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SQUEEZED STATES AND PATH INTEGRALS

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ABSTRACT

The continuous-time regularization scheme for defining phase-space path integrals is briefly reviewed as a method to define a quantization procedure that is completely covariant under all smooth canonical coordinate transformations. As an illustration of this method, a limited set of transformations is discussed that have an image in the set of the usual squeezed states. It is noteworthy that even this limited set of transformations offers new possibilities for stationary phase approximations to quantum mechanical propagators.

1. INTRODUCTION

For many years now it has been customary to define path integrals with the aid of coherent states [1]. Such formulations have been developed not only for the canonical coherent states suitable for the Weyl group (i.e., the Heisenberg algebra), but for coherent states based on other groups as well, notably the unitary and orthogonal groups with (in)definite signature, the affine group, etc. However, for the sake of convenience and to focus on the relation with standard squeezed states, attention in this paper will be confined to path integrals constructed with the aid of canonical coherent states. The construction of coherent state path integrals is generally carried out in one of two standard procedures [2]. To illustrate these two procedures let us first introduce a few standard definitions involving coherent states [2]:

$$1 = \int |p,q\rangle\langle p,q| \, dpdq/2\pi,$$
$$H(p,q) = \langle p,q|\mathcal{H}|p,q\rangle,$$
$$H(p_2,q_2;p_1,q_1) = \langle p_2,q_2|\mathcal{H}|p_1,q_1\rangle/\langle p_2,q_2|p_1,q_1\rangle,$$
$$\mathcal{H} = \int h(p,q)\, |p,q\rangle\langle p,q| \, dpdq/2\pi,$$

where $|p,q\rangle$, $(p,q)\,\epsilon I\!R^2$, denotes one of a collection of coherent states defined by

$$|p,q\rangle = e^{-iqP}e^{ipQ}|0\rangle, \quad [Q,P]=i, \quad (Q+iP)\,|0\rangle = 0,$$

all of which are normalized, $\langle p,q|p,q\rangle = 1$. In addition, we have introduced two "symbols" associated with a fairly general operator $\mathcal{H}$, namely H and h as functions on phase space. In terms of these quantities, the propagator form time t' to time $t'' = t' + T$, $T > 0$, is given by either of the two expressions

$$\langle p'',q''|e^{-i\mathcal{H}T}|p',q'\rangle = \lim_{N\to\infty}\int\cdots\int \times$$
$$\prod_{n=0}^{n=N}\langle p_{n+1},q_{n+1}|p_n,q_n\rangle\, e^{-i\varepsilon H(p_{n+1},q_{n+1};p_n,q_n)}\prod_{n=1}^{n=N} dp_n dq_n/2\pi.$$

$$\langle p'',q''|e^{-i\mathcal{H}T}|p',q'\rangle = \lim_{N\to\infty}\int\cdots\int \times$$
$$\prod_{n=0}^{n=N}\langle p_{n+1},q_{n+1}|p_n,q_n\rangle\prod_{n=1}^{n=N} e^{-i\varepsilon h(p_n,q_n)}dp_n dq_n/2\pi,$$

where we have introduced the notation $\varepsilon = T/(N+1)$, $p'',q'' = p_{N+1},q_{N+1}$, and $p',q' = p_0,q_0$. In a formal limit, in which the order of integration and the limit are interchanged and the integrand is evaluated for continuous and differential paths, the formal result emerges, respectively, that

$$\langle p'',q''|e^{-i\mathcal{H}T}|p',q'\rangle = \mathcal{M}\int e^{i\int[p\dot{q}-H(p,q)]dt}\mathcal{D}p\mathcal{D}q,$$
$$\langle p'',q''|e^{-i\mathcal{H}T}|p',q'\rangle = \mathcal{M}\int e^{i\int[p\dot{q}-h(p,q)]dt}\mathcal{D}p\mathcal{D}q,$$

where, as conventional, we use a single standard integral sign here to represent a (formal) functional integration. Since, in the general case, $H(p,q) \neq h(p,q)$, we are seemingly led to a paradox, namely that two generally *different* expressions can be given for the *same* quantity. That these two expressions are different is just a dramatic reflection of the very formal nature of such "equations" in the first place; each is correct if interpreted in the manner indicated in the lattice regularized form given above.

