# Time-frequency localisation operators-a geometric phase space approach: II. The use of dilations



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# Time–frequency localisation operators—a geometric phase space approach: II. The use of dilations

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**Abstract.** Operators which localise both in time and frequency are constructed. They restrict to a finite time interval and cut off low as well as high frequencies (band-pass filters). Explicit expressions for eigenvalues and eigenfunctions (Laguerre functions) are given.

## 1. Introduction

In a preceding paper [1], one of us defined operators which localised both in time and in frequency (i.e., low-pass filters which also restrict to a finite time interval), by means of a phase-space technique borrowed from quantum physics. The advantage of the localisation operators constructed in [1] was the simplicity of their eigenfunctions (Hermitean functions) and their eigenvalues (simple and explicit expressions involving incomplete gamma functions). The present paper again concerns operators localising in time and in frequency simultaneously, but constructed using a different procedure. The resulting operators restrict to a finite time interval, and cut off low as well as high frequencies, i.e. are band-pass filters rather than low-pass filters (see figure 1). Again it will be possible to write explicit expressions for the eigenvalues and eigenfunctions.

In order to point out the similarities and the differences between the present approach and [1], we start by recalling the essential steps of the construction in [1]. There are four such steps.

1. The definition of phase space. In the case of signals depending on a one-dimensional variable $x$ (e.g. time), the phase space is $\mathbb{R}^2$ (corresponding, for time-dependent signals, to the two variables time and frequency). We shall denote elements of phase space by $(p, q)$. The same framework is adapted to other types of signals, depending on $n$-dimensional variables (as needed, e.g., in vision analysis) [1].

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2. The definition of special functions $\varphi_{p,q}$ localised around $(p, q)$ in phase space. That is

$$\int \mathrm{d}x\, x|\varphi_{p,q}(x)|^2 = q \qquad \int \mathrm{d}k\, k|\hat{\varphi}_{p,q}(k)|^2 = p$$

where the hat denotes the Fourier transform. The functions $\varphi_{p,q}$ are defined from one generating function $\varphi$ by translation in $x$ and in $k$, i.e.

$$\varphi_{p,q}(x) = \exp(\mathrm{i}px)\varphi(x-q). \tag{1.1}$$

A special choice [1] is $\varphi^0(x) = \pi^{-1/4}\exp(-x^2/2)$, resulting in $\varphi^0_{p,q}(x) = \pi^{-1/4}\exp[\mathrm{i}px - (x-q)^2/2]$. These functions $\varphi^0_{p,q}(x)$ minimise the uncertainty relation inequality, i.e. they achieve the best possible localisation around $(p, q)$ allowed by the uncertainty principle.

3. The basic identity

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}p \int_{-\infty}^{\infty} \mathrm{d}q\, \varphi^0_{p,q}\langle\varphi^0_{p,q}, f\rangle = f \tag{1.2}$$

where $\langle , \rangle$ denotes the standard $L^2$ product,

$$\langle f, g\rangle = \int \mathrm{d}x\, \overline{f(x)}g(x).$$

This equation holds for *all* $f \in L^2(\mathbb{R})$, i.e. for all time-dependent signals with finite energy. (The identity (1.2) can even be extended to larger classes of functions.) In view of the localisation properties of $\varphi^0_{p,q}$, (1.2) can be considered as

(i) for every $(p, q)$, a 'projection' of $f$ onto the best possible localised function around $(p, q)$, i.e.

$$f \to \varphi^0_{p,q}\langle\varphi^0_{p,q}, f\rangle,$$

followed by

(ii) the recovery of the unadulterated signal $f$ by integrating over all possible phase space points $(p, q)$.

4. The restriction, in (1.2), of the integral to a *subset* $S$ of phase space (or time–frequency space; one can take, e.g., $S = [-\Omega, \Omega] \times [-T, T]$). This then defines an operator $P_S$,

$$P_S f = \frac{1}{2\pi}\int_{(p,q)\in S} \mathrm{d}p \int \mathrm{d}q\, \varphi^0_{p,q}\langle\varphi^0_{p,q}, f\rangle. \tag{1.3}$$

This is a positive operator (i.e. its eigenvalues are all positive), with norm smaller than one (i.e. the total energy of $P_S f$ is smaller than that of $f$, $\int \mathrm{d}x|(P_S f)(x)|^2 \leq \int \mathrm{d}x|f(x)|^2$). The eigenvalues of $P_S$ are therefore all between 0 and 1.

Moreover, $P_S f$ is essentially localised, in phase space, in the set $S$, i.e., $\langle\varphi^0_{p',q'}, P_S f\rangle$ will be negligibly small if $p'$, $q'$ is some distance from $S$ [1]. All these properties correspond to the idea of a filter operator associated to the phase space subset $S$.

For general $S$ the computation of the eigenvalues and eigenfunctions of $P_S$ is not easy. This computation simplifies dramatically for some special choices of $S$, however. In particular, if $S$ is a disc in phase space,

$$S_R=\{(p, q); p^2+q^2\leqslant R^2\},$$

then the eigenfunctions of $P_R=P_{S_R}$ are Hermitean functions (independently of $R$),

$$P_R H_n=\lambda_n(R)H_n$$

while the eigenvalues $\lambda_n(R)$ are given by incomplete gamma functions,

$$\lambda_n(R)=\frac{1}{n!}\int_0^{R^2/2} \mathrm{d}t\, t^n\, \mathrm{e}^{-t}.$$

For more details and proofs, we refer to [1].

The general procedure in this paper is the same as outlined in the four steps above. The main difference is that we deal with a set of functions different from the $\varphi_{p,q}$ defined by (1.1) in step 2. As a result the phase space filters we construct will be of a different nature. We shall still use a family of functions $\Phi_{p,q}$ indexed by phase space points, and generated from one single function $\Phi$, but the procedure by which the $\Phi_{p,q}$ are constructed will be different from (1.1). Nevertheless, the $\Phi_{p,q}$ are localised around the phase space point $(p, q)$ and a projection + integration formula of type (1.2) will again hold. One can therefore again restrict, as in step 4, the integral to a subset $S$ of phase space and thus define time–frequency localisation operators which will be different from those in [1]. In the present construction there will again exist special choices for the set $S$ such that the eigenfunctions and eigenvalues of the

