Discrete sets of coherent states and their use in signal analysis. *

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Abstract. We discuss expansions of L^2-functions into $\{\phi_{mn} ; m,n \in Z\}$, where the ϕ_{mn} are generated from one function ϕ, either by translations in phase space, i.e. $\phi_{mn}(x) = e^{imp_0x}\phi(x-nq_0)$, ($p_0$, q_0 fixed), or by translations and dilations, i.e. $\phi_{mn}(x) = a_0^{-m/2}\phi(a_0^{-m}x-nb_0)$. These expansions can be used for phase space localization.

1. Introduction.

We present here some recent results concerning expansions of functions $f \in L^2(\mathbb{R})$ with respect to discrete sets of coherent states. We shall distinguish two cases, the Weyl-Heisenberg case, where

$$g_{mn}(x) = e^{imp_0x}\, g(x-nq_0) \quad ,$$

and the affine case, where

$$h_{mn}(x) = a_0^{-m/2}\, h(a_0^{-m}x-nb_0) \quad .$$

In both cases the parameters m,n range over all of Z, and we shall discuss the maps T_{WH}, T_{AFF} from $L^2(\mathbb{R})$ to $l^2(Z^2)$ defined by

$$(T_{WH}f)_{mn} = \langle g_{mn}, f\rangle \quad , \quad (T_{AFF}f)_{mn} = \langle h_{mn}, f\rangle \quad .$$

These maps depend on the parameters $p_0,q_0>0$, $a_0>1$ and $b_0>0$, respectively, as well as on the functions g, $h \in L^2(\mathbb{R})$. The function h should satisfy the additional condition $\int |k|^{-1}|\hat{h}|^2 < \infty$, where $\hat{}$ denotes the Fourier transform.

The g_{mn} , h_{mn} are in fact coherent states associated with respectively the Weyl-Heisenberg group and the affine or $ax+b$-group, with labels restricted to discrete subsets of the parameter range. See [1] for more information concerning coherent states in general; a more extensive discussion of their connection to this paper can be found in the introduction of [2].

The maps T_{WH} , T_{AFF} and their properties are of interest for signal analysis. In engineering literature, the map T_{WH} is known as the "short-term Fourier transform". This is a procedure aimed at defining and computing a time-dependent frequency representation for a signal f. To do this,

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the signal f is multiplied by a "window function" g (often of compact support), and the Fourier coefficients of this product are computed. The process is repeated for different positions of the window g, leading to a frequency profile for f at different times. The resulting coefficients constitute exactly the series ($T_{WH} f$), with appropriately chosen p_0 , q_0 .

The use of T_{AFF} in signal analysis is not as wide-spread. It was first proposed by J.Morlet for the analysis of seismografic signals, where it seems to lead to better numerical results than T_{WH} [3]. The map T_{AFF} can probably be used for many other types of signals as well. Since the human ear analyses frequency in the same logarithmic way as T_{AFF} does (the number of m-levels needed to cover the region $\nu_1 < \nu < \nu_2$ is the same as for the region $2\nu_1 < \nu < 2\nu_2$), signal analysis based on T_{AFF} may be more efficient than the short-time Fourier transform for the analysis, filtering and reconstruction of speech or music.

It is a remarkable coincidence that the map T_{AFF} is also of interest to harmonic analysis, where techniques using dilations and translations have been extensively used for years (see e.g. [4]). Special choices for h_{mn} can be found in e.g. [5]; in [6] Y.Meyer constructs a function h, with $\hat{h} \in C_0^\infty$, such that the associated h_{mn} are an orthonormal base for $L^2(\mathbb{R})$ (this is generalized to more than one dimension in [7]); this base turns out to be an unconditional base for almost all useful function spaces [7][8].

We shall here discuss some mathematical properties of T_{WH} , T_{AFF} that are relevant for signal analysis. In remarks valid for both maps we shall drop the index WH or AFF, and we shall use the notation ϕ_{mn} in all formulas valid for both g_{mn} and h_{mn}.