In recent years, a very different regularization and formulation of coherent state path integrals has been developed that is both rigorous in construction and does not exhibit the paradox outlined above [3]. In this formulation, a *continuous-time regularization* scheme is found that takes the form

$$\langle p'', q''|e^{-i\mathcal{H}T}|p', q'\rangle = \lim_{\nu \to \infty} 2\pi e^{\nu T/2} \int e^{i\int[p(t)dq(t)-h(p(t),q(t))dt]} d\mu_W^\nu(p,q),$$

where μ_W^ν denotes a planar two-dimensional Wiener measure with diffusion constant ν that is pinned so that $p(t'), q(t') = p', q'$ and $p(t''), q(t'') = p'', q''$. Since $p(t)$ and $q(t)$ are (independent) Brownian motion paths, the integral $\int p(t)\, dq(t)$ is properly understood as a well-defined stochastic integral [4]. In the present form the Ito and Stratonovich formulations yield the same result; however, under coordinate transformations the Stratonovich form is chosen and so it is convenient to adopt the Stratonovich form from the outset. It must be appreciated that the expression above involving the Wiener measure is rigorous and unambiguous; the marvel is that this genuine, i.e., continuous time, path integral formulation actually provides the correct propagator for the Hamiltonian $\mathcal{H}$ provided one adopts the symbol $h(p,q)$ to use as the classical Hamiltonian in the action even though it may, in general, contain a nonzero $\hbar$. In a formal, but nevertheless suggestive language, one may also say that

$$\langle p'', q''|e^{-i\mathcal{H}T}|p', q'\rangle = \lim_{\nu \to \infty} \mathcal{M} \int e^{i\int[p\dot{q}-h(p,q)]dt} e^{-\frac{1}{2\nu}\int[\dot{p}^2+\dot{q}^2]dt} \mathcal{D}p\mathcal{D}q,$$

which shows the continuous-time regulatory nature of the indicated expression inasmuch as the ν–dependent factor in the integrand formally goes to unity as $\nu \to \infty$. Although formal in nature, the last equation may be understood as a short hand expression for the former one when it is accepted that the various terms do not have independent meaning but only in combination with one another. Thus they may be recombined into the proper mathematical form at any time. (This is similar to how the "quotient" dy/dx should be understood for the derivative.)

The expression for the propagator given above is not only well defined mathematically, but it also enjoys a covariance under generally time-dependent canonical coordinate transformations. Let two canonical coordinate systems be related according to the equation

$$p\, dq - h(p,q) = \overline{p}\, d\overline{q} + dG(\overline{p}, \overline{q}, t) - k(\overline{p}, \overline{q}, t)$$

that not only holds in the classical case *but for Brownian motion paths* as well thanks to the choice of the Stratonovich rule [4]. Under the same canonical coordinate transformation, the metric on flat space that supports the two-dimensional Brownian motion changes to

$$dp^2 + dq^2 = d\sigma^2(\overline{p}, \overline{q}, t) = A(\overline{p}, \overline{q}, t)\, d\overline{p}\, d\overline{p} + B(\overline{p}, \overline{q}, t)\, d\overline{p}\, d\overline{q} + C(\overline{p}, \overline{q}, t)\, d\overline{q}\, d\overline{q}$$

and as a consequence the propagator takes on the form in the new coordinates, which for convenience we relabel p, q again,

$$\lim_{\nu \to \infty} \mathcal{M} e^{i(G''-G')} \int e^{i\int [p\dot{q} - k(p,q,t)]dt} e^{-\frac{1}{2\nu}\int [d\sigma^2(p,q,t)/dt^2]dt} \mathcal{D}p\mathcal{D}q$$