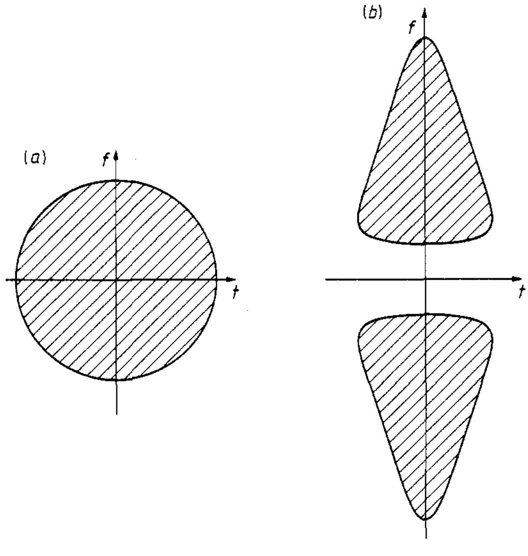


**Figure 1.** The difference between the filters constructed in [1] and here. (*a*) The filters in [1] localise in a disc in time–frequency space. (*b*) The filters constructed here cut off low frequencies as well as high frequencies.

corresponding localisation operator are known explicitly. Typically (see, e.g. figure 1($b$)) these sets consist of two parts (one for positive frequencies, one for negative frequencies) which are separated by a gap around zero frequency. They are thus very different from the discs in [1] (see figure 1($a$)). The associated eigenvalues involve incomplete beta functions, and the eigenfunctions can be written in terms of Laguerre functions.

This paper is organised as follows. In §2 we define the $\Phi_{p,q}$, and show how to build localisation operators with these functions. In fact, we can find a two-parameter family of functions $\Phi$, for each of which the $\Phi_{p,q}$ and localisation operators can be defined. For the sake of simplicity, we shall expose the whole construction for one particular $\Phi$ in this family, leaving the formulae for the general case to the appendix. In §3 we restrict ourselves to a special family of subsets $S$ of phase space, which we shall characterise explicitly. The corresponding localisation operators commute with a second-order differential operator. This is exploited, in §4, in order to write explicit expressions for the eigenfunctions and eigenvalues of the localisation operators. We end by a short discussion of the properties of these eigenvalues.

## 2. Definition of the localisation operators

The construction of the $\varphi_{p,q}$ in [1], and as defined by (1.1), corresponds, in fact, to the action of an irreducible representation of the Weyl–Heisenberg group on the function $\varphi$. The reconstruction formula (1.2) can be viewed as the expression of the square integrability of this representation [2]. The $\varphi_{p,q}$ are, in fact, the coherent states associated with the Weyl–Heisenberg group. The construction of the $\Phi_{p,q}$ we shall use here is related to the irreducible representations of a different group, namely the $ax+b$ group. In this case the group consists of $\mathbb{R}_+^* \times \mathbb{R}$ (where $\mathbb{R}_+^* = \{x \in \mathbb{R};\ x>0\}$), with group multiplication law

$$(a, b)(a', b') = (aa', b + ab').$$

This group has two nonequivalent unitary representations, both on $H_2 = \{f \in L^2(\mathbb{R}) \operatorname{supp} \hat{f} \subset \mathbb{R}_+\}$. These representations are

$$[U_\pm(a, b) f](x) = a^{-1/2} f\left(\frac{x \pm b}{a}\right) \tag{2.1}$$

or

$$[U_\pm(a, b) f]\hat{\ }(k) = a^{1/2} \exp(\pm \mathrm{i}bk)\ \hat{f}(ak). \tag{2.2}$$

Both these representations are square integrable. One finds that if $h \in H_2$ satisfies

$$\int_0^\infty \mathrm{d}k\, k^{-1} |\hat{h}(k)|^2 < \infty$$

then, for every $g \in H_2$,

$$\int_{-\infty}^\infty \frac{\mathrm{d}a}{a^2} \int_{-\infty}^\infty \mathrm{d}b\ U_+(a, b)h\, \langle U_+(a, b)h, g\rangle = Cg \tag{2.3}$$

where the value of $C$ only depends on $h$. This can easily be checked by straightforward computation. The same is true for $U_-$. It is clear that (2.3) is the analogue, for the present situation, of (1.2), where the $U_\pm(a, b)h$ play the role of the $\varphi_{p,q}$. If the function $h$ is reasonably well localised in time ($x$) and frequency ($k$), then the action of the $U_\pm(a, b)$ on $h$ results in a reasonably well localised function in both time and frequency, with respect to a shifted time centre and a dilated frequency centre. For $(a, b) \in \mathbb{R}_+^* \times \mathbb{R}$, the localisation centres of the $U_\pm(a, b)h$ sweep of all phase space, in a sense to be made precise below. Again this is completely analogous to what happens with the $\varphi_{p,q}$ [1]. This is why we shall be able to rewrite, below, (2.3) as an integral over phase space from which localisation operators can be defined.

We shall not dwell on the general framework here. For more details, the reader is referred to [3], where the $U_\pm(a, b)h$ were first introduced and named 'affine coherent states', and to [2] and [4]. However, we need very little of this general framework here. In order to make this paper entirely self-contained, we shall therefore rederive the results we need by direct computation. We wanted to at least mention the group theoretic background to show that the constructions made here and in [1] are not sheer magic.

We start by restricting ourselves to the choice of a special $h$. This is analogous to the choice $\varphi = \varphi^0$ in [1]; the choice we make here is again dictated by convenience as well as ease of interpretation. Note however that we present in this section only one representative of a whole family of possibly useful choices; the definition of this family, and the corresponding computations are given in the appendix. The explicitly characterisable phase space localisation operators are accordingly different, as shown by a comparison of figures 3 and 6.

We define $h^+ \in L^2(\mathbb{R})$ by

$$(h^+)^\wedge(k) = \begin{cases} 2k\,\mathrm{e}^{-k} & k \geqslant 0 \\ 0 & k \leqslant 0 \end{cases} \tag{2.4a}$$

or

$$h^+(x) = (2/\pi)^{1/2}(1 + \mathrm{i}x)^{-2}. \tag{2.4b}$$

Here, and in what follows, we normalise the Fourier transform so that it is unitary from $L^2(\mathbb{R})$ to $L^2(\mathbb{R})$,

$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int \mathrm{d}x \exp(\mathrm{i}kx) f(x).$$

For the sake of convenience, we combine $(a, b) \in \mathbb{R}_+^* \times \mathbb{R}$ into one complex number, $z = b + \mathrm{i}a \in \mathbb{C}$, $\operatorname{Im} z > 0$.

