $f_{j,k}^{[1]} = (f_{j,k+1} - f_{j,k})/(\xi_{j,k+1} - \xi_{j,k})$ are uniformly bounded, i.e., $|f_{j,k}^{[1]}| \le C_1$ for some $C_1 > 0$ independent of j and k, then the sequence $\theta_{\{\mathbf{x}_j, f_j\}}$ converges uniformly.

Proof. For any $l \in \mathbf{Z}$ we have

$$|f_{j+1,l} - f_{j,\lfloor l/2 \rfloor}| \le \sum_{k=\lfloor l/2 \rfloor - L}^{\lfloor l/2 \rfloor + L} |S_{j,l,k}|\,|f_{j,k} - f_{j,\lfloor l/2 \rfloor}|$$
$$\le (2L+1) C_S \sup_k |f_{j,k} - f_{j,\lfloor l/2 \rfloor}|.$$

From the definition of $f_{j,k}^{[1]}$ we have (without loss of generality, assume that $k > \lfloor l/2 \rfloor$):

$$f_{j,k} - f_{j,\lfloor l/2 \rfloor} = \sum_{s=\lfloor l/2 \rfloor}^{k-1} f_{j,s}^{[1]}(\xi_{j,s+1} - \xi_{j,s}).$$

It is easy to see that in our case $\lfloor l/2 \rfloor - L \le k \le \lfloor l/2 \rfloor + L$ so that the above sum contains no more than $L-1$ terms. Hence,

$$|f_{j,k} - f_{j,\lfloor l/2 \rfloor}| \le (L-1) C_1 d_j^{\xi},$$

and we obtain

$$|f_{j+1,l} - f_{j,\lfloor l/2 \rfloor}| \le C_S(2L+1)(L-1) d_j^{\xi} C_1 \le C d_j^{\xi}.$$

Since $\mathbf{X}$ is a two-nested grid, it follows that for any $y \in \mathbf{R}$ we have $k_j(y) = \lfloor k_{j+1}(y)/2 \rfloor$.

Therefore the summability of the grid sizes of the grid Ξ implies that for any y sequence $\{f_{j,k_j(y)}\}$ is Cauchy and hence converges. ■

Lemma 12. *Let E_y be a neighborhood of the point $y \in \mathbf{R}$.*

Suppose that $\lim_{j\to\infty} \theta_{\{\mathbf{x}_j, f_j\}}(x) = F(x)$ *for all* $x \in E_y$ *and that* F *is bounded on* E_y, *and let* $\{\xi_j\}$ *be a multilevel grid such that* $\lim_{j'\to\infty} \xi_{j+j',2^{j'}k} = \mathbf{x}_{j,k}$ *for all* $j \in \mathbf{Z}_+$, $k \in \mathbf{Z}$. *Define*

$$f_{j,k}^{[1]} := \frac{f_{j,k+1} - f_{j,k}}{\xi_{j,k+1} - \xi_{j,k}}.$$

Suppose also that $\theta_{\{\mathbf{x}_j, f_j^{[1]}\}} \to F^{[1]}$ *as* $j \to \infty$ *uniformly on* E_y, *and* $F^{[1]}$ *is continuous at* y.

Then F *is differentiable at* y *and* $F'(y) = F^{[1]}(y)$ *on* E_y.

Proof. For any $\epsilon > 0$ there exists $\delta > 0$ such that

$$|F^{[1]}(y+s) - F^{[1]}(y)| < \epsilon \qquad \text{for all} \quad s : |s| < \delta.$$

Without loss of generality we assume that $t > 0$. Now fix any t such that $0 < t < \delta$. There exist $j = j(t)$ such that all of the following is true

$$|F(y) - f_{j,k_j(y)}| < \epsilon |t|,$$
$$|F(y+t) - f_{j,k_j(y+t)}| < \epsilon |t|,$$

$$
\begin{aligned}
|y - \xi_{j,k_j(y)}| &< \min\left\{\frac{\epsilon t^2}{M}, \frac{|t|}{4}\right\}, \\
|(y+t) - \xi_{j,k_j(y+t)}| &< \min\left\{\frac{\epsilon t^2}{M}, \frac{|t|}{4}\right\}, \qquad \text{with} \quad M = \max\{F(y), F(y+t)\}, \\
|F^{[1]}(y) - f^{[1]}_{j,k}| &< \epsilon \qquad \text{for} \quad k_j(y) \le k \le k_j(y+t).
\end{aligned}
$$

The last inequality follows from the uniform convergence of sequence $f^{[1]}_j$ and continuity of its limit function $F^{[1]}$ at the point y.

Consider the difference

$$
\begin{aligned}
\left|\frac{F(y+t)-F(y)}{t} - F^{[1]}(y)\right| &= \left|\frac{F(y+t)-F(y)}{t} - \frac{F(y+t)-F(y)}{\xi_{j,k_j(y+t)} - \xi_{j,k_j(y)}}\right| \\
&\quad + \left|\frac{F(y+t)-F(y)}{\xi_{j,k_j(y+t)} - \xi_{j,k_j(y)}} - F^{[1]}(y)\right| \\
&\le 4\epsilon + \left|\frac{F(y+t) - f_{j,k_j(y+t)}}{\xi_{j,k_j(y+t)} - \xi_{j,k_j(y)}}\right| + \left|\frac{F(y) - f_{j,k_j(y)}}{\xi_{j,k_j(y+t)} - \xi_{j,k_j(y)}}\right| \\
&\quad + \left|\frac{f_{j,k_j(y+t)} - f_{j,k_j(y)}}{\xi_{j,k_j(y+t)} - \xi_{j,k_j(y)}} - F^{[1]}(y)\right| \\
&\le 8\epsilon + \left|\frac{f_{j,k_j(y+t)} - f_{j,k_j(y)}}{\xi_{j,k_j(y+t)} - \xi_{j,k_j(y)}} - F^{[1]}(y)\right|.
\end{aligned}
$$

Then $k_j(y+t) > k_j(y)$, and

$$
\begin{aligned}
f_{j,k_j(y+t)} - f_{j,k_j(y)} &= \sum_{l=k_j(y)}^{k_j(y+t)-1} f^{[1]}_{j,l}(\xi_{j,l+1} - \xi_{j,l}) \\
&= \sum_{l=k_j(y)}^{k_j(y+t)-1} (f^{[1]}_{j,l} - F^{[1]}(y))(\xi_{j,l+1} - \xi_{j,l}) \\
&\quad + F^{[1]}(y)(\xi_{j,k_j(y+t)} - \xi_{j,k_j(y)}).
\end{aligned}
$$

Therefore, we have

$$
\left|\frac{F(y+t)-F(y)}{t} - F^{[1]}(y)\right| < 9\epsilon,
$$

for all t satisfying $0 < |t| < \delta$, which proves the lemma. ■

Remark. Throughout this section we have used forward differences. Clearly, the results hold equally for backward differences. When we later combine commutation with biorthogonality we will need both forward and backward differences.

7. Commutation

For simplicity we let $\xi_{j,k} = \xi^{[1]}_{j,k}$. Given an affine subdivision scheme $(S^{[0]}, X)$:

$$
f^{[0]}_{j+1} = S^{[0]}_j f^{[0]}_j,
$$

we define a local subdivision scheme $(S^{[1]}, X)$ for the general divided differences

$$f_{j+1}^{[1]} = S_j^{[1]} f_j^{[1]},$$

so that

$$D_{j+1}^{[1]} S_j^{[0]} = S_j^{[1]} D_j^{[1]}. \tag{12}$$

Given that $S^{[0]}$ is affine we know from the previous section we can always find a scheme T for the differences. Because divided differences are simply scaled differences, the elements of the $S^{[1]}$ scheme are rescaled versions of the corresponding elements of the T scheme

$$S_{j,l,m}^{[1]} = \frac{\xi_{j,m+1} - \xi_{j,m}}{\xi_{j+1,l+1} - \xi_{j+1,l}} T_{j,l,m} = \frac{\xi_{j,m+1} - \xi_{j,m}}{\xi_{j+1,l+1} - \xi_{j+1,l}} \sum_{k=m+1}^{\lfloor (l+1)/2 \rfloor + L} (S_{j,l+1,k}^{[0]} - S_{j,l,k}^{[0]}),$$

again with the convention that $\sum_{l=l_1}^{l_2}$ is zero if $l_2 < l_1$.

So far this works for any choice of $\xi_{j,k}$ that are monotone in k for all j. However, we want to repeat this process to define $S^{[2]}$ and we thus need $S^{[1]}$ to be affine. This leads to the following proposition:

Proposition 13. *If the ξ_j are related through subdivision, i.e., $\xi_{j+1} = S_j \xi_j$, then the $S^{[1]}$ scheme is affine .*

Proof. Note that by definition $D_j^{[1]} \xi_j = \mathbf{1}$, i.e., a sequence of all ones. Then

$$\mathbf{1} = D_{j+1}^{[1]} \xi_{j+1} = D_{j+1}^{[1]} S_j^{[0]} \xi_j = S_j^{[1]} D_j^{[1]} \xi_j = S^{[1]} \mathbf{1}.$$

Thus $S^{[1]}$ is affine. ■

This shows why, in general, the ξ_j often are not simply the x_j or $\mathbf{x}_j$ because neither of them are related through subdivision. If the scheme is interpolating, then the $\mathbf{x}_j$ are interpolating and we can choose $\xi_{j,k} = \mathbf{x}_{j,k}$.

We still have not said what the $\xi_{j,k}$ actually should be. Somehow the D_j operators in the limit still have to correspond to differentiation. We next show that suitable ξ_j can be found in case $S^{[0]}$ produces linear functions. Remember that this implies the existence of sequences a_j with $a_{j+1} = S_j a_j$ that converge to the limit function x.

If our D_j correspond to differentiation in the limit, then the $a_j^{[1]}$ should converge to $x' \equiv 1$. Since $S^{[1]}$ is already affine, we know that it produces constants. If $S^{[1]}$ is stable we therefore have $a_{j,k}^{[1]} = 1$ for all j, k, implying that $\xi_{j,k} = a_{j,k} + C$. It then follows from (7) and (11) that $C = 0$, or $\xi_{j,k} = a_{j,k}$. Thus $D_j^{[1]} = \mathcal{A}^{-1} \Delta$, where $\mathcal{A}_j = \operatorname{diag}(\Delta a_j)$.

Definition 14. We shall say that a subdivision scheme (S, X) is *differencible* if it is affine, and it produces linear functions.

We note that the word *differencible* is not an English word. We chose this neologism because differencible relates to differentiable in the same way differences relate to differentiation.