This formula expresses —for the first time and after over sixty years of the theory of quantum mechanics—*a fully canonically coordinate covariant formulation of the process of quantization* [5]. This expression clarifies the role of the Schroedinger quantization rule in which coordinates act as multiplication while momenta act as derivatives; this rule of quantization is valid in and only in Cartesian coordinates as often noted, but Cartesian coordinates in *phase space* rather than in p-space and q-space *separately* as commonly stated [6]. Of course, the arena for classical mechanics resides in a symplectic manifold and it does not employ a (Riemannian) metric in its formulation. On the other hand, quantum mechanics has a different and richer basis in which a metric structure appears. Indeed, it is not unreasonable from a classical viewpoint that a metric structure is appended to the classical phase-space manifold, not for purposes of defining the Hamiltonian equations of motion, but rather to keep track of just what physics a given system refers to. For example, an harmonic oscillator (centered at the origin) *appears* as an harmonic oscillator, e.g., with a Hamiltonian given by $\frac{1}{2}\left(ap^2 + 2bpq + cq^2\right)$, $a > 0$, $b > 0$, $ac > b^2$, *only* in Cartesian coordinates in phase space. In *non*-Cartesian coordinates an harmonic oscillator assumes a different form from that indicated. Just what system actually corresponds to an harmonic oscillator (or free particle, or quartic anharmonic oscillator, etc.) is coded into the classical scheme by the implicit use of an auxiliary *flat metric* on the two-dimensional phase space, and its expression in Cartesian coordinates. This same flat metric space actually enters the formulation of the quantization procedure as described in the present article through its use as a carrier for the Brownian motion. Once it is decided which sets of canonical coordinates are the Cartesian ones, so that the expression for a system which represents (say) an harmonic oscillator is unambiguous, then the quantization procedure itself is unambiguous in the approach advocated here. After the well-defined path integral is set up, then one is free to make a variety of coordinate changes within that integral, among which possibly time dependent canonical transformations are to be distinguished. Indeed, one can go so far as to introduce a Hamilton-Jacobi transformation so that the new Hamiltonian vanishes. This puts all the dynamics into the curvilinear coordinate system that is used to track the two-dimensional planar Brownian motion. As a consequence, the overall level of difficulties is conserved, as one would expect to be the case.

It is hard to illustrate this program in its full potential, but it can be shown in a sort of small scale fashion. Indeed, it is *squeezed states* that can be used to provide a limited illustration of this overall program, and it is this "miniature" illustration to which we now turn our attention. A convenient place to start the investigation is with the kinematics rather than the dynamics, and this in turn can be done simply by looking at the propagator for vanishing Hamiltonian.

2. CHANGE OF VARIABLES IN THE PATH INTEGRAL: CONSTANT ω TRANSFORMATIONS

KINEMATICS

If the Hamiltonian vanishes, or in the limit that $T \to 0$, the "propagator" reduces to the *reproducing kernel*, an integral kernel representing a projection operator onto the relevant subspace of all square integrable functions on phase space as given by

$$\langle p'', q''; 1 | p', q'; 1 \rangle = \lim_{\nu \to \infty} \mathcal{M} \int e^{i \int p\dot{q}dt} e^{-\frac{1}{2\nu} \int (\dot{p}^2 + \dot{q}^2)dt} \mathcal{D}p \mathcal{D}q,$$

which may be explicitly evaluated as

$$\langle p'', q''; 1 | p', q'; 1 \rangle = e^{i\frac{1}{2}(p''+p')(q''-q') - \frac{1}{4}\left[(p''-p')^2 + (q''-q')^2\right]}.$$

In these expressions we have added a "1" to the label to emphasize that the coherent states are those based on an harmonic oscillator ground state with a unit angular frequency, $\omega = 1$. In particular, for a general value of ω, the configuration space representation of the coherent states reads

$$\langle x | p, q; \omega \rangle = \left(\frac{\omega}{\pi}\right)^{\frac{1}{4}} e^{-\frac{1}{2}\omega(x-q)^2 + ip(x-q)},$$

and it follows that the overlap of two such states for the same value of ω is given by

$$\begin{aligned} \langle p'', q''; \omega | p', q'; \omega \rangle &= \int \langle p'', q''; \omega | x \rangle \langle x | p', q'; \omega \rangle dx \\ &= \exp\left\{ \tfrac{i}{2}\left(p'' + p'\right)\left(q'' - q'\right) - \tfrac{1}{4}\left[\omega^{-1}\left(p'' - p'\right)^2 + \omega\left(q'' - q'\right)^2\right]\right\}. \end{aligned}$$

It is clear therefore that the coherent state overlap obeys the identity

$$\langle p'', q''; \omega | p', q'; \omega \rangle = \langle \omega^{-\frac{1}{2}} p'', \omega^{\frac{1}{2}} q''; 1 | \omega^{-\frac{1}{2}} p', \omega^{\frac{1}{2}} q'; 1 \rangle$$

which relates dilation of the angular frequency to a corresponding dilation of the coherent state labels, i.e., an expansion of one phase space coordinate and a contraction of the other.