For such $z$ we define

$$(h_z^+)(x) = [U_+(a, b)h](x) = [2/(\pi a)]^{1/2}\,[1 + \mathrm{i}(x - b)/a]^{-2} \tag{2.5a}$$

or

$$(h_z^+)^\wedge(k) = \begin{cases} 2a^{3/2}k\exp(-\mathrm{i}\bar{z}k) & k \geqslant 0 \\ 0 & k \leqslant 0. \end{cases} \tag{2.5b}$$

One finds, by direct computation, that, for all $f$, $g \in L^2(\mathbb{R})$,

$$\frac{1}{\pi}\int_0^\infty \mathrm{d}a\, a^{-2}\int_{-\infty}^\infty \mathrm{d}b\langle f, h_z^+\rangle\langle h_z^+ g\rangle = \int_0^\infty \mathrm{d}k\, \hat{f}(k)^*\hat{g}(k). \tag{2.6}$$

(In fact this is exactly (2.3) applied to $h^+$.) We shall also use the functions:

$$(h_z^-)(x) = [2/(\pi a)]^{1/2}[1 - \mathrm{i}(x-b)/a]^{-2} \tag{2.7a}$$

or

$$(h_z^-)^\wedge(k) = \begin{cases} 0 & k \geqslant 0 \\ 2a^{3/2}|k|\exp(\mathrm{i}zk) & |k| \leqslant 0. \end{cases} \tag{2.7b}$$

One then finds, for all $f$, $g \in L^2(\mathbb{R})$

$$\frac{1}{\pi}\int_0^\infty \mathrm{d}a\, a^{-2}\int_{-\infty}^\infty \mathrm{d}b\langle f, h_z^-\rangle\langle h_z^-, g\rangle = \int_{-\infty}^0 \mathrm{d}k\,\mathrm{d}k\ \hat{f}(k)^*\hat{g}(k). \tag{2.8}$$

Together (2.6) and (2.8) lead to:

$$\frac{1}{\pi}\sum_{\varepsilon = +\text{ or }-}\int_0^\infty \mathrm{d}a\, a^{-2}\int_{-\infty}^\infty \mathrm{d}b\, h_z^\varepsilon\langle h_z^\varepsilon, f\rangle = f. \tag{2.9}$$

Hence the $h_z^\varepsilon$ give rise to a decomposition + reconstruction formula of type (1.2).
Let us now investigate the localisation in phase space of the $h_z^\varepsilon$,

$$\langle x\rangle_{h_z^\varepsilon} = \int \mathrm{d}x\, x|h_z^\varepsilon(x)|^2 = b \tag{2.10a}$$

$$\langle p\rangle_{h_z^\varepsilon} = \int \mathrm{d}x\, x|(h_z^\varepsilon)^\wedge(k)|^2 = 3\varepsilon/2a. \tag{2.10b}$$

In view of (2.10) we therefore define, for $(p, q) \in \mathbb{R}^2$,

$$\Phi_{p,q}(x) = \begin{cases} h^+_{q+\mathrm{i}3/2p} & \text{if } p > 0 \\ 0 & \text{if } p = 0 \\ h^-_{q-\mathrm{i}3/2p} & \text{if } p < 0. \end{cases} \tag{2.11}$$

The function $\Phi_{p,q}$ is clearly localised, in phase space, around the point $(p, q)$ (except if

$p=0$, but since this is a set of measure zero in phase space, this does not matter).

Moreover, (2.9) can be rewritten as

$$\frac{2}{3\pi}\int_{-\infty}^{\infty} \mathrm{d}p \int_{-\infty}^{\infty} \mathrm{d}q\, \Phi_{p,q}\langle\Phi_{p,q}, f\rangle = f \tag{2.12}$$

which holds for all $f \in L^2(\mathbb{R})$.

We have therefore achieved our goal. Formula (2.12) is, as (1.2) was, a projection onto functions localised in phase space followed by an integral over all the localisation centres; the two operations together restore any given function. We can now again restrict the integral to a subset $S$ of phase space in order to define a localisation operator,

$$P_S f = \frac{2}{3\pi}\int\limits_{(p,q)\in S} \mathrm{d}p \int \mathrm{d}q\, \Phi_{p,q}\langle\Phi_{p,q}, f\rangle. \tag{2.13}$$

The operator $P_S$ defined by (2.13) is positive and bounded by one, i.e., all its eigenvalues are between 0 and 1. If $S$ has finite measure, i.e.,

$$\mathrm{Area}(S) = \int\limits_{(p,q)\in S} \mathrm{d}p \int \mathrm{d}q < \infty$$

then $P_S$ is trace class, i.e., the sum of the eigenvalues of $P_S$ is finite. This means, in particular, that the eigenvalues of $P_S$ tend to zero. This is the behaviour expected of the eigenvalues of a localisation operator.

On the other hand, explicit computation leads to

$$|\langle\Phi_{p_1,q_1}, \Phi_{p_2,q_2}\rangle| = \begin{cases} 0 & \text{if } \operatorname{sgn} p_1 \neq \operatorname{sgn} p_2 \\ 8[(\sqrt{p_1/p_2} + \sqrt{p_2/p_1})^2 + \tfrac{4}{9}p_1p_2(q_1-q_2)^2]^{-3/2} & \text{if } \operatorname{sgn} p_1 = \operatorname{sgn} p_2. \end{cases} \tag{2.14}$$

This scalar product clearly shows that the decomposition (2.13) completely separates positive and negative frequencies. Moreover, (2.14) implies:

$$|\langle\Phi_{p_1,q_1}, \Phi_{p_2,q_2}\rangle| \le C|p_1p_2|^{-3/2}|q_1-q_2|^{-3} \tag{2.15}$$

or

$$|\langle\Phi_{p_1,q_1}, \Phi_{p_2,q_2}\rangle| \le C\,(\min(|p_1|, |p_2|))^{3/2}|p_1-p_2|^{-3/2}. \tag{2.16}$$

The scalar product (2.14) is therefore small if $(p_1, q_1)$ and $(p_2, q_2)$ are far apart. The 'distance' defined on phase space by the right-hand side of (2.14) is however not uniform in $p$. With respect to this distance, one finds that

$$\langle\Phi_{p,q}, P_S f\rangle$$

is small if $(p, q)$ is far from $S$, which again corresponds to the notion of a localisation operator.