The above discussion can now be summarized in the following theorem:

Theorem 15. *If a local subdivision scheme* (S, X) *is differencible, we define the derived scheme* $(S^{[1]}, X)$ *by*

$$S^{[1]}_{j,l,m} = \frac{a_{j,m+1} - a_{j,m}}{a_{j+1,l+1} - a_{j+1,l}} \sum_{k=m+1}^{\lfloor (l+1)/2 \rfloor + L} (S_{j,l+1,k} - S_{j,l,k}), \tag{13}$$

where $a_{j,k}$ *for each* j *are the increasing sequences producing the function* x *for the subdivision scheme* (S, X). *Then* $(S^{[1]}, X)$ *likewise is a local and affine subdivision scheme and the following commutation formula holds*:

$$D^{[1]}_{j+1} S_j = S^{[1]}_j D^{[1]}_j.$$

Remark. If the a_j are not increasing, one can always reorder the indexing of the scaling functions to make a_j increasing. Given the fact that each scaling function "lives" around $a_{j,k}$ this is the natural ordering for the $\varphi_{j,k}$. In case $a_{j,k+1} = a_{j,k}$ for some k, then we see from (13) that $S^{[1]}_{j-1,l,m}$ is not defined for some l, m. However, it turns out that the corresponding $S^{[1]}_{j,n,l}$ (at the next level) are then zero (unless $a_{j+1,n}$ are coinciding again), so that the subdivision algorithm could still be implementable, and converge, as a whole.

Theorem 16. *Suppose* (S, X) *is a local differencible subdivision scheme. Then the following is true*:

If for some initial sequence $f_0 =: f^{[0]}_0$, *the subdivision scheme* (S, X) *leads to a limit function* $F(x)$, *and the subdivision scheme* $(S^{[1]}, X)$ *applied to the generalized finite differences* $f^{[1]}_0$ *converges uniformly on compact sets to a continuous limit function* $F^{[1]}(x)$, *then* F *is differentiable, and* $F'(x) = F^{[1]}(x)$.

Proof. This fact is an easy consequence of Lemma 12. ∎

As with differentiability, we define p times differencibility recursively:

Definition 17. We shall say that a subdivision scheme (S, X) is *p times differencible* if $(S^{[1]}, X)$ is $p - 1$ times differencible.

If a subdivision scheme is differencible, then it can, starting from appropriate initial sequences, construct arbitrary linear polynomials: to obtain the limit function $F(x) = \alpha x + \beta$, one simply starts from the initial data $f_0 = \alpha a_0 + \beta$. The following theorem generalizes this observation to p times differencible subdivision schemes:

Theorem 18. *If* (S, X) *is* p *times differencible, then any polynomial* $\Pi(x)$ *of degree* p *can be obtained as the limit of subdivision starting from an appropriate initial sequence* π_0.

Proof. By induction on p.

Note that, by Definition 7, $a^{[q]}_{j,k}$ constitute multilevel grids for $q = 1, \dots, p$.

Suppose the theorem holds for $p-1$. If (S, X) is p times differencible, then $(S^{[1]}, X)$ is $(p-1)$ times differencible, so that there exists a sequence $\widetilde{\pi}_0$ so that the $\widetilde{\pi}_{j+1} = S^{[1]}_j \widetilde{\pi}_j$ satisfy

$$\lim_{j\to\infty} \sum_k \widetilde{\pi}_{j,k} \chi_{(a_{j,k-1}, a_{j,k}]}(x) = \Pi'(x).$$

Now define

$$\omega_{0,k} = \sum_{l=0}^{k} \widetilde{\pi}_{0,l}(a_{0,l+1} - a_{0,l}) \qquad \text{for} \quad k \geq 0,$$

so that $D^{[1]}_j \omega_0 = \widetilde{\pi}_0$. It then follows from applying $S^{[1]}_j$ to both sides that

$$D^{[1]}_j \omega_j = \widetilde{\pi}_j \qquad \text{for} \quad j \geq 0.$$

It follows that (ω_j, x_j) converges to some limit function F_ω. From Lemma 12 above it follows that F_ω is differentiable and $F'_\omega = \Pi'$, hence $F_\omega = C_\omega + \Pi$ with some constant C_ω.

Therefore, the affine scheme (S, X) with initial sequence $\pi_0 := \omega_0 - C_\omega$ will converge to the polynomial Π. ■

8. Inverse Commutation

It is also possible to use the commutation formula in the other direction, i.e., to start with a local subdivision scheme $S^{[0]}$ and construct a local scheme $S^{[-1]}$ so that $S^{[0]}$ is its derived scheme. (We use the convention that the scheme we start with gets superscript [0] independent of whether we use forward or inverse commutation.) This means we need to find divided difference operators $D^{[0]}_j$ so that

$$D^{[0]}_{j+1} S^{[-1]}_j = S^{[0]}_j D^{[0]}_j. \tag{14}$$

As before, $D^{[0]}_j$ will be of the form $\mathcal{B}_j^{-1}\Delta$, where $\mathcal{B}_j = \operatorname{diag}(\Delta b_j)$; the problem reduces to finding, if they exist, monotone b_j so that (14) holds for a local scheme $S^{[-1]}$. This will require a condition on $S^{[0]}$. Let us analyze what it means for $S^{[0]}$ to impose that for any compactly supported sequence $f^{[-1]}_j$ the resulting $f^{[-1]}_{j+1} = S^{[-1]}_j f^{[-1]}_j$ be compactly supported too. We first compute $f^{[0]}_j$ as $D^{[-1]}_j f^{[-1]}_j$ and then use $S^{[0]}_j$ to find $f^{[0]}_{j+1}$. Given that $S^{[0]}$ is local, $f^{[0]}_{j+1}$ is compactly supported; for $f^{[-1]}_{j+1}$ to be compactly supported, we need that

$$\sum_l (f^{[-1]}_{j+1,l+1} - f^{[-1]}_{j+1,l}) = \sum_l \Delta b_{j+1,l}\, f^{[0]}_{j+1,l} = 0$$

or, in short,

$$\mathbf{1}^t \mathcal{B}_{j+1} f^{[0]}_{j+1} = 0.$$

The left-hand side is equal to

$$\mathbf{1}^t \mathcal{B}_{j+1} f_{j+1}^{[0]} = \mathbf{1}^t \mathcal{B}_{j+1} S_j^{[0]} f_j^{[0]} = \mathbf{1}^t \mathcal{B}_{j+1} S_j^{[0]} \mathcal{B}_j^{-1} \Delta f_j^{[-1]}.$$

This is zero for all compactly supported sequences $f_j^{[-1]}$ if and only if

$$\mathbf{1}^t \mathcal{B}_{j+1} S_j^{[0]} \mathcal{B}_j^{-1} = C_j \mathbf{1}^t, \tag{15}$$

for some constant C_j or, equivalently,

$$\sum_l \Delta b_{j+1,l} S_{j,l,m}^{[0]} = C_j \Delta b_{j,m}.$$

Thus if such sequences b_j exist then the inverse commutation formula can be applied.

Remark. Using a technique similar to [7, Appendix B] it is possible to show that if the scheme $S^{[-1]}$ is affine, C_j must be equal to 1.

From (14) we see that $S^{[-1]}$ has to be defined so as to satisfy

$$-\frac{S_{j,l,m}^{[0]}}{\Delta b_{j,m}} + \frac{S_{j,l,m-1}^{[0]}}{\Delta b_{j,m-1}} = \frac{S_{j,l+1,m}^{[-1]} - S_{j,l,m}^{[-1]}}{\Delta b_{j+1,l}}, \tag{16}$$

implying

$$S_{j,l,m}^{[-1]} = \sum_{n=l}^{2m+2L+2} \Delta b_{j+1,n} \left(\frac{S_{j,n,m}^{[0]}}{\Delta b_{j,m}} - \frac{S_{j,n,m-1}^{[0]}}{\Delta b_{j,m-1}} \right), \tag{17}$$

where we assume that $S_{j,n,m}^{[-1]} = 0$ if $n \geq 2m + 2L + 2$ (see Definition 5).

If $S^{[0]}$ is affine invariant then we can say more about the relationship between b_j and $S^{[-1]}$. Since

$$D_{j+1}^{[0]} S_j^{[-1]} b_j = S_j^{[0]} D_j^{[0]} b_j = S_j^{[0]} \mathbf{1} = \mathbf{1}, \tag{18}$$

we find

$$\Delta S_j^{[-1]} b_j = \Delta b_{j+1}.$$

This defines the b_j only up to a constant which depends on j. Equation (18) implies that these constants can be chosen so that $S_j^{[-1]} b_j = b_{j+1}$. If, in addition, the limit grid of the b_j is $\mathbf{X}$, then the b_j are the linear producing sequences for $S^{[-1]}$. We shall come back to this later. When we discuss the relationship between commutation and biorthogonality below (Section (11)), we will see how the b_j can be computed.

9. Scaling Functions

We next define *scaling functions* or fundamental solutions of the subdivision scheme.

Definition 19. For a converging subdivision scheme (S, X) we define, for $j \in \mathbf{N}$ and $k \in \mathbf{Z}$, the *scaling function* $\varphi_{j,k}(x)$ to be the limit function when the scheme is started on level j with a Kronecker sequence $f_{j,k'} = \delta_{k-k'}$.

The following properties of scaling functions now follow easily:

1. The scaling functions are compactly supported. Because of the locality of the subdivision scheme (see Definition 5), the support of the scaling function $\varphi_{j,k}$ on level j is contained in $[\mathbf{x}_{j,k-2L}, \mathbf{x}_{j,k+2L+1}]$. Given a fixed x only a finite number of scaling functions on level j are nonzero at x.
2. The limit function F of a subdivision scheme started on level j can be written as a linear combination of scaling functions on level j' if $j' > j$:

$$F(x) = \sum_k f_{j',k}\varphi_{j',k}(x).$$

   Because of the compact support only a finite number of terms are nonzero for a given x.
3. The scaling functions are *refinable*, i.e., a scaling function on level j can be written as a linear combination of scaling functions on level $j+1$ where the coefficients come from the subdivision matrix

$$\varphi_{j,k}(x) = \sum_l S_{j,l,k}\varphi_{j+1,l}(x). \tag{19}$$

   Often we will use φ_j for the row of scaling functions $\varphi_{j,k}$. Then the refinement equation can be written as

$$\varphi_j = \varphi_{j+1}S_j. \tag{20}$$

4. If the subdivision scheme is affine invariant, then the scaling functions form a partition of unity, i.e.,

$$\sum_k \varphi_{j,k}(x) = 1.$$

5. If the subdivision scheme is p times differencible, then any polynomial $\pi(x)$ of degree strictly less than p can be written as a linear combination of scaling functions on any level, i.e., coefficients $m_{j,k}$ exist so that

$$\pi(x) = \sum_k m_{j,k}\,\varphi_{j,k}(x).$$

6. If the subdivision scheme S is interpolating, then the scaling functions are interpolating, in the sense that

$$\varphi_{j,k}(\mathbf{x}_{j,k'}) = \delta_{k,k'}.$$

   The $m_{j,k}$ for polynomial production are then simply $\pi(\mathbf{x}_{j,k})$. Also, from the refinement relation we see that

$$\varphi_{j,k}(\mathbf{x}_{j+1,l}) = S_{j,l,k}.$$

7. Consider a subdivision scheme $(S^{[0]}, X)$ and its derived scheme $(S^{[1]}, X)$ and assume both converge. Let $\varphi^{[0]}_{j,k}$ and $\varphi^{[1]}_{j,k}$, respectively, be the associated scaling functions. Then

$$\frac{d\varphi^{[0]}_{j,k}}{dx}(x) = \frac{\varphi^{[1]}_{j,k-1}}{a_{j,k} - a_{j,k-1}} - \frac{\varphi^{[1]}_{j,k}}{a_{j,k+1} - a_{j,k}}, \tag{21}$$

or

$$\frac{d}{dx}\varphi_j^{[0]^t} = D_j^{[1]}\varphi_j^{[1]^t}.$$

8. As we shall see later, one can use the commutation formula to compute the integral of the scaling function

$$\int_{\mathbf{R}} \varphi_{j,k}^{[1]}(x)\,dx = a_{j,k+1} - a_{j,k}.$$

9. We say that a collection of scaling functions $\widetilde{\varphi}_{j,k}$ generated by a subdivision scheme $\widetilde{S}$ is *biorthogonal* to the scaling functions $\varphi_{j,k}$ in case

$$\langle \varphi_{j,k}, \widetilde{\varphi}_{j,k'} \rangle = \int_{\mathbf{R}} \varphi_{j,k}\widetilde{\varphi}_{j,k'}\,dx = w_{j,k}\delta_{k,k'} \qquad \text{with} \quad \inf_k w_{j,k} > 0 \quad \text{for all } j.$$