This relation may be codified another way as well. Let

$$D = \tfrac{1}{2}(PQ + QP)$$

denote the self-adjoint dilation operator with commutation properties $[Q, D] = iQ$, $[P, D] = -iP$, then it follows that coherent states for different angular frequencies are connected by

$$|p, q; \omega\rangle = e^{i\frac{1}{2}\ln(\omega)D}|\omega^{-\frac{1}{2}}p, \omega^{\frac{1}{2}}q; 1\rangle.$$

Thus the unitary transformation generated by D is nothing other than the squeeze operator relating coherent states and squeezed states, or relating two sets of squeezed states with different squeezing values. In forming the overlap illustrated above, the squeezing operator drops out leading to the indicated relation.

A path integral expression for the coherent state overlap at angular frequency ω can be readily obtained just by a coordinate change of the path integral appropriate for a unit angular frequency. In particular, if one makes the change of integration variables given by

$$p(t) \rightarrow \omega^{-\frac{1}{2}}p(t), \qquad q(t) \rightarrow \omega^{\frac{1}{2}}q(t),$$

then it immediately follows that

$$\langle p'', q''; \omega | p', q'; \omega\rangle = \lim_{\nu \rightarrow \infty} \mathcal{M} \int e^{i\int p\dot{q}dt} e^{-\frac{1}{2\nu}\int(\omega^{-1}\dot{p}^2 + \omega\dot{q}^2)dt} \mathcal{D}p\mathcal{D}q$$

showing quite clearly the connection of the relative scale factor in the two-dimensional Brownian motion and the parametric dependence in the coherent state representation. All this has assumed that ω has been constant throughout; next we take up the case of a time variable ω.

3. CHANGE OF VARIABLES IN THE PATH INTEGRAL: NONCONSTANT ω TRANSFORMATIONS

The overlap of two coherent states for two *different* values of ω is given by

$$\langle p'', q''; \omega'' | p', q'; \omega'\rangle = \int \langle p'', q''; \omega'' | x\rangle \langle x | p', q'; \omega'\rangle dx$$

$$= \frac{\sqrt{2}}{\sqrt{\sqrt{\frac{\omega''}{\omega'}} + \sqrt{\frac{\omega'}{\omega''}}}} \exp\left[i\frac{(p''\omega' + p'\omega'')(q'' - q')}{\omega'' + \omega'} - \frac{(p'' - p')^2}{2(\omega'' + \omega')} - \frac{\omega''\omega'(q'' - q')^2}{2(\omega'' + \omega')} \right].$$

This expression also exhibits an alternative form given by

$$\begin{aligned}\langle p'',q'';\omega''|p',q';\omega'\rangle &= \langle \omega''^{-\frac{1}{2}}p'',\omega''^{\frac{1}{2}}q'';1|e^{-i\frac{1}{2}\ln\left(\frac{\omega''}{\omega'}\right)D}|\omega'^{-\frac{1}{2}}p',\omega'^{\frac{1}{2}}q';1\rangle \\ &= \langle \omega''^{-\frac{1}{2}}p'',\omega''^{\frac{1}{2}}q'';1|e^{-i\int\frac{d\omega}{2\omega}D}|\omega'^{-\frac{1}{2}}p',\omega'^{\frac{1}{2}}q';1\rangle \\ &= \langle \omega''^{-\frac{1}{2}}p'',\omega''^{\frac{1}{2}}q'';1|e^{-i\int\frac{\dot\omega}{2\omega}dtD}|\omega'^{-\frac{1}{2}}p',\omega'^{\frac{1}{2}}q';1\rangle\end{aligned}$$

where we have introduced a smooth but otherwise arbitrary function $\omega(t)$, $t' \le t \le t''$,which interpolates between $\omega'' = \omega(t'')$ and $\omega' = \omega(t')$. Of course, this expression holds as well even in the special case that $\omega'' = \omega'$ in which case $\omega(t)$ goes smoothly between equal initial and final values, but is otherwise arbitrary.