It is clear that T should be injective if we want to be able to reconstruct f from Tf. In order to avoid instabilities in numerical computations, we impose the stronger condition that T should have a bounded inverse on its range. If we assume that T is defined on all of $L^2(\mathbb{R})$, this implies that T is bounded, by the closed graph theorem. All this means that we require, for some constants A, B , $0 < A \leq B < \infty$, that

$$A \leq T^*T \leq B \ , \tag{1}$$

or, equivalently, that

$$\forall f \in L^2(\mathbb{R}) \ : \ A\,||f||^2 \leq \sum_{m,n} |<\phi_{mn}, f>|^2 \leq B\,||f||^2 \ . \tag{2}$$

A set of vectors ϕ_{mn} satisfying (2) is called a *frame* (after Duffin and Schaeffer [9]). If the ϕ_{mn} are a frame, then, for all $f \in L^2(\mathbb{R})$,

$$f = \sum_{m,n} \psi_{mn} <\phi_{mn}, f> \ , \tag{3}$$

with $\psi_{mn} = (T^*T)^{-1}\phi_{mn}$, where the series converges in L^2-norm :

$$f = L^2\!-\!\lim_{K\to\infty} \sum_{\substack{m,n \\ |m|,|n| \leq K}} \psi_{mn} <\phi_{mn}, f>$$

Note that while (3) resembles the expansion with respect to biorthogonal bases, it definitely is not the same: the ϕ_{mn} need not be a base, and $<\psi_{mn}, \phi_{m'n'}> \neq \delta_{mm'}\ \delta_{nn'}$ in general.

We shall call the constants A, B in (1), (2) *frame bounds*. For some special frames, the largest lower frame bound and the smallest upper frame bound coincide, i.e.

$$T^*T = A\ \mathbf{1} \quad \text{or} \quad \forall f \in L^2(\mathbb{R}) : \sum_{m,n} |<\phi_{mn}, f>|^2 = A\ ||f||^2 \ .$$

In this case we say that the ϕ_{mn} constitute a *tight frame*. For a tight frame the expansion formula (3) becomes

$$f = A^{-1} \sum_{m,n} \phi_{mn} <\phi_{mn}, f> \quad . \tag{4}$$

Note again that the ϕ_{mn} in (4) need not be an orthogonal base.

This paper is organized as follows. Section 2 addresses frame questions. Basically we show two things, for the affine case as well as for the Weyl-Heisenberg case : 1) how to construct tight frames , 2) how to determine, for a *given* function g or h a range of parameters (p_0, q_0) or (a_0, b_0) such that the resulting g_{mn}, h_{mn} constitute a frame. The same methods can also be used to compute frame bounds, and we give a few numerical examples. In section 3 we discuss phase space localization. Let us illustrate what this means by an example in the Weyl-Heisenberg case. Take $g(x) = \pi^{-1/4} \exp(-x^2/2)$, $q_0 = p_0 = \pi^{1/2}$. In this case (see [10]) the g_{mn} do constitute a frame. Since $\int dx\ x\,|g(x)|^2 = 0 = \int dy\ y\,|\hat{g}(y)|^2$, it follows that g_{mn} is concentrated, in phase space, around the point (mp_0, nq_0) , i.e. $\int dx\ x\,|g_{mn}(x)|^2 = nq_0$, $\int dy\ y\,|(g_{mn})^\wedge(y)|^2 = mp_0$. Therefore the inner product $<g_{mn}, f>$ "measures" the phase space content of the signal f around the point (mp_0,nq_0) . If one knows a priori that f is mostly localized in phase space, i.e.

$$\int_{|t| \geq T} dt\ |f(t)|^2 \leq \delta \quad , \quad \int_{|\omega| \geq \Omega} d\omega\ |\hat{f}(\omega)|^2 \leq \delta \ , \ \text{for some } \delta \ll ||f||^2 \ ,$$

then it seems reasonable to expect that the partial reconstruction of f, using only those $<g_{mn}, f>$ for which $|m\, p_0| \leq \Omega$, $|n\, q_0| \leq T$, would be close to f. This is in fact correct; a precise statement is given in section 3. A similar result holds for h_{mn}-frames.

Part of the results in section 2 have been published before [2][10]; the rest of the material has not been published yet. An extended version of these results will appear elsewhere [11].

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2. Frames and frame bounds.

2A. The Weyl-Heisenberg case

In this case there exists a critical value for the product p_0q_0. The following theorem states that if $p_0q_0 = 2\pi$, only functions g which are either not very smooth or don't decay very fast, give rise to a frame.