Note that the functions $\Phi_{p,q}$ are less well localised in phase space than the $\varphi_{p,q}$ used in [1]. The function $\hat{\Phi}_{p,q}$ has exponential fall-off, so that the localisation in frequency is not bad; the function $\Phi_{p,q}$ itself, however, only falls off as $|x|^{-2}$. Other similar examples given in the appendix have faster fall-off, but never better than an inverse polynomial.

It is generally impossible, for arbitrary sets $S$, to write down explicit expressions for the eigenfunctions and eigenvalues of $P_S$. As in [1], it turns out that it is possible to choose $S$ in such a way that the eigenfunctions and eigenvalues do become simple expressions. These special choices for $S$ are different from the discs in [1]; we shall see in the next section that the different construction of the functions $\Phi_{p,q}$ makes it possible to construct band-pass filters with explicitly known eigenvalues and eigenfunctions.

## 3. Restriction to special subsets of phase space

The localisation operators on discs constructed in [1] could be completely characterised because they commuted with a second-order differential operator. This operator was the harmonic oscillator Hamiltonian; the commutation followed from the very special role that the functions $\varphi^{0}_{p,q}$ (the 'canonical coherent states') play with respect to the harmonic oscillator [1]. Something similar will happen here.

For the sake of convenience we shall work with the variables $a$, $b$ and translate our results into $p$, $q$ variables at a later stage only. We also restrict ourselves to positive frequencies here; the treatment of negative frequencies is completely similar.

We proceed in several steps.

First we define a time-dependent flow on the half-plane $\mathbb{R}^*_+ \times \mathbb{R}$ by

$$b(t)+\mathrm{i}a(t)=z(t)=\frac{z\cos t-\sin t}{z\sin t+\cos t}. \tag{3.1}$$

This flow has the peculiarity that it preserves the circles

$$|z-\mathrm{i}c|^2=C^2-1 \tag{3.2}$$

or

$$a^2+b^2+1=2aC$$

and this for all $C \geqslant 1$ (see figure 2).

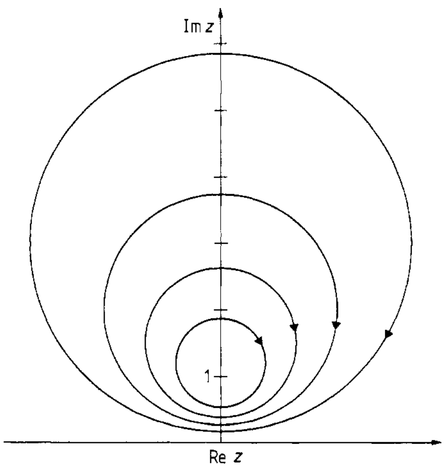


**Figure 2.** The flow $z(t)=(z\cos t-\sin t)/(z\sin t+\cos t)$.

This family of circles covers the whole half-plane. The circles are non-intersecting; for any given point $(b, a) \in \mathbb{R} \times \mathbb{R}_+^*$ one finds the unique circle, of the bundle, which goes through $(b, a)$ by taking

$$C = \frac{a^2 + b^2 + 1}{2a}.$$

For later reference, we also point out that the measure $a^{-2} \mathrm{d}a \, \mathrm{d}b$ on the half-plane $\mathbb{R}_+^* \times \mathbb{R}$ is left invariant by the flow $z(t)$, i.e.

$$a(t)^{-2} \, \mathrm{d}[a(t)] \, \mathrm{d}[b(t)] = a^{-2} \, \mathrm{d}a \, \mathrm{d}b \tag{3.3}$$

as can be checked by direct calculation.

Next we introduce the second-order differential operator $T$,

$$T = -k \frac{\mathrm{d}^2}{\mathrm{d}k^2} - \frac{\mathrm{d}}{\mathrm{d}k} + k + \frac{1}{k}. \tag{3.4}$$

We have used the notation '$k$' to stress that $T$ acts as a second-order differential operator on the *Fourier transform* of the functions under consideration, i.e. 'on the frequency side'. More explicitly,

$$(Tf)\hat{}(k) = -k(\hat{f})''(k) - (\hat{f})'(k) + k\hat{f}(k) + \frac{1}{k}\hat{f}(k).$$

One can of course also write an expression for the action of $T$ without having to introduce the Fourier transform. The operator $T$ is then the following (less transparent) integro-differential operator,

$$(Tf)(x) = \mathrm{i}(x^2 + 1) f'(x) + \mathrm{i}x f(x) + \mathrm{i} \int_{-\infty}^{x} \mathrm{d}y \, f(y).$$

For $z = b + \mathrm{i}a$, $z(t) = b(t) + \mathrm{i}a(t)$ as in (3.1), one finds

$$\exp(\mathrm{i}Tt)(h_z^+)\hat{} = a^{3/2} a(t)^{-3/2} (\bar{z} \sin t + \cos t)^{-3} [h_{z(t)}^+], \tag{3.5}$$

where $(h_z^+)\hat{}$ is defined as in (2.5*b*). It is easy to verify (3.5) by differentiation, i.e., by checking that $(\mathrm{d}/\mathrm{d}t - \mathrm{i}T)$, when applied to the right-hand side of (3.5), gives zero. Since for $t = 0$ the right-hand side of (3.5) is obviously equal to $(h_z^+)\hat{}$, this proves (3.5). Note that the coefficient $\gamma(z, t)$ in (3.5),

$$\gamma(z, t) = a^{3/2} a(t)^{-3/2} (\bar{z} \sin t + \cos t)^{-3}$$

can be rewritten as

$$\gamma(z, t) = \left( \frac{|\bar{z} \sin t + \cos t|}{\bar{z} \sin t + \cos t} \right)^3. \tag{3.6}$$

This is a pure phase factor, $|\gamma(z, t)| = 1$, as was to be expected since $\exp(\mathrm{i}Tt)$ is unitary, and $\|h_{z(t)}\| = 1$ for all $t$.