In the latter form the overlap of two coherent states for differing angular frequencies has been expressed in terms of the matrix element of a kind of propagator between coherent states of the same angular frequency. But the latter form admits a path integral expression. In particular, it follows that

$$\langle p'',q'';1|e^{-i\int\frac{\dot\omega}{2\omega}Ddt}|p',q';1\rangle$$

$$= \lim_{\nu\to\infty}\mathcal{M}\int e^{i\int[p\dot q-\frac{\dot\omega}{2\omega}pq]dt}e^{-\frac{1}{2\nu}\int[\dot p^2+\dot q^2]dt}\mathcal{D}p\mathcal{D}q.$$

Now, much as was the case earlier when ω was constant, we next make a time-dependent change of variables of the form

$$p(t) \to \omega(t)^{-\frac{1}{2}}p(t),$$
$$q(t) \to \omega(t)^{\frac{1}{2}}q(t),$$

where $\omega'' = \omega(t'')$, $\omega' = \omega(t')$. In making a time-dependent substitution of variables, additional terms will arise in the path integral integrand on the right hand side. In particular, the term

$$\int[p\dot q - \tfrac{\dot\omega}{2\omega}pq]\,dt \;\to\; \int[p\dot q]\,dt,$$

the formal flat measure remains unchanged,

$$\mathcal{D}p\mathcal{D}q \;\to \mathcal{D}p\mathcal{D}q,$$

while the all important formal weighting factor

$$e^{-\frac{1}{2\nu}\int[\dot p^2+\dot q^2]dt} \;\to\; e^{-\frac{1}{2\nu}\int[\omega(t)^{-1}\dot p^2+\omega(t)\dot q^2]dt}.$$

Other terms might be contemplated in the exponent of the final expression such as those involving time derivatives of $\omega(t)$; however all of these will be negligible in the limit that $\nu \rightarrow \infty$ since they are not as singular as the indicated terms. While we prefer this heuristic characterization of the transformed Wiener process one should bear in mind that only the *coordinate description* of the planar two-dimensional Brownian motion is being changed and the process itself is in no way effected. We are encoding this change of coordinates by means of the change of coordinates of the metric on the plane. (A rigorous analysis of such transformed Wiener processes is in progress by the present authors.) Thus it follows after such a substitution of variables that

$$\langle p'', q''; \omega'' | p', q'; \omega' \rangle = \langle \omega''^{-\frac{1}{2}} p'', \omega''^{\frac{1}{2}} q''; 1 | e^{-i\frac{1}{2}\int \frac{\dot{\omega}}{\omega} D dt} | \omega'^{-\frac{1}{2}} p', \omega'^{\frac{1}{2}} q'; 1 \rangle$$

$$= \lim_{\nu \rightarrow \infty} \mathcal{M} \int e^{i\int p\dot{q}dt} e^{-\frac{1}{2\nu}\int [\omega(t)^{-1}\dot{p}^2 + \omega(t)\dot{q}^2]dt} \mathcal{D}p\mathcal{D}q.$$

Consequently, the introduction of a smooth, time-dependent angular frequency that interpolates between the initial and final values in the Wiener measure provides just the right ingredient to yield the overlap between two different coherent states based on two different angular frequencies. This expression yields a simple but nontheless bona fide example of how the classical Hamiltonian — here just $\dot{\omega}(t)\, p(t)\, q(t) / 2\omega(t)$ — may be eliminated in favor of a change of coordinates with which to describe the two-dimensional Brownian motion on the phase space plane. Such an elimination additionally involves a change of coordinates at the endpoints, as illustrated in the central equation, but in the case of squeezed states, there is an alternative interpretation involving coherent states based on differing angular frequencies as embodied in the first part of the equation. We now turn our attention to the inclusion of dynamics in this example through the presence of a rather general nonvanishing Hamiltonian.

4. CHANGE OF VARIABLES IN THE PATH INTEGRAL: NONCONSTANT ω TRANSFORMATIONS AND A GENERAL HAMILTONIAN

INTRODUCTION OF GENERAL DYNAMICS

Based on the earlier discussion it is quite straightforward to include a rather general Hamiltonian $h(p,q)$ into the problem. In particular, based on the initial discussion, let us consider

$$\langle p'', q''; 1 | T e^{-i\int [\mathcal{H} + \frac{\dot{\omega}}{2\omega} D]dt} | p', q'; 1 \rangle$$