We can rewrite this in matrix form as

$$\int_{\mathbf{R}} \varphi_j^t \widetilde{\varphi}_j\,dx = W_j,$$

where $W_j = \operatorname{diag}(w_j)$. We will often refer to the $\varphi_{j,k}$ as the *primal* scaling functions, while the $\widetilde{\varphi}_{j,k}$ are the *dual* scaling functions. We also refer to the schemes S and $\widetilde{S}$ as being biorthogonal. Biorthogonality now implies that

$$W_j = \int_{\mathbf{R}} \varphi_j^t \widetilde{\varphi}_j\,dx = \int_{\mathbf{R}} S_j^{[0]^t} \varphi_{j+1}^t \widetilde{\varphi}_{j+1} \widetilde{S}_j^{[0]}\,dx = S_j^{[0]^t} W_{j+1} \widetilde{S}_j^{[0]}.$$

Thus a necessary condition for biorthogonality between is S and $\widetilde{S}$:

$$S_j^{[0]^t} W_{j+1} \widetilde{S}_j^{[0]} = W_j. \tag{22}$$

Note that unlike the standard definition used for wavelets, biorthogonality here does not imply that $w_k = 1$. This is because requiring both S and $\widetilde{S}$ to be affine, as we shall do below, already fixes the normalization, so that we cannot fix $w_j = 1$. In fact it would be more consistent to refer to the more restrictive case $w_k = 1$ as biortho**norm**ality.

Note. The name *scaling function* is a little misleading. For regular nested grids the different scaling functions are translated and dilated copies of $\varphi_{0,0}$. In the irregular case this is no longer true.

10. Commutation Examples

In the two families of examples considered here, we shall assume that the multilevel grid X is two-nested, i.e., $x_{j+1,2k} = x_{j,k}$ for all j, k. Our first example concerns B-splines [1]; our second example discusses Lagrange interpolation [8], [10]. Note that in the B-spline example, to be consistent with standard B-spline notation, we have to use backward instead of forward differencing. This use of both backward and forward differencing will also turn out to fit exactly in the later discussion of biorthogonality and commutation; note that this also turns up in the regular case as reviewed in Section 2.

10.1. *B-Splines*

It is well known that B-splines of order p with knots $\{x_{j,k}\}$, normalized so that they constitute a partition of unity, satisfy the following refinement equation (see [14]):

$$N_{j,k}^{[p]}(x) = \sum_{t=0}^{p} b_{k,t;p}^{j} N_{j+1,2k+t}^{[p]}(x), \tag{23}$$

where the support of $N_{j,k}^{[p]}$ is $[x_{j,k}, x_{j,k+p}]$. The $b_{k,t;p}^{j}$ satisfy several recursive relations [14], of which the following are particularly relevant to us:

$$b_{k,t;p}^{j} = \frac{x_{j+1,2k+t+p-1} - x_{j,k}}{x_{j,k+p-1} - x_{j,k}} b_{k,t;p-1}^{j} + \frac{x_{j,k+p} - x_{j+1,2k+t+p-1}}{x_{j,k+p} - x_{j,k+1}} b_{k+1,t-2;p-1}^{j} \tag{24}$$

and

$$\begin{aligned} b_{k,t;p}^{j} \;&=\; b_{k,t-1;p}^{j} + \frac{x_{j+1,2k+t+p-1} - x_{j+1,2k+t}}{x_{j,k+p-1} - x_{j,k}} b_{k,t;p-1}^{j} \\ &\quad - \frac{x_{j+1,2k+t+p-1} - x_{j+1,2k+t}}{x_{j,k+p} - x_{j,k+1}} b_{k+1,t-2;p-1}^{j}; \end{aligned} \tag{25}$$

the starting conditions for both iterations are

$$\begin{aligned} b_{k,t;1}^{j} \;&=\; 0 \qquad \text{for} \quad t < 0 \quad \text{or} \quad t > 1, \\ b_{k,0;1}^{j} \;&=\; b_{k,1;1}^{j} = 1. \end{aligned}$$

Therefore, if $f(x) = \sum_k f_{j,k} N_{j,k}^{[p]}(x)$ is a general spline of order p with knots $\{x_{j,k}\}$, then the coefficients $f_{j,k}$ will be related through a subdivision scheme

$$f_{j+1,l} = \sum_{k} S_{j,l,k}^{BS(p)} f_{j,k}, \tag{26}$$

where the subdivision matrix elements are given by

$$S_{j,l,k}^{BS(p)} = b_{k,l-2k;p}^{j}. \tag{27}$$

(We use the superscript $BS(p)$ for "B-spline of order p" throughout.)

All polynomials up to the order p can be written as linear combinations of the $N_{j,k}^{[p]}$. In particular, if we define, for $p \geq 2$:

$$a_{j,k}^{BS(p)} = \frac{1}{p-1} \sum_{u=1}^{p-1} x_{j,k+u}, \tag{28}$$

then $\sum_k a_{j,k}^{BS(p)} N_{j,k}^{[p]} = x$ [11]. In the case of B-splines the linear producing sequences $a^{BS(p)}$ are referred to as *Greville* abscissae [11]. It follows that the $a_{j,k}^{BS(p)}$ are related through subdivision (one can also prove this directly by an inductive argument using the recursion (24)); they are thus sequences that produce linear functions for the subdivision scheme $S^{BS(p)}$ if $p \geq 2$. It follows that one can define the derived scheme for $S^{BS(p)}$ by applying our earlier theory. In fact this derived scheme is already known: note that using (27) and (28), equation (25) can be rewritten as

$$\frac{S^{BS(p)}_{j,2k+t,k} - S^{BS(p)}_{j,2k+t-1,k}}{a^{BS(p)}_{j+1,2k+t} - a^{BS(p)}_{j+1,2k+t-1}} = \frac{S^{BS(p-1)}_{j,2k+t,k}}{a^{BS(p)}_{j,k} - a^{BS(p)}_{j,k-1}} - \frac{S^{BS(p-1)}_{j,2k+t,k+1}}{a^{BS(p)}_{j,k+1} - a^{BS(p)}_{j,k}},$$

which is exactly a commutation formula (compare with (16)) using backward differences. (We will come back to this in Section 11.3.)

So the derived scheme, when using backward differences, for the B-spline subdivision of order p is nothing else but the B-spline subdivision scheme of order $p-1$.

10.2. *Lagrange Interpolating Subdivision*

As another example, we consider Lagrange interpolating subdivision of order q (see [7]), labeled by superscripts $LI(q)$. In this scheme:

- function values at even locations are simply copied

(29) $$f_{j+1,2k} = f_{j,k};$$

- Lagrange interpolation is used at odd locations, that is,

(30) $$f_{j+1,2k+1} = \lambda(x_{j+1,2k+1}),$$

  where λ is the unique polynomial of degree $q-1$ which interpolates function values on the previous level: $\lambda(x_{j,k+u}) = f_{j,k+u}$ for $\lceil (1-q)/2 \rceil \le u \le \lfloor q/2 \rfloor$.

Then the corresponding subdivision scheme coefficients are given by

(31) $$S^{LI(q)}_{j,2k,k+u} = \delta_u,$$

(32) $$S^{LI(q)}_{j,2k+1,k+u} = \prod_{\substack{\lceil (1-q)/2 \rceil \le v \le \lfloor q/2 \rfloor \\ v \ne u}} \frac{x_{j+1,2k+1} - x_{j,k+v}}{x_{j,k+u} - x_{j,k+v}}.$$

It follows from the definition that this scheme reproduces polynomial sequences up to degree $q-1$, that is,

$$S^{LI(q)}_j x_j^{q'} = x_{j+1}^{q'} \qquad \text{if} \quad 0 \le q' < q,$$

where $(x_j^q)_k = (x_{j,k})^q$.

In particular, $S^{LI(q)}$ produces linear functions, and the linear-producing sequences are just the x_j. One can thus again define a derived scheme $S^{LI(q)[1]}$.

For $q > 2$ it turns out that $S^{LI(q)[1]}$ again preserves linear functions, because $S^{LI(q)}$ preserves quadratics: since applying $S^{LI(q)}$ to $x_j^2/2$ leads to $x^2/2$, a simple differentiation, together with the commutation formula, implies that the sequences $D^{LI[1]}(x_j^2/2)$ lead to the limit $(x^2/2)' = x$. So the $a_j^{LI[1]}$, defined by

$$a^{LI[1]}_{j,k} = \frac{1}{2}\frac{x^2_{j,k+1} - x^2_{j,k}}{x_{j,k+1} - x_{j,k}} = \tfrac{1}{2}(x_{j,k+1} + x_{j,k}),$$

are linear-producing sequences for $S^{LI(q)[1]}$; note that the $a_j^{LI[1]}$ do not depend on q.

Similarly, one can use that $S^{LI(q)}$ preserves cubics (when $q > 3$) to show that $S^{LI(q)[2]}$ (well defined since $S^{LI(q)[1]}$ produces linear functions) also produces linear functions, and so on. It will turn out that the linear-producing sequences are given by

$$a_{j,k}^{LI[p]} := \left(\sum_{u=0}^{p} x_{j,k+u} \right) / (p+1).$$

To show that these do indeed work, let us introduce the compound difference operator $\mathcal{D}_j^{LI[p]} := D_j^{LI[p]} \dots D_j^{LI[1]}$, where the $D_j^{LI[p']}$ are the divided difference operators associated with the $a_j^{LI[p'-1]}$:

$$D_j^{[p]} := \frac{1}{\Delta a_j^{LI[p-1]}} \Delta. \tag{33}$$

Then the following lemma holds (proved in Appendix A).

