# Differential Reassignment

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***Abstract*—A geometrical description is given for reassignment vector fields of spectrograms. These vector fields are shown to be connected with both an intrinsic phase characterization and a scalar potential. This allows for the generalization of the original reassignment process to a differential version based on a dynamical evolution of time-frequency particles.**

***Index Terms*—Reassignment method, time-frequency distributions.**

## I. Introduction and Notations

The reassignment method has been introduced first to improve the readability of time-frequency representations [1], [2]. It can be considered as a postprocessing based on the definition of a time-frequency displacement vector field $\boldsymbol{r}(t,\omega) = (\hat{t}(t,\omega) - t, \hat{\omega}(t,\omega) - \omega)^t$, where $\hat{t}(t,\omega)$ and $\hat{\omega}(t,\omega)$ stand for the coordinates of the reassignment point associated with the time-frequency point $(t,\omega)$ where the distribution has been computed. In the spectrogram case, this vector field can be related to the phase $\phi(t,\omega)$ of the short-time Fourier transform (STFT). Precisely, it admits the quasisymmetric form

$$\boldsymbol{r}(t,\omega) = (-t/2 - \partial\phi/\partial\omega, -\omega/2 + \partial\phi/\partial t)^t \quad (1)$$

provided that the definition of the STFT makes use of the Weyl operator $[W(t,\omega)h](x) = h(x-t)\exp\ i(\omega x - t\omega/2)$ according to

$$F_f(t,\omega) = \langle f, W(t,\omega)h\rangle = \int f(x)h^*(x-t)e^{-i\omega x}e^{it\omega/2}\,dx.$$

What we propose in this letter is to focus on a geometrical description of the reassignment vector field. We show in Section II how the reassignment vector field can be expressed in terms of an intrinsic "geometric phase," the reassignment vectors being tangent to level curves of this phase. The same vector field is shown in Section III to be connected to a scalar potential, which relates the reassignment process to a steepest descent method. This interpretation allows for a differential generalization of the original reassignment method based on a dynamical evolution of time-frequency particles, with possible applications to problems such as the partitioning of the time-frequency plane.

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## II. Geometric Phase and Level Curves

Equation (1) suggests that reassignment vectors follow the level curves of some two-dimensional (2-D) function related to $\phi(t,\omega)$. However, whereas such level curves are expected to be covariant to shifts of the signal in the time-frequency plane, care has to be taken in the selection of this 2-D function since, by itself, the phase of the STFT appears to depend on the choice of an origin in the time-frequency plane. Indeed, if we denote $[T_s f](x) = f(x-s)$ the time shifted version of a signal $f$, we get

$$F_{T_s f}(t,\omega) = F_f(t-s,\omega)e^{-i\omega s},$$

with the result that $\phi(t,\omega)$ is not covariant to shifts in time of the signal. Similar equations can be written with frequency shifts. This shows that the choice of the origin in the time-frequency plane influences the phase $\phi(t,\omega)$. To address this problem, we introduce a new function $\Phi_{(t_0,\omega_0)}(t,\omega)$ that we call *geometric phase*. $\Phi_{(t_0,\omega_0)}(t,\omega)$ is the phase we would have observed, if we had chosen our origin at $(t_0,\omega_0)$, at the time-frequency point with coordinates $(t,\omega)$ with respect to this new frame. Moving the origin in time from zero to $t_0$ means that the function value of $f$ at a distance $t$ fromthe new origin will be given by $f(t+t_0) = [W(-t_0,0)f](t)$. Similarly, moving the time frequency origin to $(t_0,\omega_0)$ corresponds to replacing $f$ by $W(-t_0,-\omega_0)f$. So

$$\begin{aligned}\Phi_{(t_0,\omega_0)}(t,\omega) &= \arg\{\langle W(-t_0,-\omega_0)f, W(t,\omega)h\rangle\}\\ &= \arg\{\langle f, W(t_0,\omega_0)W(t,\omega)h\rangle\}.\end{aligned}$$