$$= \lim_{\nu \rightarrow \infty} \mathcal{M} \int e^{i\int [p\dot{q} - h(p,q) - \frac{\dot{\omega}}{2\omega} pq]dt} e^{-\frac{1}{2\nu}\int [\dot{p}^2 + \dot{q}^2]dt} \mathcal{D}p\mathcal{D}q$$

which after a change of integration variables becomes

$$\begin{aligned}
&\lim_{\nu \to \infty} \mathcal{M} \int e^{i\int[p\dot{q}-k(p,q,t)]dt} e^{-\frac{1}{2\nu}\int\left[\omega(t)^{-1}\dot{p}^2+\omega(t)\dot{q}^2\right]dt} \mathcal{D}p\mathcal{D}q \\
&= \langle \omega''^{-\frac{1}{2}}p'', \omega''^{\frac{1}{2}}q''; 1|Te^{-i\int\left[\mathcal{H}+\frac{\dot{\omega}}{2\omega}D\right]dt}|\omega'^{-\frac{1}{2}}p', \omega'^{\frac{1}{2}}q'; 1\rangle \\
&= \langle \omega''^{-\frac{1}{2}}p'', \omega''^{\frac{1}{2}}q''; 1|e^{-\frac{i}{2}\ln\omega'' D} Te^{-i\int\mathcal{H}'(t)dt} e^{\frac{i}{2}\ln\omega' D}|\omega'^{-\frac{1}{2}}p', \omega'^{\frac{1}{2}}q'; 1\rangle \\
&= \langle p'', q''; \omega''|Te^{-i\int\mathcal{H}'(t)dt}|p', q'; \omega'\rangle,
\end{aligned}$$

where

$$k\left(p\left(t\right), q\left(t\right), t\right) = h\left(\omega\left(t\right)^{-\frac{1}{2}} p\left(t\right), \omega\left(t\right)^{\frac{1}{2}} q\left(t\right)\right),$$

which contains an explicit time dependence from the angular frequency as well as an implicit dependence just from the time dependence of p and q themselves, and in addition where

$$\mathcal{H}'\left(t\right) = e^{i\frac{1}{2}\ln\frac{\omega(t)}{\omega'}D} \mathcal{H} e^{-i\frac{1}{2}\ln\frac{\omega(t)}{\omega'}D}.$$

The basic significance of the preceding equations can be summarized as follows:

$$\begin{aligned}
&\langle p'', q''; \omega''|Te^{-i\int\mathcal{H}'(t)dt}|p', q'; \omega'\rangle \\
&= \langle \omega''^{-\frac{1}{2}}p'', \omega''^{\frac{1}{2}}q''; 1|Te^{-i\int\left[\mathcal{H}+\frac{\dot{\omega}}{2\omega}D\right]dt}|\omega'^{-\frac{1}{2}}p', \omega'^{\frac{1}{2}}q'; 1\rangle \\
&= \left[\lim_{\nu \to \infty} \mathcal{M} \int e^{i\int\left[p\dot{q}-h(p,q)-\frac{\dot{\omega}}{2\omega}pq\right]dt} e^{-\frac{1}{2\nu}\int[\dot{p}^2+\dot{q}^2]dt} \mathcal{D}p\mathcal{D}q\right]_{\substack{p(t)\to\omega(t)^{-\frac{1}{2}}p(t) \\ q(t)\to\omega(t)^{\frac{1}{2}}q(t)}} \\
&= \lim_{\nu \to \infty} \mathcal{M} \int e^{i\int[p\dot{q}-k(p,q,t)]dt} e^{-\frac{1}{2\nu}\int\left[\omega(t)^{-1}\dot{p}^2+\omega(t)\dot{q}^2\right]dt} \mathcal{D}p\mathcal{D}q.
\end{aligned}$$

This is the most general form we are able to offer using squeezed states, and it shows, in the first of the equalities, how part of the Hamiltonian can be absorbed quantum mechanically by a change of the fiducial vectors — indeed just like going to the interaction picture in ordinary quantum mechanics, which is then responsible for the introduction of the time-dependent "interaction" picture Hamiltonian $\mathcal{H}'(t)$. The second pair of equalities just uses the original form of the path integral as modified by a change of variables that effects the end point conditions as well. The final equality just accounts for that very change of variables as requested in the line above.