Lemma 20.

$$\mathcal{D}_j^{LI[p]}[(x_j)^{p+1}] = a_j^{LI[p]}. \tag{34}$$

We will use this inductively. Assume that we have established that the $a_j^{LI[p']}$ are the linear-producing sequences for $S^{LI(q)[p']}$ for $p' < p < q$. Then $S^{LI(q)[p]}$ is well defined, and satisfies the "compound commutation" formula

$$S_j^{LI(q)[p]} \mathcal{D}_j^{LI[p]} = \mathcal{D}_{j+1}^{LI[p]} S_j^{LI(q)}. \tag{35}$$

Then

$$\begin{aligned} S_j^{LI(q)[p]} a_j^{LI[p]} &= S_j^{LI(q)[p]} \mathcal{D}_j^{LI[p]}(x_j^{p+1}) = \mathcal{D}_{j+1}^{LI[p]} S_j^{LI(q)}(x_j^{p+1}) \\ &= \mathcal{D}_{j+1}^{LI[p]}(x_{j+1}^{p+1}) = a_{j+1}^{LI[p]}. \end{aligned}$$

Since, on the other hand, the $a_j^{LI[p]}$ constitute a multilevel grid that obviously has X as its limit, it follows that $S^{LI(q)[p]}$ produces linear functions. We can then use this to define the affine scheme $S^{LI(q)[p+1]}$ and continue the argument.

Let us look at a few special cases:

- For $q = 2$, the Lagrange interpolating scheme leads to piecewise linear functions; in this case $S^{LI(2)}$ is a "shifted" version of $S^{BS(2)}$, and thus converges to a linear spline.
- For the cubic case, $q = 4$, we showed in [7] that the subdivision scheme converges to a $C^{2-\epsilon}$ limit function under some restrictions on the grid. Our analysis involved proving properties for the derived schemes $S_p^{LI(4)}$; in particular, we proved growth limits on the subdivision scheme $S_3^{LI(4)}$, which is a well-defined subdivision scheme that does not converge.
- For the quintic case, $q = 6$, we showed under similar restrictions that the limit functions are at least C^2.

10.3. *Orthogonal Scaling Functions on the Interval*

As we will later show, the B-spline and Lagrangian commutation examples are closely connected. We here give a third, unrelated example: commutation for orthogonal scaling functions on the interval as introduced in [4]. Consider the D4 orthogonal scaling function supported on [0, 3]:

$$\varphi(x) = \sqrt{2}\sum_{n=0}^{3} h_n \varphi(2x-n).$$

We know that

$$\sum_k \varphi(x-k) = 1 \qquad \text{and} \qquad \sum_k (x-k)\varphi(x-k) = M_1 = \frac{3-\sqrt{3}}{2}$$

and hence

$$\sum_k (k+M_1)\varphi(x-k) = x.$$

We only consider the left endpoint of the interval. Take, on $[0, \infty)$:

$$\begin{aligned}\widehat{\varphi}^{\,0}(x) &= 1\chi_{x\geq 0} - \sum_{k=1}^{\infty}\varphi(x-k) = \sum_{k=0}^{2}\varphi(x+k)\chi_{x\geq 0} \quad \text{(supported on [0, 3])},\\ \widehat{\varphi}^{\,-1}(x) &= \sum_{k=1}^{2} k\varphi(x+k)\chi_{x\geq 0} \quad \text{(supported on [0, 2])}.\end{aligned}$$

and the family $\{\varphi(x-k) = \varphi_k \mid k \geq 1\}$. We know that these boundary functions satisfy the following refinement equations:

$$\begin{aligned}\widehat{\varphi}^{\,0}(x) &= \widehat{\varphi}^{\,0}(2x) + \varphi_1(2x) + \frac{3-\sqrt{3}}{4}\varphi_2(2x) + \frac{1-\sqrt{3}}{4}\varphi_3(2x),\\ \widehat{\varphi}^{\,-1}(x) &= \tfrac{1}{2}\widehat{\varphi}^{\,-1}(2x) + \frac{3-\sqrt{3}}{4}\widehat{\varphi}^{\,0}(2x) + \frac{3-\sqrt{3}}{4}\varphi_1(2x).\end{aligned}$$

The sequence that produces $1\chi_{x\geq 0}$ is given by $0, 1, 1, 1, \ldots$ as

$$1\chi_{x\geq 0} = 0\widehat{\varphi}^{\,-1} + 1\widehat{\varphi}^{\,0} + \sum_{k\geq 1}\varphi_k.$$

The sequence $a_0 = (a_{0,k})_{k\geq -1}$ that produces $x\,\chi_{x\geq 0}$ is given by

$$-1+M_1 \quad 0+M_1 \quad 1+M_1 \quad 2+M_1 \quad 3+M_1 \ldots.$$

The boundary functions $\widehat{\varphi}^{\,-1}$ and $\widehat{\varphi}^{\,0}$ are orthogonal to the rest of the φ_k, but not orthogonal to each other. Also the associated subdivision scheme is not affine as starting with the all one sequence does not generate a constant. Indeed the constant generating sequence has it first coefficient equal to zero.

To make the basis functions orthogonal and the scheme affine, we propose to use a linear combination at the boundary given by

$$\begin{pmatrix} \varphi^0 \\ \varphi^{-1} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \widehat{\varphi}^0 \\ \widehat{\varphi}^{-1} \end{pmatrix}.$$

In order for the scheme to be affine we need that

$$1 = \varphi^{-1} + \varphi^0 + \sum_{k\geq 1} \varphi_k = (a+c)\widehat{\varphi}^0 + (b+d)\widehat{\varphi}^1 + \sum_{k\geq 1} \varphi_k = \widehat{\varphi}^0 + \sum_k \varphi_k.$$

Thus the scheme is affine in case $a + c = 1$ and $b + d = 0$. The transformation matrix is then given by

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & -d \\ 1-a & d \end{pmatrix}.$$

Its determinant is d, so we cannot choose d to be zero. The linear producing sequence $a_{j,0}$ after transformation is given by

$$\left[M_1 - \frac{a}{d}\right]\varphi^{-1} + \left[M_1 - \frac{a-1}{d}\right]\varphi^0 + \sum_{k=1}^{\infty}(M_1 + k)\varphi_k = x\,\chi_{x\geq 0}.$$

From this we see that if $a = d = 1$, the a_j form an arithmetic progression as $a_{j,k} = M_1 + k$. However this choice does not lead to orthogonality. Remember that $\widehat{\varphi}^0$ and $\widehat{\varphi}^{-1}$ are orthogonal to the φ_k, but not to each other and they are not normalized:

$$\begin{aligned} \langle\widehat{\varphi}^0, \widehat{\varphi}^0\rangle &= \widehat{p}_{0,0} = \frac{168 - 42\sqrt{3}}{84}, \\ \langle\widehat{\varphi}^0, \widehat{\varphi}^1\rangle &= \widehat{p}_{0,1} = \frac{189 - 104\sqrt{3}}{84}, \\ \langle\widehat{\varphi}^1, \widehat{\varphi}\rangle &= \widehat{p}_{1,1} = \frac{119 - 63\sqrt{3}}{84}. \end{aligned}$$

The inner products after transformation can be computed from

$$\begin{pmatrix} p_{0,0} & p_{0,1} \\ p_{0,1} & p_{1,1} \end{pmatrix} = A \begin{pmatrix} \widehat{p}_{0,0} & \widehat{p}_{0,1} \\ \widehat{p}_{0,1} & \widehat{p}_{1,1} \end{pmatrix} A^t.$$

This yields

$$\begin{aligned} p_{0,0} &= a^2\widehat{p}_{0,0} - 2ad\,\widehat{p}_{0,1} + d^2\widehat{p}_{1,1}, \\ p_{0,1} &= a(1-a)\widehat{p}_{0,0} + d(2a-1)\widehat{p}_{0,1} - d^2\widehat{p}_{1,1}, \\ p_{1,1} &= (1-a)^2\widehat{p}_{0,0} + 2d(1-a)\widehat{p}_{0,1} + d^2\widehat{p}_{1,1}. \end{aligned}$$

Two degrees of freedom is not sufficient to make the φ^0 and φ^1 orthonormal as that would require $p_{0,0} = 1 = p_{1,1}$, $p_{0,1} = 0$. We instead try to make $p_{0,1} = 0$ and $p_{0,0} = p_{1,1}$. Note that

$$\begin{aligned} p_{0,0} + p_{0,1} &= a\,\widehat{p}_{0,0} + d\,\widehat{p}_{0,1}, \\ p_{0,1} + p_{1,1} &= (1-a)\widehat{p}_{0,0} + d\,\widehat{p}_{0,1}. \end{aligned}$$

Thus, if $p_{0,1} = 0$, then

$$\begin{aligned} p_{0,0} &= a\,\widehat{p}_{0,0} + d\,\widehat{p}_{0,1}, \\ p_{1,1} &= (1-a)\,\widehat{p}_{0,0} + d\,\widehat{p}_{0,1}. \end{aligned}$$

For these to be equal, we need $a = \frac{1}{2}$; $p_{0,1} = 0$ then implies $\widehat{p}_{0,0}/4 = d^2\widehat{p}_{1,1}$ from which we see that

$$d = \frac{\sqrt{300 + 114\sqrt{3}}}{8\sqrt{13}} \approx 0.7732.$$

These choices make φ^0 and φ^1 orthogonal and equal in norm. We now have an orthogonal set of basis functions and an affine subdivision scheme. The linear producing sequence is not an arithmetic progression so the traditional commutation does not apply and our irregular commutation rule is needed. With it one can now build a derived subdivision scheme and associated boundary functions.

11. Biorthogonality and Commutation

11.1. *Setting*

In this section we explore how the commutation formula interacts with biorthogonality. The ambition is to obtain the same behavior as in the regular case where a commutation step on the primal side with forward differencing corresponds to an inverse commutation step on the dual side with backward differencing. This implies that, given one pair of biorthogonal scaling functions, we can construct others by means of differentiation and integration.

Suppose we start out with two convergent subdivision schemes $(S^{[0]}, X)$ and $(\widetilde{S}^{[0]}, X)$ with biorthogonal scaling functions $\varphi_{j,k}^{[0]}$ and $\widetilde{\varphi}_{j,k}^{[0]}$:

$$\int \varphi_{j,k}^{[0]}\widetilde{\varphi}_{j,l}^{[0]}\,dx = \delta_{k,l}\,w_{j,k}. \tag{36}$$

We assume that the primal scheme $(S^{[0]}, X)$ is differencible and, as before, we denote the linear-producing sequences by a_j; this means (see Definition 7) that the monotone sequences a_j are related through subdivision and have $\mathbf{X}$ as limit grid. As before, we define a divided difference operator as $D_j = \mathcal{A}_j^{-1}\Delta$, where $\mathcal{A}_j = \operatorname{diag}(\Delta a_j)$. By Theorem 15 we know that a derived subdivision scheme $S^{[1]}$ exists for which the following commutation formula holds:

$$D_{j+1}S_j^{[0]} = S_j^{[1]}D_j. \tag{37}$$

The purpose of this section is to find the scheme $\widetilde{S}^{[-1]}$ which is the "natural" dual to $S^{[1]}$ and show that it is related through inverse commutation to $\widetilde{S}^{[0]}$.