Theorem 1. *Choose $g \in L^2(\mathbb{R})$, $p_0q_0 = 2\pi$. If the associated g_{mn} constitute a frame, then either $xg \notin L^2$ or $g' \notin L^2$.*

This theorem was first formulated by R.Balian [12]; his proof contains a gap that was filled recently by R.Coifman and S.Semmes [13]. We give here a sketch of their proof.

Proof. The proof uses the Zak transform U_Z which maps $L^2(\mathbb{R})$ unitarily onto $L^2([0,1]^2)$,

$$(U_Zf)(t,s) = q_0^{1/2} \sum_{l \in Z} e^{2\pi itl} f(q_0(s-l)) \quad .$$

U_Zf can be defined for $(t,s) \in \mathbb{R}^2$ (instead of $[0,1]^2$) ; in that case it satisfies

$$(U_Zf)(t+1,s) = (U_Zf)(t,s) \quad , \tag{5a}$$

$$(U_Zf)(t,s+1) = e^{2\pi it} (U_Zf)(t,s) \quad . \tag{5b}$$

Using $(U_Zg_{mn})(t,,s) = e^{-2\pi itn} e^{2\pi ims} (U_Zg)(t,s)$, one finds that the g_{mn} constitute a frame, with frame bounds A, B , if and only if

$$A^{1/2} \leq |(U_Zg)(t,s)| \leq B^{1/2} \quad . \tag{6}$$

If U_Zg were continuous, then (5), together with (6), would lead to a contradiction (see [14]; a very detailed proof can be found in [15]). Basically the argument is as follows. If U_Zg were continuous, then $\log U_Zg$ would be a continuous, one-valued function (because of (6)). But (5) allows one to compute two *different* values for $\log(U_Zg)(1,1)$, corresponding to the two paths $(0,0) \to (0,1) \to (1,1)$ and $(0,0) \to (1,0) \to (1,1)$. This is clearly a contradiction.
Unfortunately, U_Zg need not be continuous. One easily checks that if g' , $xg \in L^2$, then $\partial_t U_Zg$, $\partial_s U_Zg \in L^2_{loc}(\mathbb{R}^2)$. This does not imply that U_Zg is continuous (this is where Balian's proof failed : he implicitly assumes that $\nabla G \in L^2$ implies that G is continuous, which is true in 1 but not in 2 dimensions), but one can use this to show that there exists $r > 0$, with r small enough to ensure that the *continuous* function G_r ,

$$G_r(t,s) = r^{-2} \int_{|t'-t| \leq r} dt' \int_{|s'-s| \leq r} ds' \, (U_Zg)(t',s') \quad ,$$

is close enough to U_Zg to satisfy

$$A^{1/2}/2 \leq |G_r(t,s)| \leq 2B^{1/2}$$

$$G_r(t+1,s) = G_r(t,s)$$

$$G_r(t,s+1) = e^{2\pi it}\, G_r(t,s) + R(t,s)$$

with $|R(t,s)| \leq A^{1/2}\,\pi/16$. This then leads to a contradiction.

□

This theorem shows that it is impossible, starting from a reasonably smooth and decreasing function g, to construct a frame if $p_0q_0 = 2\pi$. An example of this phenomenon is the von Neumann lattice, also called the set of Gabor wave functions, which correspond to $p_0q_0 = 2\pi$, $g(x) = \pi^{-1/4}\exp(-x^2/2)$. The corresponding g_{mn} (which were, in fact, proposed by Gabor [16] for signal analysis purposes) lead to very bad numerical reconstruction results [17], which is due to the fact that the g_{mn} do not constitute a frame (see also [10]), even though T_{WH} is injective [18][19][14].

For $p_0q_0 > 2\pi$ it is known that T_{WH} is not even injective if $g(x) = \pi^{-1/4}\exp(-x^2/2)$ [18][19]. I believe this result to be generally true, i.e. T_{WH} should not be injective, if $p_0q_0 > 2\pi$, for *any* $g \in L^2$. For rational values of $p_0q_0/2\pi$ this can easily be shown, again by means of the Zak transform. The proof for irrational values of $p_0q_0/2\pi$ seems to be harder.