Similarly one shows

$$\exp(\mathrm{i}Tt)(h_z^-)\hat{} = \overline{\gamma(z, t)} (h_z^-)\hat{}.$$

We are now in a position to define sets $S$ such that the associated localisation operator $P_S$ commutes with $T$. Choose any $C > 1$. We define $S_C \subset \mathbb{R}^2$ by

$$S_C = \left\{ (p, q) \in \mathbb{R}^2; \frac{9}{4p^2} + q^2 + 1 \leq \frac{3}{|p|} C \right\}. \tag{3.7}$$

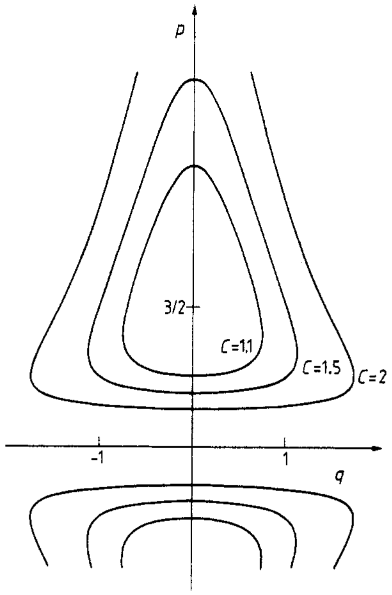


**Figure 3.** The boundary of the set $S_C=\{(p,q);|(q+\mathrm{i}3/2|p|)-\mathrm{i}C|^2\leqslant C^2-1\}$ for different values of $C$. For $C\to 1$ the set $S_C$ shrinks to the points $(0,\pm\frac{3}{2})$.

A few examples of such sets, for different values of $C$, are given in figure 3. Under the correspondence $a=3/2|p|$, $b=q$, as used in (2.11), the set $S_C$ corresponds exactly to the disc

$$|z-\mathrm{i}C|=(C^2-1)^{1/2}$$

in the half-plane $\mathbb{R}\times\mathbb{R}^*_+$; $z=b+\mathrm{i}a$. As was pointed out earlier, this disc is invariant under the flow $z\to z(t)$. This is one of the crucial elements in our proof that $T$ and $P_{S_C}$ commute. More concretely, we have

$$P_{S_C}=\frac{1}{\pi}\int\limits_{|z-\mathrm{i}C|^2\leqslant C^2-1}\frac{\mathrm{d}a}{a^2}\int\mathrm{d}b[h_z^+\langle h_z^+,f\rangle+h_z^-\langle h_z^-,f\rangle]. \tag{3.8}$$

Hence

$$(\exp(\mathrm{i}Tt)P_{S_C}f)\hat{\ }(k)$$

$$=\frac{1}{\pi}\int\limits_{|z-iC|^2\leqslant C^2-1}\frac{\mathrm{d}a}{a^2}\int\mathrm{d}b\{\gamma(z,t)[h^+_{z(t)}]\hat{\ }(k)\langle h_z^+,f\rangle+\overline{\gamma(z,t)}[h^-_{z(t)}]\hat{\ }(k)\langle h_z^-,f\rangle\}. \tag{3.9}$$

Using the invariance of the set $\{z;\ |z-\mathrm{i}C|^2\leqslant C^2-1\}$ and of the measure $a^{-2}\,\mathrm{d}a\,\mathrm{d}b$ under the flow $z\to z(t)$, we make the substitution $z'=z(t)$, and find

$$(3.9)=\frac{1}{\pi}\int\limits_{|z-\mathrm{i}C|^2\leqslant C^2-1} a^{-2}\,\mathrm{d}a\int\mathrm{d}b$$

$$\times\{\gamma[z(-t),t](h_z^+)\hat{}(k)\langle h^+_{z(-t)},f\rangle+\overline{\gamma[z(-t),t]}(h_z^-)\hat{}(k)\langle h^-_{z(-t)},f\rangle\}. \quad (3.10)$$

On the other hand, one easily checks from the expression (3.6) for $\gamma(z,t)$ that

$$\gamma[z(-t),t]=\overline{\gamma(z,-t)};$$

resulting in

$$\gamma[z(-t),t]\langle h^+_{z(-t)},f\rangle=\langle\gamma(z,-t)h^+_{z(-t)},f\rangle=\langle\exp(-\mathrm{i}Tt)h_z^+,f\rangle.$$

Hence

$$(3.10)=\frac{1}{\pi}\int\limits_{|z-\mathrm{i}C|^2\leqslant C^2-1} a^{-2}\,\mathrm{d}a\int\mathrm{d}b[(h_z^+)\hat{}(k)\langle h_z^+,\exp(\mathrm{i}Tt)f\rangle+(h_z^-)\hat{}(k)\langle h_z^-,\exp(\mathrm{i}Tt)f\rangle].$$

Putting everything together, we have thus, for all $f\in L^2(\mathbb{R})$

$$\exp(\mathrm{i}Tt)P_{S_C}f=P_{S_C}\exp(\mathrm{i}Tt)f$$

i.e. $P_{S_C}$ commutes with the second-order differential operator $T$. Note that, as in [1], and unlike the prolate spheroidal case [5], the commuting differential operator $T$ does not depend on the size of the localisation set $S_C$, which varies with $C$. It follows that the eigenfunctions of $P_{S_C}$, which are nothing but the eigenfunctions of $T$, will also be independent of $C$; the $C$ dependence is completely absorbed in the eigenvalues.

It is of interest to compute the area of $S_C$: according to standard (semiclassical) arguments the number of eigenvalues of $P_{S_C}$ larger than $\frac{1}{2}$ should be approximately equal to the area $|S_C|$ of $S_C$, multiplied by $(2\pi)^{-1}$ (the 'Nyquist density' in information theory).

One finds

$$|S_C|=\int\limits_{(p,q)\in S}\mathrm{d}p\int\mathrm{d}q=3\int\limits_{|b+\mathrm{i}(a-C)|^2\leqslant C^2-1}\mathrm{d}a\,a^{-2}\int\mathrm{d}b=6\pi(C-1). \quad (3.11)$$

In the next section we compute the eigenvalues and eigenfunctions of $P_{S_C}$.