Note that in the regular setting the commutation on the primal side uses forward differences while the inverse commutation on the dual side uses backward differences. We shall likewise use backward differences on the dual side in the present irregular setting. We thus introduce the backward differencing operator $\widetilde{\Delta}\, f_k = f_k - f_{k-1}$. (Note that in every commutation formula and inverse commutation formula that we encountered, we can replace Δ by $\widetilde{\Delta}$, and obtain an equally valid formula for backward differences. The need for backward differencing on the dual side is quite natural given that $\widetilde{\Delta} = -\Delta^t$.) As before we will use a divided differencing operator given by $\widetilde{D}_j = \mathcal{B}_j^{-1}\widetilde{\Delta}$, where $\mathcal{B}_j = \operatorname{diag}(\widetilde{\Delta} b_j)$ and the b_j are monotone sequences yet to be determined. The inverse commutation we are looking for is now of the form

$$\widetilde{D}_{j+1}\widetilde{S}_j^{[-1]} = \widetilde{S}_j^{[0]}\widetilde{D}_j. \tag{38}$$

Note that in this section we use D and $\widetilde{D}$ as a shorthand notation for $D^{[1]}$ and $\widetilde{D}^{[0]}$.

11.2. *General Result*

Remember from (15) that the condition on $\widetilde{S}^{[0]}$ for the existence of inverse commutation is given by

$$\mathbf{1}^t\mathcal{B}_{j+1}\widetilde{S}_j^{[0]}\mathcal{B}_j^{-1} = C_j\mathbf{1}^t, \tag{39}$$

for some sequences b_j and constants C_j. Note that the change from Δ to $\widetilde{\Delta}$ has to be carried out consistently: the $\mathcal{B}_j$ in (15) were diagonal matrices with $\Delta b_{j,k}$ on the diagonal; here the entries are $\widetilde{\Delta} b_{j,k}$.

On the other hand, we can multiply (22) first on the right with W_j^{-1} and then on the left with $\mathbf{1}^t$; because the primal subdivision scheme $S_j^{[0]}$ is affine we obtain

$$\mathbf{1}^t S_j^{[0]^t} W_{j+1}\widetilde{S}_j^{[0]} W_j^{-1} = \mathbf{1}^t W_{j+1}\widetilde{S}_j^{[0]} W_j^{-1} = \mathbf{1}^t. \tag{40}$$

For $\mathcal{B}_j = W_j$ and $C_j = 1$, (39) and (40) coincide and we thus have shown the following lemma:

Lemma 21. *If S and $\widetilde{S}$ are dual with normalization W_j and S is affine, then $\widetilde{S}$ satisfies the condition for inverse commutation with $\mathcal{B}_j = W_j$ and $C_j = 1$.*

We stick with the choice $\mathcal{B}_j = W_j$ in what follows:

Remark. Note that the inverse is not true: if S and $\widetilde{S}$ are dual, and $\widetilde{S}$ satisfies the condition for inverse commutation with $\mathcal{B}_j = W_j$ and $C_j = 1$, this does not imply that S is affine.

It remains to check whether the subdivision schemes $S^{[1]}$ and $\widetilde{S}^{[-1]}$ are biorthogonal. Namely, we would like to find diagonal matrices $M_j = \operatorname{diag}(m_j)$ so that

$$S_j^{[1]^t} M_{j+1}\widetilde{S}_j^{[-1]} = M_j. \tag{41}$$

The diagram below shows the desired situation. Vertical arrows indicate forward commutation, and the horizontal links show the biorthogonality with the corresponding normalization. Lemma 21 showed that we should choose $\widetilde{\Delta} b_j = w_j$. The symmetries in the diagram suggest then that $m_j = \Delta a_j$ is a natural choice; this choice would turn the triangle of $S_j^{[0]}$, $\widetilde{S}_j^{[-1]}$, $S_j^{[1]}$ into another instance of Lemma 21.

$$\begin{array}{ccccc} S_j^{[0]} & \longleftarrow & w_j = \widetilde{\Delta} b_j & \longrightarrow & \widetilde{S}_j^{[0]} \\ {\scriptstyle D_j}\downarrow{\scriptstyle \Delta a_j} & & & & {\scriptstyle \widetilde{D}_j}\uparrow{\scriptstyle \widetilde{\Delta} b_j} \\ S_j^{[1]} & \longleftarrow & m_j \stackrel{?}{=} \Delta a_j & \longrightarrow & \widetilde{S}_j^{[-1]} \end{array}$$

We thus need to prove that

$$S_j^{[1]t} \mathcal{A}_{j+1} \widetilde{S}_j^{[-1]} = \mathcal{A}_j. \tag{42}$$

We first prove the equation obtained from multiplying (42) on the left by D_j^t, i.e.,

$$D_j^t S_j^{[1]t} \mathcal{A}_{j+1} \widetilde{S}_j^{[-1]} = \Delta^t. \tag{43}$$

Using the commutation formulas, the left-hand side of (43) reduces to

$$\begin{aligned} D_j^t (S_j^{[1]})^t \mathcal{A}_{j+1} \widetilde{S}_j^{[-1]} &= S_j^{[0]t} D_{j+1}^t \mathcal{A}_{j+1} \widetilde{S}_j^{[-1]} && \text{(forward commutation)} \\ &= -S_j^{[0]t} \widetilde{\Delta} \widetilde{S}_j^{[-1]} && \text{(definition of } D_j\text{)} \\ &= -S_j^{[0]t} \mathcal{B}_{j+1} \widetilde{D}_{j+1} \widetilde{S}_j^{[-1]} && \text{(definition of } \widetilde{D}_j\text{)} \\ &= -S_j^{[0]t} \mathcal{B}_{j+1} \widetilde{S}_j^{[0]} \widetilde{D}_j && \text{(inverse commutation)} \\ &= -\mathcal{B}_j \widetilde{D}_j && \text{(definition of } \widetilde{D}_j\text{)} \\ &= \Delta^t, \end{aligned}$$

which proves (43). To obtain (42), we observe that if U is a banded matrix and $\Delta^t U = \Delta^t$, then $U = \mathrm{Id}$.

11.3. *Construction of* $\widetilde{S}^{[-1]}$

This construction is similar to what is done in Section 8, except that we use backward differencing and correspondingly backward summations. We now can construct $\widetilde{S}^{[-1]}$ starting from (38):

$$\frac{\widetilde{S}_{j,l,m}^{[0]}}{b_{j,m} - b_{j,m-1}} - \frac{\widetilde{S}_{j,l,m+1}^{[0]}}{b_{j,m+1} - b_{j,m}} = \frac{\widetilde{S}_{j,l,m}^{[-1]} - \widetilde{S}_{j,l-1,m}^{[-1]}}{b_{j+1,l} - b_{j+1,l-1}}, \tag{44}$$

or

$$\frac{\widetilde{S}_{j,l,m}^{[0]}}{\widetilde{\Delta} b_{j,m}} - \frac{\widetilde{S}_{j,l,m+1}^{[0]}}{\widetilde{\Delta} b_{j,m+1}} = \frac{\widetilde{S}_{j,l,m}^{[-1]} - \widetilde{S}_{j,l-1,m}^{[-1]}}{\widetilde{\Delta} b_{j+1,l}}. \tag{45}$$

Since the scheme $\widetilde{S}^{[0]}$ is local, that is, $\widetilde{S}_{j,l,k}^{[0]} = 0$ for $|k - \lfloor l/2 \rfloor| > \widetilde{L}$, we can define the scheme $\widetilde{S}^{[-1]}$ to be local as well:

$$\widetilde{S}_{j,l,m}^{[-1]} = \sum_{t \le l} \widetilde{\Delta} b_{j+1,t} \left(\frac{\widetilde{S}_{j,t,m}^{[0]}}{\widetilde{\Delta} b_{j,m}} - \frac{\widetilde{S}_{j,t,m+1}^{[0]}}{\widetilde{\Delta} b_{j,m+1}} \right), \tag{46}$$

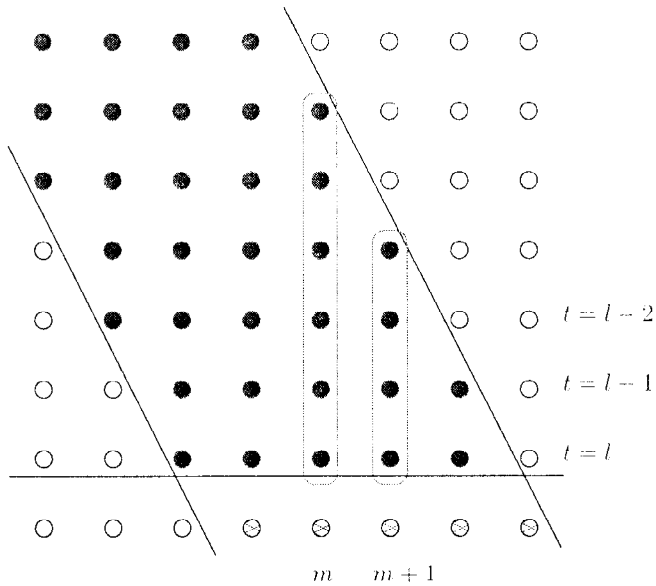

Fig. 2. Inverse commutation: the black dots represent all the nonzero $\widetilde{S}^{[0]}_{j,t,n}$ with $t \le l$; the two circled columns select the $\widetilde{S}^{[0]}_{j,t,n}$ that play a role in the definition of $\widetilde{S}^{[-1]}_{j,l,m}$ (see (47)).

or

$$(47)\qquad \widetilde{S}^{[-1]}_{j,l,m} = \frac{1}{\widetilde{\Delta} b_{j,m}} \sum_{t=2m-2\widetilde{L}}^{l} \widetilde{\Delta} b_{j+1,t} \widetilde{S}^{[0]}_{j,t,m} - \frac{1}{\widetilde{\Delta} b_{j,m+1}} \sum_{t=2(m+1)-2\widetilde{L}}^{l} \widetilde{\Delta} b_{j+1,t} \widetilde{S}^{[0]}_{j,t,m+1}.$$

11.4. *Finding the Linear Producing b_j*

We have identified a suitable inverse commutation leading to a scheme biorthogonal to $S^{[1]}_j$. The sequences of locations $b_{j,k}$ are not completely defined; we have fixed only the $\widetilde{\Delta} b_j$ thus far.