5. EQUAL END POINT ANGULAR FREQUENCIES

Let us return to the path integrals discussed at the beginning of this article, namely to those for which the initial and final angular frequencies are the same. For the sake of convenience, let us choose that value to be unity, i.e., $\omega'' = \omega' = 1$, and return to the original notation for the coherent states with unit ω, namely that $|p,q\rangle = |p,q;1\rangle$. However, this time we will retain the option of using a time-dependent angular frequency $\omega(t)$ to interpolate smoothly between the original and final values of unity. In this case the formulas developed above simplify to become

$$\begin{aligned}
&\langle p'',q''|Te^{-i\int\mathcal{H}'(t)dt}|p',q'\rangle \\
&= \langle p'',q''|Te^{-i\int[\mathcal{H}+\frac{\dot\omega}{2\omega}D]dt}|p',q'\rangle \\
&= \left[\lim_{\nu\to\infty}\mathcal{M}\int e^{i\int[p\dot q-h(p,q)-\frac{\dot\omega}{2\omega}pq]dt}e^{-\frac{1}{2\nu}\int[\dot p^2+\dot q^2]dt}\,\mathcal{D}p\mathcal{D}q\right]_{\substack{p(t)\to\omega(t)^{-\frac12}p(t)\\ q(t)\to\omega(t)^{\frac12}q(t)}} \\
&= \lim_{\nu\to\infty}\mathcal{M}\int e^{i\int[p\dot q-k(p,q,t)]dt}e^{-\frac{1}{2\nu}\int[\omega(t)^{-1}\dot p^2+\omega(t)\dot q^2]dt}\,\mathcal{D}p\mathcal{D}q.
\end{aligned}$$

With a slight generalization, we can now turn this equation around to read

$$\begin{aligned}
&\langle p'',q''|e^{-i\mathcal{H}T}|p',q'\rangle \\
&= \lim_{\nu\to\infty}\mathcal{M}\int e^{i\int[p\dot q-h(p,q)]dt}e^{-\frac{1}{2\nu}\int[\dot p^2+\dot q^2]dt}\,\mathcal{D}p\mathcal{D}q. \\
&= \left[\lim_{\nu\to\infty}\mathcal{M}\int e^{i\int[p\dot q-h(p,q)]dt}e^{-\frac{1}{2\nu}\int[\dot p^2+\dot q^2]dt}\,\mathcal{D}p\mathcal{D}q\right]_{\substack{p(t)\to\omega(t)^{-\frac12}p(t)\\ q(t)\to\omega(t)^{\frac12}q(t)}} \\
&= \lim_{\nu\to\infty}\mathcal{M}\int e^{i\int[p\dot q-k(p,q,t)+\frac{\dot\omega}{2\omega}pq]dt}e^{-\frac{1}{2\nu}\int[\omega(t)^{-1}\dot p^2+\omega(t)\dot q^2]dt}\,\mathcal{D}p\mathcal{D}q.
\end{aligned}$$

In this expression the third line holds because it simply corresponds to a change of integration variables that does not have any effect on the values of the boundary labels since $\omega'' = \omega' = 1$. The last line represents, just as before, the consequences of that very change of variables.

Now observe that on the *left* side of this equation there is no reference to the function $\omega(t)$, $t' \le t \le t''$, $\omega(t'') = \omega(t') = 1$, while on the *right* side of the equation, in the last part of the equation in particular, the function $\omega(t)$ enters in a prominent way. This becomes especially significant when an *approximate* evaluation of the path integral is admitted, such as that which arises from a *stationary phase approximation.* Stationary phase approximations for coherent state path integrals with Wiener measure regularization of the kind considered here have been worked out previously [7] and we do not repeat that discussion here. The point we wish to emphasize, however, is that the choice of the angular frequency $\omega(t)$ *will* enter most probably in the form of the approximate solution, and naturally some expressions will be better approximations to the real answer than other expressions will. Just which will be the best approximation is, of course, not too easy to establish. Perhaps one scheme is to ask that the result be stationary with respect to small changes of the functional form of $\omega(t)$. In practice one might want to let $\omega(t)$ depend on just a few discrete parameters and to seek stationary variations with respect to just these few parameters. This certainly seems easier to do than to ask for an extremal variation with respect to the entire function $\omega(t)$.