For any choice of $p_0q_0 < 2\pi$, it is possible to construct functions $g \in C_0^\infty(\mathbb{R})$ such that the associated g_{mn} constitute a tight frame. The following construction works if $\pi \leq p_0q_0 \leq 2\pi$. Similar constructions can be made if $p_0q_0 < \pi$ [2]. Let ν be an increasing C^∞-function from $\mathbb{R}$ to $\mathbb{R}$ such that $\nu(x) = 0$ if $x \leq 0$, $\nu(x) = 1$ if $x \geq 1$. Define $g(x)$ by

$$g(x) = \begin{cases} \sin\left[\dfrac{\pi}{2}\,\nu\left(\dfrac{p_0x}{2\pi-p_0q_0}\right)\right] & \text{if } x \leq \dfrac{2\pi}{p_0} - q_0 \\ 1 & \text{if } \dfrac{2\pi}{p_0} - q_0 \leq x \leq q_0 \\ \cos\left[\dfrac{\pi}{2}\,\nu\left(\dfrac{p_0(x-q_0)}{2\pi-p_0q_0}\right)\right] & \text{if } x \geq q_0 \, . \end{cases}$$

Since $\operatorname{supp} g = [0,\frac{2\pi}{p_0}]$, one finds, $\forall f \in L^2(\mathbb{R})$: $\sum_m |\langle g_{mn}, f\rangle|^2 = \frac{2\pi}{p_0}\int dx\; |g(x-nq_0)|^2\; |f(x)|^2$. Since, by construction, $\sum_n |g(x-nq_0)|^2 = 1$, this implies that $\sum_{m,n} |\langle g_{mn}, f\rangle|^2 = \frac{2\pi}{p_0}\; \|f\|^2$, i.e. the g_{mn} constitute a tight frame.

In the above construction, p_0 and q_0 are fixed, and an appropriate g is constructed. In situations where g is already fixed, the following theorem gives sufficient conditions on p_0, q_0 (and g) ensuring that the g_{mn} are a frame.

Theorem 2.

If 1. $m(q_0) = \inf_{x \in [0,q_0)} \sum_n |g(x-nq_0)|^2 > 0$

2. $\sup_{s \in \mathbb{R}} \left[(1+s^2)^{(1+\epsilon)/2} B(s,q_0) \right] = C_\epsilon < \infty$ *for some* $\epsilon > 0$,
where $B(s) = \sup_{x \in [0,q_0)} \sum_n |g(x-nq_0)| \, |g(x+s-nq_0)|$,

then there exists a $p_0^c > 0$ *such that*

$\forall p_0 < p_0^c$: *the* g_{mn} *are a frame*

$\forall p_0 > p_0^c$: $\exists\, p'_0 \in [p_0^c, p_0]$ *for which the* g_{mn} *are not a frame* .

Proof. Using the Poisson formula one finds

$$\sum_{m,n} |\langle g_{mn}, f\rangle|^2 = \frac{2\pi}{p_0} \sum_{n,k} \int dx \; g(x-nq_0)\, g(x-nq_0-\frac{2\pi}{p_0}k)^* f(x)^* f(x-\frac{2\pi}{p_0}k) \ . \qquad (7)$$

Via the Cauchy-Schwarz inequality this leads to

$$\sum_{m,n} |\langle g_{mn}, f\rangle|^2 \geq \frac{2\pi}{p_0} \|f\|^2 \left\{ m(q_0) - 2C_\epsilon \sum_{k=1}^{\infty} \left[1 + \left(\frac{2\pi k}{p_0} \right)^2 \right]^{-(1+\epsilon)/2} \right\} .$$

Since this lower bound is positive for p_0 small enough, we find
$p_0^c = \inf\{ p_0 \,;$ *the* g_{mn} *do not constitute a frame* $\} > 0$.

□

Remarks

1. If $xg, g' \in L^2$, then $p_0^c \leq 2\pi/q_0$, by theorem 1.
2. Equation (7) can also be used to show that T_{WH} is bounded for all p_0 .
3. Condition 2. is satisfied if e.g. $(1+x^2)^{(1+\epsilon)/2} (|g| + |g'|) \in L^2(\mathbb{R})$.
4. If $\sum_n |g(x-nq_0)|^2$ is continuous, then condition 1. is necessary.
5. One can use the argument in the proof to compute frame bounds. The following table lists some such frame bounds, for $g(x) = \pi^{-1/4} \exp(-x^2/2)$.

q_0	p_0	A	B
1.	π	.60	3.55
	$3\pi/2$	.03	3.55
2.	$\pi/2$	1.60	2.43
	$3\pi/4$	.58	2.09

2B. The affine case.