Now we would like to check whether this makes it possible to define the b_j so that they generate linear functions for $\widetilde{S}^{[-1]}$. Since we have only used the differences of b_j's so far, we can freely translate b_j within one level. As observed at the end of Section 8, we can choose these translations so that the b_j themselves, not only the $\widetilde{\Delta}\, b_j$, are related via subdivision, i.e., $\widetilde{S}^{[-1]}_j b_j = b_{j+1}$; we shall show that they also converge. At this point there will still be one overall translation degree of freedom left; this can be fixed, as we shall see below, so that the limit grid of the b_j is **X**.

In this section, we shall assume that not only $S^{[0]}$ but also $\widetilde{S}^{[0]}$ is affine. We start by deriving some useful properties of the $\widetilde{\Delta} b_{j,k}$. We have

$$\langle \varphi^{[0]}_{j,k}, \widetilde{\varphi}^{[0]}_{j,l} \rangle = \widetilde{\Delta} b_{j,k} \delta_{k,l}.$$

Because $S^{[0]}$ and $\widetilde{S}^{[0]}$ are affine the $\varphi_{j,k}^{[0]}$ and the $\widetilde{\varphi}_{j,k}^{[0]}$ form a partition of unity:

$$\sum_k \varphi_{j,k}^{[0]} = \sum_k \widetilde{\varphi}_{j,k}^{[0]} = 1. \tag{48}$$

It then follows that

$$\int \varphi_{j,k}^{[0]}\, dx = \int \widetilde{\varphi}_{j,k}^{[0]}\, dx = \widetilde{\Delta} b_{j,k}. \tag{49}$$

Summing equations (49) on level $j + j'$ for k from $2^{j'} K_1 + 1$ to $2^{j'} K_2$ leads to

$$\sum_{2^{j'} K_1+1 \le k \le 2^{j'} K_2} \int \varphi_{j+j',k}^{[0]}\, dx = b_{j+j',2^{j'}K_2} - b_{j+j',2^{j'}K_1}.$$

On the other hand, assuming that the $\varphi_{j,k}^{[0]}$ are uniformly bounded, the partition of unity together with the fact that $\varphi_{j,k}^{[0]}$ is supported in $[\mathbf{x}_{j,k-2L}, \mathbf{x}_{j,k+2L+1}]$, implies

$$\sum_{2^{j'} K_1+1 \le k \le 2^{j'} K_2} \int \varphi_{j+j',k}^{[0]}\, dx \longrightarrow \mathbf{x}_{j,K_2} - \mathbf{x}_{j,K_1} \qquad \text{as} \quad j' \to \infty.$$

We therefore have

$$\lim_{j'\to\infty} (b_{j+j',2^{j'}K_2} - b_{j+j',2^{j'}K_1}) = \mathbf{x}_{j,K_2} - \mathbf{x}_{j,K_1}. \tag{50}$$

We next prove that the sequence $b_{j,0}$ converges as $j \to \infty$. Note that the scheme $\widetilde{S}^{[-1]}$ is local, that is, $\widetilde{S}_{j,l,m}^{[-1]} = 0$ for $m \ge \lfloor l/2 \rfloor + \widetilde{L} + 1$ and for $m \le \lfloor l/2 \rfloor - \widetilde{L} - 2$. Then we have

$$\begin{aligned}
b_{j+1,0} &= \sum_{m=-\widetilde{L}-1}^{\widetilde{L}} \widetilde{S}_{j,0,m}^{[-1]} b_{j,m} \\
&= \sum_{m=-\widetilde{L}-1}^{\widetilde{L}} \left[\frac{1}{\widetilde{\Delta} b_{j,m}} \sum_{t=2m-2\widetilde{L}}^{0} \widetilde{\Delta} b_{j+1,t} \widetilde{S}_{j,t,m}^{[0]} \right. \\
&\qquad \left. - \frac{1}{\widetilde{\Delta} b_{j,m+1}} \sum_{t=2(m+1)-2\widetilde{L}}^{0} \widetilde{\Delta} b_{j+1,t} \widetilde{S}_{j,t,m+1}^{[0]} \right] b_{j,m} \\
&= \sum_{m=-\widetilde{L}-1}^{\widetilde{L}} \left[\sum_{t=2m-2\widetilde{L}}^{0} \frac{\widetilde{\Delta} b_{j+1,t} \widetilde{S}_{j,t,m}^{[0]} b_{j,m}}{\widetilde{\Delta} b_{j,m}} - \sum_{t=2(m+1)-2\widetilde{L}}^{0} \frac{\widetilde{\Delta} b_{j+1,t} \widetilde{S}_{j,t,m+1}^{[0]} b_{j,m}}{\widetilde{\Delta} b_{j,m+1}} \right] \\
&= \sum_{m=-\widetilde{L}-1}^{\widetilde{L}} \left[\sum_{t=2m-2\widetilde{L}}^{0} \frac{\widetilde{\Delta} b_{j+1,t} \widetilde{S}_{j,t,m}^{[0]} b_{j,m}}{\widetilde{\Delta} b_{j,m}} \right] - \sum_{m=-\widetilde{L}}^{\widetilde{L}+1} \left[\sum_{t=2m-2\widetilde{L}}^{0} \frac{\widetilde{\Delta} b_{j+1,t} \widetilde{S}_{j,t,m}^{[0]} b_{j,m-1}}{\widetilde{\Delta} b_{j,m}} \right]
\end{aligned}$$

$$= b_{j,\lfloor l/2\rfloor-\widetilde{L}-1} + \sum_{m=\lfloor l/2\rfloor-\widetilde{L}}^{\lfloor l/2\rfloor+\widetilde{L}} \left[\sum_{t=2m-2\widetilde{L}}^{0} \widetilde{\Delta} b_{j+1,t}\, \widetilde{S}^{[0]}_{j,t,m} \right].$$

It follows that

$$(51) \qquad b_{j+1,0} - b_{j,0} = (b_{j,0-\widetilde{L}-1} - b_{j,0}) + \sum_{m=-\widetilde{L}}^{\widetilde{L}} \left[\sum_{t=2m-2\widetilde{L}}^{0} \widetilde{\Delta} b_{j+1,t} \widetilde{S}^{[0]}_{j,t,m} \right].$$

Since the scheme $\widetilde{S}^{[0]}$ is bounded, and

$$|\widetilde{\Delta} b_{j,k}| \leq \int |\varphi^{[0]}_{j,k}| dx \leq \|\varphi^{[0]}_{j,k}\|_\infty |\mathbf{x}_{j,k+2L+1} - \mathbf{x}_{j,k-2L}| \leq C' d_j,$$

we can conclude from (51) that the sequence $b_{j,0}$ converges as $j \to \infty$. Remember that we still have one free overall translation parameter. We can fix it now so that $\lim_{j\to\infty} b_{j,0} = x_{0,0}$. It then follows from (50) $\lim_{j'\to\infty} b_{j+j',2^{j'}k} = \mathbf{x}_{j,k}$ for all j and k, and we have achieved convergence of the b_j to the limit grid $\mathbf{X}$, as desired. The following theorem summarizes the main results of this section:

Theorem 22. *Let* $(S^{[0]}, X)$ *and* $(\widetilde{S}^{[0]}, X)$ *be a pair of biorthogonal, local, bounded, and convergent subdivision schemes with normalization sequences* w_j *for* $j \in \mathbf{N}$. *Assume that* $S^{[0]}$ *is differencible, with linear producing sequences* a_j. *Then*:

- *We can define a derived scheme* $(S^{[1]}, X)$ *as in* (13); *this is a local affine subdivision scheme*; *it satisfies the commutation formula*

$$D_{j+1} S^{[0]}_j = S^{[1]}_j D_j.$$

- *We can also define a scheme* $(\widetilde{S}^{[-1]}, X)$ *for which* $(\widetilde{S}^{[0]}, X)$ *is the derived scheme, by means of* (17). *This is a local affine subdivision scheme that satisfies the inverse commutation formula*

$$\widetilde{D}^{[0]}_{j+1} \widetilde{S}^{[-1]}_j = \widetilde{S}^{[0]}_j \widetilde{D}^{[0]}_j.$$

  The differencing operator from $\widetilde{S}^{[-1]}$ *to* $\widetilde{S}^{[0]}$ *is linked to the normalization sequences*: $\widetilde{D}_j = W_j^{-1} \widetilde{\Delta}$.
- *The schemes* $S^{[1]}$ *and* $\widetilde{S}^{[-1]}$ *are biorthogonal with normalization sequences* Δa_j. *Moreover, the associated scaling functions satisfy*

$$\int \varphi^{[1]}_{j,k}\, dx = \int \widetilde{\varphi}^{[-1]}_{j,k}\, dx = m_{j,k} = \Delta a_{j,k}.$$

- *If, in addition,* $\widetilde{S}^{[0]}$ *is affine, and the* $\varphi^{[0]}_{j,k}$ *are uniformly bounded, then* $\widetilde{S}^{[-1]}$ *produces linear functions and the linear producing sequences* b_j *satisfy* $\widetilde{\Delta} b_j = w_j$.

12. Example of Commutation and Biorthogonality

Let us revisit our earlier examples of Section 10. We denote the corresponding scaling functions by $\varphi_{j,k}^{LI(q)}$ for the Lagrange interpolation schemes, and by $\varphi_{j,k}^{BS(p)} = N_{j,k}^{[p]}$ for the spline schemes. We start by showing that the $\varphi_{j,k}^{LI(q)[1]}$ and the $\varphi_{j,k}^{BS(1)}$ are biorthogonal.