Since we have

$$\begin{aligned}&[W(t_0,\omega_0)W(t,\omega)h](x)\\ &\quad= W(t_0,\omega_0)(h(x-t)e^{i\omega x}e^{-it\omega/2})\\ &\quad= h(x-(t-t_0))e^{i[2(\omega+\omega_0)x-(\omega+\omega_0)(t+t_0)+(t\omega_0-t_0\omega)]/2}\\ &\quad= [W(t_0+t,\omega_0+\omega)h](x)e^{i[(t_0,\omega_0)\,(t,\omega)]/2}\end{aligned}$$

we can express $\Phi_{(t_0,\omega_0)}(t,\omega)$ with the help of the symplectic form $[(t_0,\omega_0)(t,\omega)] = t\omega_0 - t_0\omega$ as

$$\Phi_{(t_0,\omega_0)}(t,\omega) = \phi(t+t_0,\omega+\omega_0) - [(t_0,\omega_0)(t,\omega)].$$

This shows in a straightforward manner that the reassignment vector at $(t_0,\omega_0)$ is tangent to the level curve through $(t_0,\omega_0)$ of $\Phi_{(t_0,\omega_0)}(t,\omega)$, as follows:

$$\begin{aligned}\boldsymbol{r}(t_0,\omega_0) = (&\partial_\omega\Phi_{(t_0,\omega_0)}(t,\omega),\\ &-\partial_t\Phi_{(t_0,\omega_0)}(t,\omega))^t_{(t,\omega)=(0,0)}.\end{aligned}$$

This justifies our name of "geometric" phase for $\Phi_{(t_0,\omega_0)}(t,\omega)$, in that its local behavior near $(t_0,\omega_0)$ has a geometric interpretation.

## III. Scalar Potential and Steepest Descent

Let us rewrite the STFT (we will now denote $F(t,\omega)$ for short) in a Bargmann-like way [3] with $\mathcal{F}$ a function of the complex variable $z=\omega+it$ and of its complex conjugate $z^*$, as follows:

$$F(t,\omega)=\mathcal{F}(z,z^*)\ \exp(-|z|^2/4). \tag{2}$$

The phases of $F$ and $\mathcal{F}$ are obviously equal. Thus, by noticing that $\log\mathcal{F}(t,\omega)=\log|\mathcal{F}(t,\omega)|+i\phi(t,\omega)$ and using (1), it is possible to express the reassignment vectors with the partial derivatives of $\mathcal{F}$

$$\begin{aligned}\hat{t}(t,\omega)-t &= t/2-\mathrm{Im}\{\partial_\omega\mathcal{F}/\mathcal{F}\}\\ &= -t/2-\mathrm{Im}\{(\partial_z\mathcal{F}+\partial_{z^*}\mathcal{F})/\mathcal{F}\}, \end{aligned}\tag{3}$$

$$\begin{aligned}\hat{\omega}(t,\omega)-\omega &= -\omega/2+\mathrm{Im}\{\partial_t\mathcal{F}/\mathcal{F}\}\\ &= -\omega/2+\mathrm{Re}\{(\partial_z\mathcal{F}-\partial_{z^*}\mathcal{F})/\mathcal{F}\}. \end{aligned}\tag{4}$$

The differentiation of $\log F(t,\omega)$

$$\begin{aligned}\partial_t F/F &= -t/2+(i\partial_z\mathcal{F}-i\partial_{z^*}\mathcal{F})/\mathcal{F}\\ \partial_\omega F/F &= -\omega/2+(\partial_z\mathcal{F}-\partial_{z^*}\mathcal{F})/\mathcal{F}\end{aligned}$$

gives rise to another couple of equations, as follows:

$$\begin{aligned}\mathrm{Re}\{\partial_t F/F\} &= -t/2-\mathrm{Im}\{(\partial_z\mathcal{F}-\partial_{z^*}\mathcal{F})/\mathcal{F}\}\\ &= \partial_t|F|/|F|,\\ \mathrm{Re}\{\partial_\omega F/F\} &= -\omega/2+\mathrm{Re}\{(\partial_z\mathcal{F}+\partial_{z^*}\mathcal{F})/\mathcal{F}\}\\ &= \partial_\omega|F|/|F|\end{aligned}$$