There is no analog of the Zak transform for the affine case, and therefore no analog of theorem 1. For any $a_0 > 1$, $b_0 > 0$ it is possible to construct h such that the associated h_{mn} constitute a frame. One such example is given by the following construction. We take the same function ν as in § 2A, and we define $l = 2\pi/[b_0(a_0^2-1)]$. Then $\hat{h}$ is defined by

$$\hat{h}(y) = \begin{cases} 0 & \text{if } y \le l \\ \sin\left[\frac{\pi}{2}\,\nu\left(\frac{y-l}{l(a_0-1)}\right)\right] & \text{if } l \le y \le a_0 l \\ \cos\left[\frac{\pi}{2}\,\nu\left(\frac{y-a_0 l}{a_0 l(a_0-1)}\right)\right] & \text{if } y \ge a_0 l \end{cases}$$

Clearly $\operatorname{supp}\hat{h} = [l, a_0^2 l]$. Define $h^+(y) = h(y)$, $h^-(y) = h(y)^*$. Then the $h_{mn}^{\pm}$ constitute a frame. Explicit calculation shows that

$$\sum_{+,-}\sum_{n} |\langle h_{mn}^{\pm}, f\rangle|^2 = \frac{2\pi}{b_0}\int_0^\infty dy\; |\hat{h}(a_0^m y)|^2\left[\,|\hat{f}(y)|^2 + |\hat{f}(-y)|^2\,\right].$$

By construction $\sum_m |\hat{h}(a_0^m y)|^2 = \chi_{(0,\infty)}(y)$. Hence $\sum_{+,-}\sum_{m,n} |\langle h_{mn}^{\pm}, f\rangle|^2 = \frac{2\pi}{b_0}\,\|f\|^2$.

Remark

One can also choose $h^1 = \operatorname{Re} h$, $h^2 = \operatorname{Im} h$; the set $\{ h_{mn}^{\epsilon}\ ; m,n \in Z\ , \epsilon = 1 \text{ or } 2 \}$ again constitutes a tight frame.

As in § 2A, one can again consider the situation in which h itself is fixed, and determine a_0, b_0 such that the associated h_{mn} constitute a frame. This allows, in particular, choices where h rather than $\hat{h}$ has compact support.

Theorem 3.

If

1. $m(a_0) = \inf_{1\le |y| \le a_0} \sum_m |\hat{h}(a_0^m y)|^2 > 0$
2. $\sup_{s\in\mathbb{R}} \left[(1+s^2)^{(1+\epsilon)/2}\, B(s) \right] = C_\epsilon < \infty$ for *some* $\epsilon > 0$,
   with $B(s) = \sup_{1\le |y| \le a_0} \sum_m |\hat{h}(a_0^m y)|\; |\hat{h}(a_0^m y + s)|$,

then there exists $b_0^c > 0$ *such that, for all* $b_0 < b_0^c$, *the associated* h_{mn} *constitute a frame.*

Proof. The proof is entirely analogous to the WH-case : one uses $(h_{mn})^\wedge(y) = a_0^{m/2}\exp[inb_0 a_0^m y]\,\hat{h}(a_0^m y)$, and applies the Poisson formula.

□

Again this can be used to compute frame bounds for given h, a_0, b_0. The following two tables give the values of b_0^c for a few a_0-values, and give frame bounds for $b_0 = 1. < b_0^c$, for the two functions $h(x) = 2\, 3^{-1/2}\, \pi^{-1/4}\, (1-x^2)\exp(-x^2/2)$ and $h(x) = \sin x\; \chi_{|x|\le\pi}$.

$$h(x) = 2\,3^{-1/2}\,\pi^{-1/4}\,(1-x^2)\,\exp(-x^2/2)$$

a_0	b_0^c	A for $b_0=1.$	B for $b_0=1.$	B/A for $b_0=1.$
1.25	2.05	10.53	10.65	1.01
1.5	2.05	5.79	5.86	1.01
2.0	1.91	3.25	3.57	1.10

$$h(x) = \pi^{-1/2}\,\sin x\;\chi_{|x|\leq\pi}$$

a_0	b_0^c	A for $b_0=1.$	B for $b_0=1.$	B/A for $b_0=1.$
1.25	2.95	18.51	20.81	1.12
1.5	2.91	10.09	11.55	1.14
2.0	2.80	5.70	6.98	1.22

3. Phase space localization.

We explained in the introduction the intuition behind the phase space localization results we present here. The precise statement is :

Theorem 4.