For the case of Lagrange interpolating subdivision, (21) gives

$$\frac{d}{dx}\varphi_{j,k}^{LI(q)[0]} = \frac{\varphi_{j,k-1}^{LI(q)[1]}}{\Delta x_{j,k-1}} - \frac{\varphi_{j,k}^{LI(q)[1]}}{\Delta x_{j,k}},$$

Integrating both sides from $x_{j,k'}$ to $x_{j,k'+1}$ and using the fact that $\varphi_{j,k}^{LI(q)[0]}(x_{j,k'}) = \delta_{k,k'}$ we obtain

$$\delta_{k,k'+1} - \delta_{k,k'} = \frac{1}{\Delta x_{j,k-1}} \int_{x_{j,k'}}^{x_{j,k'+1}} \varphi_{j,k-1}^{LI(q)[1]}\,dx - \frac{1}{\Delta x_{j,k}} \int_{x_{j,k'}}^{x_{j,k'+1}} \varphi_{j,k}^{LI(q)[1]}\,dx. \tag{52}$$

Since $\varphi_{j,k'}^{BS(1)} = \chi_{[x_{j,k'},x_{j,k'+1})}$, we can rewrite the integrals as inner products of $\varphi^{LI(q)[1]}$ and $\varphi^{BS(1)}$ functions. Since $\langle\varphi_{j,l}^{LI(q)[1]}, \varphi_{j,k'}^{BS(1)}\rangle = 0$, if l and k' are sufficiently far apart, we can conclude, by summing (52) telescopically, that

$$\int_{\mathbf{R}} \varphi_{j,k}^{LI(q)[1]} \varphi_{j,k'}^{BS(1)}\,dx = \delta_{k,k'}\,\Delta x_{j,k}. \tag{53}$$

Thus the schemes $S^{LI(q)[1]}$ and $S^{BS(1)}$ constitute a biorthogonal pair and we can apply Theorem 22 which yields a biorthogonal pair $S^{LI(q)[2]}$ and $S^{BS(2)}$; note that the use of forward differencing on the primal side and backward differencing on the dual side, as dictated by Theorem 22, is entirely consistent with our discussion of the *LI* and *BS* examples in Section 10. We can repeat this procedure: as long as $p < q$, we can construct the derived schemes $S^{LI(q)[p]}$, and they will remain biorthogonal to the $S^{BS(p)}$ in the sense of (22):

$$S_j^{LI(q)[p]^t} W_{j+1}^p S_j^{BS(p)} = W_j^p,$$

with $w_{j,k}^p = \Delta a_{j,k}^{LI[p-1]} = \widetilde{\Delta} a_{j,k}^{BS(p+1)}$. Not all of these biorthogonal subdivision scheme pairs correspond to biorthogonal scaling functions, because as p increases, $S^{LI(q)[p]}$ will stop converging. For $q = 4$, for instance, we established in [7] that the scaling functions $\varphi_{j,k}^{LI(4)}$ are in $C^{2-\epsilon}$; $S^{LI(4)}$ and $S^{LI(4)[1]}$ converge, but $S^{LI(4)[p]}$ for $p = 2, 3$ do not. Although $\varphi_{j,k}^{LI(4)[2]}$ is not continuous it may well be square integrable, for some multilevel grids (it is for regular grids [6]); if the piecewise constant functions $\theta_{\{x_j,\, f_j\}}$ converge in L^2, then we would still have biorthogonality of $\varphi_{j,k}^{LI(4)[2]}$ and $\varphi_{j,k}^{BS(2)}$ in these cases. We conjecture that this will be true for multilevel grids that are not "too irregular."

Formally, we obtain the following diagram of biorthogonal affine subdivision pairs:

$$
\begin{array}{ccccc}
S_j^{LI(q)[0]} & & & & \\
{\scriptstyle a^{LI[0]}}\big\downarrow & & & & \\
S_j^{LI(q)[1]} & \longleftarrow & \Delta a^{LI[0]} = \widetilde{\Delta} a^{BS(2)} & \longrightarrow & S_j^{BS(1)} \\
{\scriptstyle a^{LI[1]}}\big\downarrow & & & & \big\uparrow{\scriptstyle a^{BS(2)}} \\
S_j^{LI(q)[2]} & \longleftarrow & \Delta a^{LI[1]} = \widetilde{\Delta} a^{BS(3)} & \longrightarrow & S_j^{BS(2)} \\
{\scriptstyle a^{LI[2]}}\big\downarrow & & & & \big\uparrow{\scriptstyle a^{BS(3)}} \\
S_j^{LI(q)[3]} & \longleftarrow & \Delta a^{LI[2]} = \widetilde{\Delta} a^{BS(4)} & \longrightarrow & S_j^{BS(3)} \\
{\scriptstyle a^{LI[3]}}\big\downarrow & & & & \big\uparrow{\scriptstyle a^{BS(4)}} \\
\ldots & & & & \ldots \\
{\scriptstyle a^{LI[q-2]}}\big\downarrow & & & & \big\uparrow{\scriptstyle a^{BS(q-1)}} \\
S_j^{LI(q)[q-1]} & \longleftarrow & \Delta a^{LI[q-2]} = \widetilde{\Delta} a^{BS(q)} & \longrightarrow & S_j^{BS(q-1)} \\
 & & & & \big\uparrow{\scriptstyle a^{BS(q)}} \\
 & & & & \ldots
\end{array}
$$

13. Wavelets and Commutation

In this section we study how commutation affects wavelets. We assume here that the reader is familiar with the basic concepts of wavelets and multiresolution analysis. Unlike classical wavelets, the wavelets we consider here are not translates and dilates of one fixed function. Rather they are instances of so-called second-generation wavelets [17].

Consider local, convergent, and differencible subdivision schemes $S^{[0]}$ and $\widetilde{S}^{[0]}$ which are biorthogonal and generate scaling functions $\varphi_{j,k}^{[0]}$ and dual scaling functions $\widetilde{\varphi}_{j,k}^{[0]}$. A collection of compactly supported functions $\widetilde{\psi}_{j,k}^{[0]}$ are called *dual wavelets* if

$$\widetilde{\psi}_{j,k}^{[0]} = \sum_l G_{j,l,k}^{[0]} \widetilde{\varphi}_{j+1,l}^{[0]}, \tag{54}$$

and

$$\langle \varphi_{j,k}^{[0]}, \widetilde{\psi}_{j,k'}^{[0]} \rangle = 0 \tag{55}$$

for $j \in \mathbf{N}$ and $k, k' \in \mathbf{Z}$. Note that since the $\varphi_{j,k}^{[0]}$ form a partition of unity, the wavelets integrate to zero,

$$\int_{\mathbf{R}} \widetilde{\psi}_{j,k}^{[0]} \, dx = 0. \tag{56}$$

After commutation, we need to find which are the right wavelets to pair with the $\varphi_{j,k}^{[1]}$. Consistent with earlier notation, we will denote them by $\widetilde{\psi}_{j,k}^{[-1]}$; as in the regular case, we define them to be the antiderivative of the $\widetilde{\psi}_{j,k}^{[0]}$:

$$\widetilde{\psi}_{j,k}^{[-1]}(x) = \int_{-\infty}^{x} \widetilde{\psi}_{j,k}^{[0]}(t)\,dt. \tag{57}$$

Given that the $\widetilde{\psi}_{j,k}^{[0]}$ are compactly supported and integrate to zero, the $\widetilde{\psi}_{j,k}^{[-1]}$ are compactly supported as well. To justify our definition, we need to show two things: that

$$\langle \varphi_{j,k}^{[1]}, \widetilde{\psi}_{j,k'}^{[-1]} \rangle = 0, \tag{58}$$

and that the $\widetilde{\psi}_{j,k}^{[-1]}$ can be written as a finite linear combination of the $\widetilde{\varphi}_{j+1,l}^{[-1]}$.

We first prove (58). Starting from (55) and using partial integration and (21) we obtain

$$\begin{aligned} 0 &= \langle \varphi_{j,k}^{[0]}, \widetilde{\psi}_{j,k'}^{[0]} \rangle = \langle \varphi_{j,k}^{[0]}, d/dx\, \widetilde{\psi}_{j,k'}^{[-1]} \rangle = -\langle d/dx\, \varphi_{j,k}^{[0]}, \widetilde{\psi}_{j,k'}^{[-1]} \rangle \\ &= \frac{\langle \varphi_{j,k}^{[1]}, \widetilde{\psi}_{j,k'}^{[-1]} \rangle}{\Delta a_{j,k}} - \frac{\langle \varphi_{j,k-1}^{[1]}, \widetilde{\psi}_{j,k'}^{[-1]} \rangle}{\Delta a_{j,k-1}}. \end{aligned}$$

Summing this up in a telescoping manner yields

$$\frac{\langle \varphi_{j,k}^{[1]}, \widetilde{\psi}_{j,k'}^{[-1]} \rangle}{\Delta a_{j,k}} - \frac{\langle \varphi_{j,k-l}^{[1]}, \widetilde{\psi}_{j,k'}^{[-1]} \rangle}{\Delta a_{j,k-l}} = 0.$$

Because of compact support, there exists a value of l for which the second term is zero. Consequently, (58) holds.

Next, we show that the $\widetilde{\psi}_{j,k}^{[-1]}$ can be written as a finite combination of $\widetilde{\varphi}_{j+1,l}^{[-1]}$. We can rewrite (54) as $\widetilde{\psi}_{j}^{[0]} = \widetilde{\varphi}_{j+1}^{[0]} G_j^{[0]}$. From (55), (20), (22) we then find

$$S_j^t W_{j+1} G_j^{[0]} = 0.$$

Multiplying on the left with $\mathbf{1}^t$ then gives

$$\sum_k w_{j+1,l} G_{j,l,k}^{[0]} = 0 \qquad \text{or} \qquad \sum_k \widetilde{\Delta} b_{j+1,l} G_{j,l,k}^{[0]} = 0. \tag{59}$$

Define now $G_{j,l,k}^{[-1]} = -\sum_{m \le l} \widetilde{\Delta} b_{j+1,m} G_{j,m,k}^{[0]}$. Because of (59), only finitely many of these are nonzero, for fixed j and k. On the other hand,

$$\begin{aligned} \frac{d}{dx}\left[\sum_l G_{j,l,k}^{[-1]} \widetilde{\varphi}_{j+1,l}^{[-1]}\right] &= \sum_l G_{j,l,k}^{[-1]} \left(\frac{\widetilde{\varphi}_{j+1,l+1}^{[0]}}{\widetilde{\Delta} b_{j+1,l+1}} - \frac{\widetilde{\varphi}_{j+1,l}^{[0]}}{\widetilde{\Delta} b_{j+1,l}} \right) \\ &= \sum_l (G_{j,l-1,k}^{[-1]} - G_{j,l,k}^{[-1]}) \frac{\widetilde{\varphi}_{j+1,l}^{[0]}}{\widetilde{\Delta} b_{j+1,l}} = \sum_l G_{j,l,k}^{[0]} \widetilde{\varphi}_{j+1,l}^{[0]} \\ &= \widetilde{\psi}_{j,k}^{[0]} = \frac{d}{dx} \widetilde{\psi}_{j,k}^{[-1]}. \end{aligned}$$

Since $\widetilde{\psi}_{j,k}^{[-1]}$ and $\sum_l G_{j,l,k}^{[-1]} \widetilde{\varphi}_{j+1,l}^{[-1]}$ both have compact support, equality follows.