## 4. Eigenfunctions and eigenvalues

Since $P_{S_C}$ and $T$ commute, finding the eigenfunctions of $P_{S_C}$ reduces to finding the eigenfunctions of $T$, with

$$(Tf)\hat{}(k)=\left(-k\frac{\mathrm{d}^2}{\mathrm{d}k^2}-\frac{\mathrm{d}}{\mathrm{d}k}+k+\frac{1}{k}\right)\hat{f}(k).$$

The operator $T$ decomposes, in fact, into a positive and a negative frequency part. More precisely, under the decomposition

$$L^2(\mathbb{R}) = H_2 \oplus H_2^- = \{f: \operatorname{supp} \hat{f} \subset \mathbb{R}_+\} \oplus \{f; \operatorname{supp} \hat{f} \subset \mathbb{R}_-\}$$

the domain $D(T)$ of $T$ decomposes also

$$D(T) = D_+ \oplus D_-$$

with $D_+ = D(T) \cap H_2$, $D_- = D(T) \cap H_2^-$. One sees indeed from (3.5) that

$$\begin{aligned} f \in H_2^- &\Rightarrow \langle f, h_z^+ \rangle = 0 \qquad &&\forall z \\ &\Rightarrow \langle \exp(-\mathrm{i}Tt)f, h_z^+ \rangle = \gamma(z,t) \langle f, h_{z(t)}^+ \rangle = 0 \qquad &&\forall z, \forall t \\ &\Rightarrow \operatorname{supp}(\exp(-\mathrm{i}Tt)f)^\wedge \subset \mathbb{R}_- \qquad &&\forall t \\ &\Rightarrow \exp(-\mathrm{i}Tt) f \in H_2^- \qquad &&\forall t. \end{aligned}$$

Similarly $f \in H_2 \to \exp(-\mathrm{i}Tt) f \in H_2 \, \forall t$, which shows that $\exp(-\mathrm{i}Tt)$ and therefore $T$ leave $H_2$ and $H_2^-$ invariant.

On the other hand, every term in $T$ is odd in the variable $k$. This implies that the parity operator $\Pi$ intertwines $T|_{D_+}$ and $-T|_{D_-}$. Concretely, for all $f \in D_+$, one has $(\Pi f)(k) = f(-k) \in D_-$ and $T\Pi f = -\Pi T f$.

It suffices, therefore, to compute the spectral decomposition of $T_+ = T|_{D_+}$. Note that $P_{S_C}$ is a compact operator (since $S_C$ has a finite surface). On the other hand $P_{S_C}$ is nondegenerate, because $P_{S_C} f = 0$ would imply $\langle h_z^+, f \rangle = 0$ for all $z$ in the set $|z - \mathrm{i}C|^2 \leqslant C^2 - 1$; since $(\operatorname{Im} z)^{-3/2} \langle h_z^+, f \rangle$ is an entire function in $z$, this would imply $\langle h_z^+, f \rangle$ for all $z$, hence $f = 0$.

The (unbounded) operator $T_+$ therefore commutes with a compact nondegenerate operator. This implies that $T_+$ has a purely discrete spectrum; the spectral decomposition of $T_+$ reduces to the computation of eigenfunctions.

One such eigenfunction is easy to find from (3.5).

The flow $z \to z(t)$ conserves all the circles $|z - \mathrm{i}C|^2 = C^2 - 1$, and in particular the degenerate circle $|z - \mathrm{i}|^2 = 0$, i.e., the point $z = \mathrm{i}$ or $b = 0$, $a = 1$. Substituting $z = \mathrm{i}$ into (3.6) gives $\gamma(\mathrm{i}, t) = \exp(3\mathrm{i}t)$. We find therefore

$$(\exp(\mathrm{i}Tt)h^+)^\wedge = \exp(3\mathrm{i}t)(h^+)^\wedge$$

which means that $\hat{h}^+(k) = 2k\,\mathrm{e}^{-k} \in D_+$ and

$$(Th^+)^\wedge = 3(h^+)^\wedge.$$

It turns out that all the eigenfunctions of $T_+$ have a structure similar to that of $h^+$. If we substitute for $f \in H_2$, $\hat{f}(k) = k\,\mathrm{e}^{-k} g(k)$, then we find

$$(T_+ f)^\wedge(k) = k\,\mathrm{e}^{-k}\left(-k\frac{\mathrm{d}^2}{\mathrm{d}k^2} + (2k-3)\frac{\mathrm{d}}{\mathrm{d}k} + 3\right) g(k).$$

The equation

$$-kg'' + (2k-3)g' + 3g = \lambda g \tag{4.1}$$

is, up to a dilation, the Laguerre equation. It has a solution $g$ leading to a square integrable $\hat{f}$ only if $\lambda = 3 + 2n$, where $n$ is a non-negative integer. For $\lambda = 3 + 2n$, the regular solution of (4.1) is the Laguerre polynomial $L_n^2(2k)$ (where the supercript 2 is an index, and not a power), with

$$L_n^\alpha(x)=\frac{1}{n!}\,\mathrm{e}^x x^{-\alpha}\frac{\mathrm{d}^n}{\mathrm{d}x^n}(\mathrm{e}^{-x}x^{n+\alpha})=\sum_{m=0}^{n}(-1)^m\frac{\Gamma(n+\alpha+1)}{\Gamma(n-M=1)\Gamma(\alpha+m+1)}\frac{1}{m!}x^m.$$

Substitution into $\hat f$ leads to

$$\hat f_n(k)=kL_n^2(2k)\,\mathrm{e}^{-k} \tag{4.2}$$

and

$$(T_+f_n)\hat{}=(2n+3)\hat f_n. \tag{4.3}$$

Since it is well known that for any $\alpha>0$ the functions $x^{\alpha/2}L_n^\alpha\exp(-x/2)$ constitute an orthogonal basis for $L^2(\mathbb{R}_+)$, (4.2) and (4.3) imply that we have the complete spectral decomposition of $T_+$.

The complete set of orthonormal eigenfunctions of $T$ is thus given by the functions (the normalisation follows from the standard normalisation of Laguerre polynomials)

$$(f_n^+)\hat{}=\begin{cases}2\sqrt{2}[(n+2)(n+1)]^{-1/2}k\,L_n^2(2k)\,\mathrm{e}^{-k} & k\geqslant 0\\ 0 & k\leqslant 0\end{cases}$$

$$(f_n^-)\hat{}=\begin{cases}0 & k\geqslant 0\\ 2\sqrt{2}[(n+2)(n+1)]^{-1/2}|k|\,L_n^2(2|k|)\exp^{-|k|} & k\leqslant 0,\end{cases}$$

with $(Tf_n^\pm)\hat{}=\pm(3+2n)(Tf_n^\pm)\hat{}$.