from which, when mixed with (3) and (4),one can deduce that

$$\boldsymbol{r}=\nabla\log|F|-2(\mathrm{Im}\{\partial_{z^*}\log\mathcal{F}\},\ \mathrm{Re}\{\partial_{z^*}\log\mathcal{F}\})^t$$

where the equality holds over $\boldsymbol{R}^2\backslash\{(t,\omega)|F(t,\omega)=0\}$. This result can be interpreted in the following way. Given the Bargmann factorization (2) of the STFT, its reassignment vector field can be decomposed in two terms, one which is the gradient of a scalar potential (namely $\log|F|$) and an additional one which is a measure of the nonanalyticity of $\mathcal{F}$.

If the observation window $h$ is a Gaussian function of unit variance (whose isocontours are circles in a Wigner representation), i.e., if we are considering the Bargmann representation of a "coherent states" space [3], $\mathcal{F}$ is an entire function of $z$, and thus $\partial_{z*}\mathcal{F}=0$ (Cauchy equations). We conclude that

$$\boldsymbol{r}(t,\omega)=\nabla\log|F| \tag{5}$$

which proves that in such a case, the reassignment vector field is the gradient of the scalar potential $\log|F|$. As a consequence, reassignment vectors are all plotting the direction of maxima of the STFT modulus. Comparing (1) and (5), we can furthermore remark that, up to a constant, the phase is in this case entirely determined by the modulus, and vice-versa. This means that the corresponding spectrogram (squared modulus of the STFT) carries as much information as the complete (complex-valued) STFT itself.

In the case of an arbitrary window, reassignment vectors follow the steepest descent direction of $-\log|F|$, modified by $G(t,\omega)=2\ (\mathrm{Im}\{\partial_{z^*}\log\mathcal{F}\},\ \mathrm{Re}\{\partial_{z^*}\log\mathcal{F}\})^t$. For instance, when analyzing with a Gaussian window of arbitrary variance $a, h(t)=\pi^{-1/4}/\sqrt{a}e^{-t^2/(2a^2)}$, the additional term can be written analytically $G(t,\omega)=-(a-1/a)\tilde{\boldsymbol{r}}(t/a,a\omega)$, with $\tilde{\boldsymbol{r}}$ the reassignment vector obtained with a unit variance Gaussian window. $G(t,\omega)$ vanishes when $a=1$, i.e., when returning to the Bargmann case.

## IV. Differential Reassignment

The fact that a link exists between the reassignment vector field and a scalar potential suggests to look at the system whose dynamical behavior is governed by this potential. From this perspective, let us consider that the reassignment vector field is the velocity field that controls the motion of each time-frequency contribution $F(t,\omega)$ considered as a particle, with $(t,\omega)$ as its starting position. We obtain the following motion equations:

$$\begin{aligned}t(0) &= t\\ \omega(0) &= \omega\\ dt(s)/ds &= \hat{t}(t(s),\omega(s))-t(s)\\ d\omega(s)/ds &= \hat{\omega}(t(s),\omega(s))-\omega(s)\end{aligned}\tag{6}$$

which define a process referred to as *differential reassignment*. In the unit variance Gaussian case, (6) describe a fully dissipative system so that each particle converges to some maximum of $\log|F|$. Differential reassignment can be viewed as a PDE-based processing of time-frequency "images," or as a generalization of the fixed point algorithm used by the ridge and skeleton approach [4]. (Let us briefly recall that such an algorithm is looking for maxima lines of Gabor (or wavelet) transforms by exploring the time-frequency (or time-scale) plane along a direction that is necessarily parallel to the frequency or time axes, whereas we have shown here that, in the unit Gaussian case, differential reassignment always uses the shortest way to reach the ridge.) Taking into account the above derived properties of the reassignment vector field, it becomes natural to describe a signal in the time-frequency plane in terms of attractors, basins of attraction and watersheds. A variety of signal characterizations can be deduced from such a parametrization. For instance, a partitioning [5] of the time-frequency plane in distinct signal components can be obtained this way, each component corresponding to the set of all time-frequency particles that converge to the same attractor.

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