Let g be a function such that, for some $\delta > 0$, $\int dx \left[|g(x)|^2 + |g(x)|\,|g'(x)| \right] (1+x^2)^{1+\delta} < \infty$, and $\int dy \left[|\hat{g}(y)|^2 + |\hat{g}(y)|\,|\hat{g}'(y)| \right] (1+y^2)^{1+\delta} < \infty$. Let $p_0, q_0 > 0$. Suppose that the g_{mn} constitute a frame, with frame bounds A, B. Then, $\forall \epsilon > 0$, $\exists$ $\alpha(\epsilon)$, $\beta(\epsilon)$ such that, for all $T, \Omega > 0$,

$$\Big\| f - \sum_{\substack{m,n \in Z \\ |mp_0| \leq \Omega+\alpha(\epsilon) \\ |nq_0| \leq T+\beta(\epsilon)}} \psi_{mn} \langle g_{mn}, f\rangle \Big\|$$

$$\leq (B/A)^{1/2} \left\{ \left[\int_{|t|\geq T} dt\; |f(t)|^2 \right]^{1/2} + \left[\int_{|\omega|\geq\Omega} d\omega\; |\hat{f}(\omega)|^2 \right]^{1/2} + \epsilon\, \|f\| \right\}$$

Proof. As in the introduction, $\psi_{mn} = (T^*T)^{-1}\, g_{mn}$. We denote $|mp_0| \leq \Omega+\alpha$, $|nq_0| \leq T+\beta$ as $(m,n) \in box$. Introducing $(P_T f)(t) = f(t)\,\chi_{|t|\leq T}$ and $(Q_\Omega f)^\wedge(\omega) = \hat{f}(\omega)\,\chi_{|\omega|\leq\Omega}$, one finds, after some manipulations,

$$\| f - \sum_{m,n \in box} \psi_{mn} \langle g_{mn}, f \rangle \| \leq \sup_{\|\phi\|=1} \sum_{m,n \notin box} |\langle \phi, \psi_{mn} \rangle| \, |\langle g_{mn}, f \rangle|$$

$$\leq (B/A)^{1/2} \Big[\|(\mathbf{1}-P_T)f\| + \|(\mathbf{1}-Q_\Omega)f\| \Big]$$

$$+ A^{-1/2} \left\{ \left(\sum_{\substack{m,n \\ |mp_0|>\Omega+\alpha}} |\langle g_{mn}, Q_\Omega f \rangle|^2 \right)^{1/2} + \left(\sum_{\substack{m,n \\ |nq_0|>T+\beta}} |\langle g_{mn}, P_T f \rangle|^2 \right)^{1/2} \right\} .$$

Using the Poisson formula and the Cauchy-Schwarz formula leads to

$$\sum_{\substack{m,n \\ |nq_0|>T+\beta}} |\langle g_{mn}, P_T f \rangle|^2 \leq \frac{2\pi}{p_0} \left[\sum_k \left(1 + k^2 \frac{4\pi^2}{p_0^2} \right)^{-(1+\delta)/2} \right] \|P_T f\|^2$$

$$\times \int_{|x|>\beta} dx \left[|g(x)|^2 (1+x^2)^{1+\delta} + 2\, q_0\, |g(x)|\, (1+x^2)^{\delta} \Big(|g'(x)|\, (1+x^2) + x\, |g(x)| \Big) \right] .$$

The Q_Ω-term can be estimated similarly. The theorem then follows easily.

□

A similar theorem holds for the affine case.

The important fact about theorem 4 is that $\alpha(\epsilon)$, $\beta(\epsilon)$ *are independent* of T, Ω . Note that $\alpha(\epsilon)$, $\beta(\epsilon) \to \infty$ as $\epsilon \to 0$, i.e. infinite precision is only reached by taking the infinite collection $(\langle g_{mn}, f \rangle)_{m,n \in Z}$. If both $g, \hat{g}$ have rapid decay at ∞ (e.g. g Gaussian), then α, β are still reasonable for fairly small ϵ (for g Gaussian, $q_0 = p_0 = \pi^{1/2}$, one has $\alpha(\epsilon) = \beta(\epsilon) \underset{\epsilon \to 0}{\sim} (C + |\ln \epsilon|)^{1/2}$).

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