Since all the eigenvalues of $T_+$, $T_-$ are nondegenerate, these are also the eigenfunctions of $P_{SC}$. It turns out that the corresponding eigenvalues of $P_{SC}$ can also be computed explicitly. The crucial ingredient is the formula,

$$\int_0^\infty \mathrm{d}x\,x^\alpha L_n^\alpha(x)\exp(-sx)=\frac{\Gamma(\alpha+n+1)}{n!}\,s^{-\alpha-n-1}(s-1)^n$$

which holds for all non-negative integers $n$, all $\alpha>0$, and $s\in\mathbb{C}$ with $\operatorname{Re}s>0$. Applying this to the case at hand we find, with $z=b+\mathrm{i}a$, $a=\operatorname{Im}z>0$

$$\langle h_z^+,f_n^+\rangle=4[2(n+1)(n+2)]^{1/2}a^{3/2}\left(\frac{z-\mathrm{i}}{z+\mathrm{i}}\right)^n(1-\mathrm{i}z)^{-3}.$$

Consequently

$$\langle f_n^+,P_{SC}f_n^+\rangle=\frac{1}{\pi}\int_{|z-\mathrm{i}C|^2\leqslant C^2-1}a^{-2}\,\mathrm{d}a\int\mathrm{d}b\,|\langle h_z^+,f_z^+\rangle|^2$$

$$=\frac{(n+1)(n+2)}{2\pi}\int_{|z-\mathrm{i}C|^2\leqslant C^2-1}\mathrm{d}a\int\mathrm{d}b\,a|z+\mathrm{i}|^{-6}\left|\frac{z-\mathrm{i}}{z+\mathrm{i}}\right|^{2n}.$$

This integral can be calculated explicitly by making the change of variables

$$\xi=\alpha+\mathrm{i}\beta=\frac{(z-\mathrm{i})}{(z+\mathrm{i})}.$$

This change of variables maps the upper half plane ($\mathrm{Im}\, z > 0$) to the unit disc ($|\xi| < 1$). The set $|z - \mathrm{i}C|^2 \leq C^2 - 1$ is transformed into the smaller disc $|\xi|^2 \leq (C-1)/(C+1)$, and one finds

$$
\begin{aligned}
\lambda_n^+(C) &= \langle f_n^+, P_{S_C} f_n^+ \rangle \\
&= \frac{(n+1)(n+2)}{\pi} \int\limits_{\alpha^2+\beta^2 \leq (C-1)/(C+1)} \mathrm{d}\alpha \int \mathrm{d}\beta \quad (\alpha^2+\beta^2)^n (1-\alpha^2-\beta^2) \\
&= (n+1)(n+2) \int_0^{(C-1)/(C+1)} \mathrm{d}t\, t^n (1-t) \qquad (4.4) \\
&= (n+1)\left(1 - \frac{2}{C+1}\right)^{n+1} \left(\frac{2}{C+1} + \frac{1}{n+1}\right).
\end{aligned}
$$

The computation of $\lambda_n^-(C) = \langle f_n^-, P_{S_C} f_n^- \rangle$ is entirely analogous; one finds

$$\lambda_n^-(C) = \lambda_n^+(C).$$

Every eigenvalue of $P_{S_C}$ has therefore multiplicity two, with associated eigenfunctions $f_n^+, f_n^-$. The two linear combinations

$$\psi_n^{\mathrm{e}} = \frac{1}{\sqrt{2}} (f_n^+ + f_n^-)$$

$$\psi_n^{\mathrm{o}} = \frac{1}{\sqrt{2}\mathrm{i}} (f_n^+ - f_n^-)$$

are therefore also eigenfunctions of $P_{S_C}$. They have the advantage of being real functions of $x$,

$$(\psi_n^{\mathrm{e}})(x) = 2\sqrt{2}[\pi(n+1)(n+2)]^{-1/2} \int_0^\infty \mathrm{d}k\, k\mathrm{e}^{-k} L_n^2(2k) \cos kx \qquad (4.5a)$$

and

$$(\psi_n^{\mathrm{o}})(x) = 2\sqrt{2}[\pi(n+1)(n+2)]^{-1/2} \int_0^\infty \mathrm{d}k\, k\mathrm{e}^{-k} L_n^2(2k) \sin kx. \qquad (4.5b)$$

The $\psi_n^{\mathrm{e}}$, $\psi_n^{\mathrm{o}}$ are respectively even, odd functions in $x$ (whence their superscripts). Graphs for the first few values of $n$ are shown in figure 4.

We have thus achieved our aim. We have constructed a special set of phase space localisation operators, which localise in time and correspond to band-pass filters in frequency. For these operators, the eigenvalues and eigenfunctions can be written in closed, analytic form; they are given, respectively, by the relatively simple expressions (4.4) and (4.5).

We mentioned earlier that the construction as given here corresponds to one example of a two-parameter family of similar constructions. We give the general formulae in the appendix, without many details, since the general construction proceeds exactly along the same lines as in the example exposed above. Note that the sets $S_C^{\beta\gamma}$ corresponding to the general case may look slightly different from the $S_C$ given above (see figure 6).

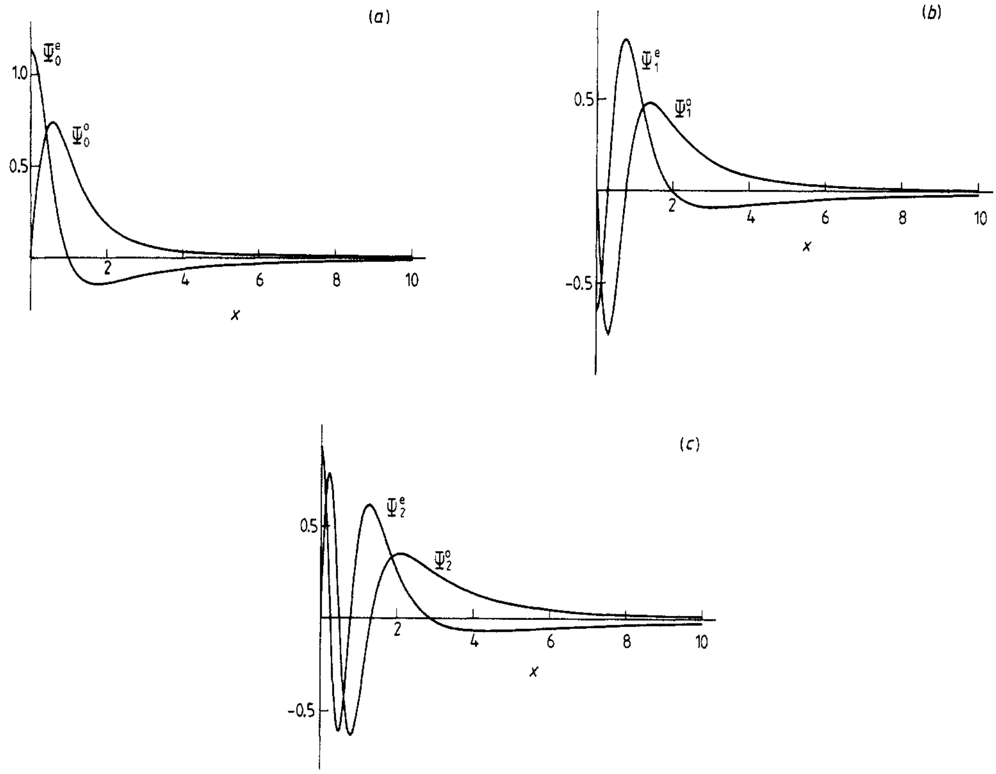


**Figure 4.** The first few eigenfunctions $\psi_m^o$, $\psi_n^e$ of $P_{S_C}$, for $n=0, 1, 2$.

We conclude this section by a discussion of the properties of the eigenvalues $\lambda_n(C)$ of $P_{S_C}$,