One can actually see a miniature working of *this* general kind of procedure in comparing the usual and the Maslov stationary phase approximations to the sharp position propagator; see, e.g., [7]. As given earlier, the configuration-space form of the coherent states given by

$$\langle x|p,q;\omega\rangle = \left(\frac{\omega}{\pi}\right)^{\frac{1}{4}} e^{-\frac{1}{2}\omega(x-q)^2+ip(x-q)},$$

makes clear that $\lim_{\omega\to\infty} \left(\frac{\omega}{4\pi}\right)^{\frac{1}{4}} \langle x|p,q;\omega\rangle = \delta(x-q)$ which converts the coherent state representation to the sharp position or configuration representation, while $\lim_{\omega\to 0} \left(\frac{1}{4\pi\omega}\right)^{\frac{1}{4}} \langle x|p,q;\omega\rangle = \frac{1}{\sqrt{2\pi}} e^{ip(x-q)}$ which converts it to the dual or momentum representation (up to an unimportant phase factor). These features can also be seen in the relation

$$\langle p'',q'';\omega''|p',q';\omega'\rangle = \int \langle p'',q'';\omega''|x\rangle\langle x|p',q';\omega'\rangle dx$$

$$= \frac{\sqrt{2}}{\sqrt{\sqrt{\frac{\omega''}{\omega'}}+\sqrt{\frac{\omega'}{\omega''}}}} \exp\left[i\frac{(p''\omega'+p'\omega'')(q''-q')}{\omega''+\omega'} - \frac{(p''-p')^2}{2(\omega''+\omega')} - \frac{\omega''\omega'(q''-q')^2}{2(\omega''+\omega')}\right].$$

that gives the overlap of two coherent states based on differing angular frequencies. Consider the limiting situation in which both $\omega'' \to \infty$, $\omega' \to 0$. In that case it follows that

$$\lim_{\substack{\omega''\to\infty \\ \omega'\to 0}} \left(\frac{\omega''}{16\pi^2\omega'}\right)^{\frac{1}{4}} \langle p'',q'';\omega''|p',q';\omega'\rangle = \frac{1}{\sqrt{2\pi}} e^{ip'(q''-q')}.$$

Thus it should be no surprise that the two standard stationary-phase type approximations are actually contained in the coherent state approach in the form of suitable limits. In a manner of speaking, the usual configuration space approach just involves choosing a constant and very large value of the angular frequency parameter ω (taken to infinity at the end of the calculation) and making a stationary phase approximation to the resulting path integral. On the other hand, the Maslov approach takes the propagator from a sharp configuration initially to a sharp momentum finally, approximates that path integral by a stationary phase approximation, and then returns the end point to configuration space by a Fourier transform. This approach can be approximated in our method by taking an angular frequency history $\omega(t)$ that is initially huge (tending toward infinity) and finally very small (tending toward zero), approximating *that* path integral by a stationary phase approximation, and finally making a change from coherent states based on a very small angular frequency to one based on a huge angular frequency just by the kinematical factor given above. The coherent state approximation developed in particular in reference [7] proceeds in yet another way, namely, starting with a sharp configuration initially, propagating to a coherent state with a finite nonzero value of the angular frequency, i.e., $\omega = O(1)$, approximating *that* path integral by a stationary phase approximation, and then passing from the final coherent state representation to a sharp configuration one. This approach can also be approximated in our scheme by having an $\omega(t)$ that initially is huge, and finally is finite and nonzero [$O(1)$], approximating, in turn, *that* path integral by a stationary phase approximation, and then passing back to a coherent state based on a huge angular frequency at the final point.

6. CONCLUSIONS

In this article we have attempted to show the reader what the authors believe is the "latest" in path integral construction — the state of the art — and illustrate how variable changes can be *rigorously* carried out within the path integral formulation itself. Squeezed coherent states have been used as convenient bases throughout in the illustration of the general program by a "miniature" subprogram involving a fairly limited change of integration variables. The resultant formalism is able to express a path integral in terms of an essentially arbitrary function, the time varying angular frequency, $\omega(t)$, which lends itself to various selections in case an approximation scheme is invoked. By illustrating that the usual and the Maslov approaches are but two small examples of how such an optimization can be used, it becomes clear that there are hidden in these formulas a whole host of differing approximation schemes some of which, in certain applications at least, may well be better than the schemes currently in use. It is left to the future to see just how to exploit the vast number of possibilities that have been opened up here.

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