A similar argument holds on the dual side. Given differentiable wavelet functions $\psi_{j,k}^{[0]}$ so that

$$\langle \widetilde{\varphi}_{j,k}^{[0]}, \psi_{j,k'}^{[0]} \rangle = 0,$$

we define new wavelets as derivatives

$$\psi_{j,k}^{[1]}(x) = -\frac{d}{dx}\psi_{j,k}^{[0]}(x);$$

if the $\psi_{j,k}^{[0]}$ are finite linear combinations of the $\varphi_{j+1,l}^{[0]}$, then this immediately implies that the $\psi_{j,k}^{[1]}$ are finite linear combinations of the $\varphi_{j+1,l}^{[1]}$. It then follows that

$$\langle \widetilde{\varphi}_{j,k}^{[-1]}, \psi_{j,k'}^{[1]} \rangle = -\frac{\langle \widetilde{\varphi}_{j,k}^{[0]}, \psi_{j,k'}^{[0]} \rangle}{\widetilde{\Delta} b_{j,k}} + \frac{\langle \widetilde{\varphi}_{j,k+1}^{[0]}, \psi_{j,k'}^{[0]} \rangle}{\widetilde{\Delta} b_{j,k+1}} = 0.$$

Note that our construction implies that if the original wavelets are biorthogonal, i.e.,

$$\langle \psi_{j,k}^{[0]}, \widetilde{\psi}_{j,l}^{[0]} \rangle = \delta_{k,l}$$

(we can take the normalization constant equal to 1 without loss of generality), then the $\psi_{j,k}^{[-1]}$, $\widetilde{\psi}_{j,l}^{[1]}$ will automatically be biorthogonal as well. The above results can be summarized in the following theorem:

Theorem 23. *Given a biorthogonal multiresolution analysis with functions* $(\varphi_{j,k}^{[0]}, \widetilde{\varphi}_{j,k}^{[0]}, \psi_{j,k}^{[0]}, \widetilde{\psi}_{j,k}^{[0]})$, *commutation allows us to build a new biorthogonal multiresolution analysis with functions* $(\varphi_{j,k}^{[1]}, \widetilde{\varphi}_{j,k}^{[-1]}, \psi_{j,k}^{[1]}, \widetilde{\psi}_{j,k}^{[-1]})$ *where the following relationships hold*:

$$\begin{aligned}
\frac{\varphi_{j,k-1}^{[1]}}{\Delta a_{j,k-1}} - \frac{\varphi_{j,k}^{[1]}}{\Delta a_{j,k}} &= \frac{d\varphi_{j,k}^{[0]}}{dx} \\
\widetilde{\varphi}_{j,k}^{[-1]}(x) &= \int_{\infty}^{x} \frac{\widetilde{\varphi}_{j,k+1}^{[0]}(t)}{\widetilde{\Delta} b_{j,k+1}} - \frac{\widetilde{\varphi}_{j,k}^{[0]}(t)}{\widetilde{\Delta} b_{j,k}}\, dt \\
\psi_{j,k}^{[1]}(x) &= -\frac{d}{dx}\psi_{j,k}^{[0]}(x) \\
\widetilde{\psi}_{j,k}^{[-1]}(x) &= \int_{-\infty}^{x} \widetilde{\psi}_{j,k}^{[0]}(t)\, dt.
\end{aligned}$$

14. Example of Wavelets and Commutation

Let us consider the two biorthogonal scaling function families $\varphi_{j,k}^{LI(q)[1]}$ and $\varphi_{j,k}^{BS(q)}$. Then we can define wavelets $\psi_{j,k}^{q[1]}$ and $\widetilde{\psi}_{j,k}^{q[1]}$ as follows:

$$\begin{aligned}
\psi_{j,k}^{q[1]} &= \frac{\varphi_{j+1,2k}^{LI(q)[1]}}{\Delta x_{j+1,2k}} - \frac{\varphi_{j+1,2k+1}^{LI(q)[1]}}{\Delta x_{j+1,2k+1}}, \\
\widetilde{\psi}_{j,k}^{q[1]} &= \sum_n \left[\frac{\Delta x_{j+1,2k+1}}{\Delta x_{j,n}} S_{j,2k+1,n}^{LI(q)[1]} \varphi_{j+1,2n}^{BS(1)} - \frac{\Delta x_{j+1,2k}}{\Delta x_{j,n}} S_{j,2k,n}^{LI(q)[1]} \varphi_{j+1,2n+1}^{BS(1)} \right].
\end{aligned}$$

These definitions are inspired by the construction of biorthogonal weighted wavelets in [16, Chapter 5]. Keep in mind though that the notation and normalization in [16] is quite different from ours. Orthogonality of the $\psi_{j,k}^{q[1]}$ to the $\varphi_{j,k}^{BS(1)}$ follows immediately from $\varphi_{j,k}^{BS(1)} = \varphi_{j+1,2k}^{BS(1)} + \varphi_{j+1,2k+1}^{BS(1)}$, combined with (53). To check orthogonality of the $\widetilde{\psi}_{j,k}^{q[1]}$ to the $\varphi_{j,k'}^{LI(q)[1]}$ we use (53) again, as well as

$$S_{j,2l,k}^{LI(q)[1]} \Delta x_{j+1,2l} + S_{j,2l+1,k}^{LI(q)[1]} \Delta x_{j+1,2l+1} = \delta_{l,k} \Delta x_{j,k}$$

which is (22) written out in detail. Using the shorthand (just for this argument) $y_{l,k} = S_{j,l,k}^{LI(q)[1]} \delta x_{j+1,l}$, we have indeed

$$\begin{aligned}
\langle \widetilde{\psi}_{j,k}^{q[1]}, \varphi_{j,k'}^{LI(q)[1]} \rangle &= \sum_{n,l} \frac{1}{\Delta x_{j,n}} (y_{2k+1,n} y_{l,k'} \delta_{l,2n} - y_{2k,n} y_{l,k'} \delta_{l,2n+1}) \\
&= \sum_{n} \frac{1}{\Delta x_{j,n}} ([\delta_{n,k} \Delta x_{j,k} - y_{2k,n}] y_{2n,k'} - [\delta_{n,k'} \Delta x_{j,k'} - y_{2n,k'}] y_{2k,n}) \\
&= 0.
\end{aligned}$$

Finally, we also check the biorthogonality of the $\psi_{j,k}^{q[1]}$ to the $\widetilde{\psi}_{j,k'}^{q[1]}$:

$$\begin{aligned}
\langle \widetilde{\psi}_{j,k}^{q[1]}, \psi_{j,k'}^{q[1]} \rangle &= \frac{1}{\Delta x_{j,k}} (S_{j,2k+1,k'}^{LI(q)[1]} \Delta x_{j+1,2k+1} + S_{j,2k,k'}^{LI(q)[1]} \Delta x_{j+1,2k}) \\
&= \frac{1}{\Delta x_{j,k}} \delta_{k,k'} \Delta x_{j,k'} = \delta_{k,k'}.
\end{aligned}$$

By the mechanisms given in the previous section, we can then define $\psi_{j,k}^{q[p]}$ and $\widetilde{\psi}_{j,k}^{q[p]}$ as long as the commutation arguments work. These $\widetilde{\psi}_{j,k}^{q[p]}$ are then compactly supported biorthogonal irregular knot B-spline wavelets, with a compactly supported biorthogonal family. In the regular case, they reduce to the compactly supported spline wavelets of [3].

Remarks.

- Formally it might have been easier to first establish biorthogonality between $LI(q)[0]$ and $BS(0)$ and then use commutation from there. Indeed $LI(q)[0]$ yields interpolating scaling functions and $BS(0)$ yields Dirac measures which are formally biorthogonal. However, the fact that $BS(0)$ is not affine and that it does not converge in the traditional sense makes the derivation quite awkward. Therefore we found it more natural to start with the $LI(q)[1]$ (average-interpolating) and $BS(1)$ (box functions).
- A different construction of compactly supported biorthogonal irregular knot B-spline wavelets and their duals is given in [5].

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Appendix A. Proof of Lemma 20

Lemma 20 is an easy consequence of the following general result:

Lemma 24. *For* $p = 1, \ldots, q-1$, *we have*

$$(\mathcal{D}_j^{[p]}[(x_j)^q])_k = \frac{1}{p!} \sum_{\substack{(\alpha_0,\ldots,\alpha_p) \\ \alpha_0+\cdots+\alpha_p=q-p}} (x_{j,k})^{\alpha_0}(x_{j,k+1})^{\alpha_1}\cdots(x_{j,k+p})^{\alpha_p}. \tag{60}$$

Proof. We prove this by induction. Equation (60) is obviously true for $p = 1$. Assume now that it holds for p. Let $b_k := D_j^{[p]} \cdots D_j^{[1]}[(x_{j,k})^q]$. By assumption we have

$$\begin{aligned}
b_{k+1} - b_k &= \frac{1}{p!}\left[\sum_{\substack{(\alpha_0,\ldots,\alpha_p) \\ \alpha_0+\cdots+\alpha_p=q-p}} (x_{j,k+1})^{\alpha_0}(x_{j,k+2})^{\alpha_1}\cdots(x_{j,k+p+1})^{\alpha_p}\right. \\
&\qquad \left. - \sum_{\substack{(\alpha_0,\ldots,\alpha_p) \\ \alpha_0+\cdots+\alpha_p=q-p}} (x_{j,k})^{\alpha_0}(x_{j,k+1})^{\alpha_1}\cdots(x_{j,k+p})^{\alpha_p}\right] \\
&= \frac{1}{p!}\left[\sum_{\alpha_p=0}^{q-p}(x_{j,k+p+1})^{\alpha_p} \sum_{\substack{(\alpha_0,\ldots,\alpha_{p-1}) \\ \alpha_0+\cdots+\alpha_{p-1}=q-p-\alpha_p}} (x_{j,k+1})^{\alpha_0}(x_{j,k+2})^{\alpha_1}\cdots(x_{j,k+p})^{\alpha_{p-1}}\right. \\
&\qquad \left. - \sum_{\alpha_0=0}^{q-p}(x_{j,k})^{\alpha_0} \sum_{\substack{(\alpha_1,\ldots,\alpha_p) \\ \alpha_1+\cdots+\alpha_p=q-p-\alpha_0}} (x_{j,k+1})^{\alpha_1}(x_{j,k+2})^{\alpha_2}\cdots(x_{j,k+p})^{\alpha_p}\right] \\
&= \frac{1}{p!}\sum_{\gamma=0}^{q-p}\left[((x_{j,k+p+1})^{\gamma} - (x_{j,k})^{\gamma})\right. \\
&\qquad \left. \times \sum_{\substack{(\beta_1,\ldots,\beta_p) \\ \beta_1+\cdots+\beta_p=q-p-\gamma}} (x_{j,k+1})^{\beta_1}(x_{j,k+2})^{\beta_2}\cdots(x_{j,k+p})^{\beta_p}\right] \\
&= \frac{1}{p!}(x_{j,k+p+1} - x_{j,k})\sum_{\gamma=0}^{q-p}\left[\sum_{\substack{(\beta_1,\ldots,\beta_p) \\ \beta_1+\cdots+\beta_p=q-p-\gamma}} (x_{j,k+1})^{\beta_1}(x_{j,k+2})^{\beta_2}\cdots(x_{j,k+p})^{\beta_p}\right. \\
&\qquad \left. \times \sum_{\beta_0=0}^{\gamma-1}(x_{j,k})^{\beta_0}(x_{j,k+p+1})^{\gamma-\beta_0}\right] \\
&= \frac{1}{(p+1)!}(a_{j,k+1}^{[p]} - a_{j,k}^{[p]}) \sum_{\substack{(\beta_0,\ldots,\beta_{p+1}) \\ \beta_0+\cdots+\beta_{p+1}=q-p-1}} (x_{j,k})^{\beta_0}(x_{j,k+1})^{\beta_1}\cdots(x_{j,k+p+1})^{\beta_{p+1}},
\end{aligned}$$

which proves the lemma. ∎