$$\lambda_n(C)=(n+1)\left(\frac{C-1}{C+1}\right)^{n+1}\left(\frac{2}{C+1}+\frac{1}{n+1}\right). \tag{4.6}$$

All these eigenvalues are between 0 and 1. For $C$ close to 1, the norm of $P_{S_C}$ is close to zero, since

$$\|P_{S_C}\|\leqslant \operatorname{Tr} P_{S_C}=\frac{2}{3\pi}|S_C|=4(C-1).$$

Consequently all the $\lambda_n(C)$ are then close to zero too, as can also be checked directly from (4.6). For reasonably large $C$ however, the $\lambda_n(C)$ behave typically as the eigenvalues of a filter operator: they are close to 1 for $n$ small, with

$$\lambda_0(C)=1-\frac{4}{(C+1)^2},$$

and they plunge to zero for a larger, $C$-dependent value of $n$. This behaviour is illustrated by figure 5, where the $\lambda_n(C)$ are plotted, as a function of $n$, for different values of $C$.

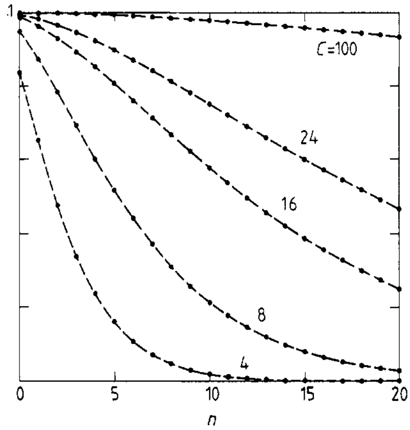


**Figure 5.** The eigenvalues $\lambda_n(C)$, $n=0, 1, \ldots, 20$ for a few values of $C$.

For any $\gamma \in (0, 1)$, the value of $n$ for which $\lambda_n(C)$ crosses $\gamma$ is asymptotically, for large $C$, determined by:

$$n = \eta C + \mathrm{O}(1)$$

with

$$\eta(2+\eta^{-1})(1-2C^{-1})^{\eta C} = \gamma$$

or

$$2\eta - \ln(1+2\eta) = -\ln\gamma + \mathrm{O}(C^{-1}).$$

This implies that

$$\begin{aligned}&\#\{\text{eigenfunctions}; \lambda_n(C) \geqslant \gamma\}\\ &= 2\#\{n; \lambda_n(C) \geqslant \gamma\}\\ &= 2CF^{-1}(-\ln\gamma) + \mathrm{O}(1),\end{aligned}$$

where $F(t) = 2t - \ln(1+2t)$. In particular,

$$\#\{\text{eigenfunctions}; \lambda_n(C) \geqslant \tfrac{1}{2}\} = 2CF^{-1}(\ln 2) + \mathrm{O}(1),$$

or, using (3.11),

$$\frac{\#\{\text{eigenfunctions}; \lambda_n(C) \geqslant \frac{1}{2}\}}{(2\pi)^{-1}|S_C|} = \tfrac{2}{3} F^{-1}(\ln 2) + \mathrm{O}(C^{-1}).$$

Since $\frac{2}{3} F^{-1}(\ln 2) \simeq 0.646 \neq 1$, this seems to be in contradiction with semiclassical arguments, or with the Nyquist density. This contradiction is only apparent, and is due to the fact that the width of the 'plunge region' is proportional to $C$, i.e. to $|S_C|$. We have indeed, for $\gamma > \delta \in (0, 1)$,

$$\#\{\text{eigenfunctions}; \delta \leqslant \lambda_n(C) < \gamma\} \times (2\pi)|S_C|^{-1}$$

$$= \tfrac{2}{3}[F^{-1}(-\ln \gamma) - F^{-1}(-\ln \delta)] + \mathrm{O}(|S_C|^{-1}).$$

In the case of the prolate spheroidal wavefunctions [5] or in [1] this ratio tends to zero, respectively as $|S|^{-1} \log |S|$ and as $|S|^{-1/2}$, for $|S| \to \infty$, which caused the ratio of the number of eigenvalues larger than $\frac{1}{2}$, and of the area $|S|$ to tend to the Nyquist density $(2\pi)^{-1}$. The fact that the width of the 'plunge region' is of the same order as $|S_C|$ itself is due to edge effects. These result from the non-uniform phase space localisation of the $\Phi_{p,q}$, as illustrated by the $p$ dependence in formulae (2.15), (2.16). A similar phenomenon occurred in a different use of the functions $\Phi_{p,q}$ for signal analysis purposes, where again it was noticed that the usual time-frequency density arguments could not be applied directly [6, §2.3.B.1].

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## Appendix. A two-parameter family of choices

The construction given in §§2–4 corresponds to the particular choice (2.5) and (2.7) for the functions $h_z^\pm$. The whole construction still works for the following two-parameter generalisations of the functions (2.5), (2.7). We define

$$(h^+_{z;\beta,\gamma})\hat{}(k) = \begin{cases} \gamma^{1/2}(2a)^{(2\beta+1)/2\gamma}\{\Gamma[(2\beta+1)/\gamma]\}^{-1/2}k^\beta \exp(-\mathrm{i}\bar{z}k^\gamma) & k \geqslant 0 \\ 0 & k \leqslant 0 \end{cases}$$

$$(h^-_{z;\beta,\gamma})\hat{}(k) = (h^+_{z;\beta,\alpha})\hat{}(-k);$$

where $z = b - ia$ and $a = \mathrm{Im}\ z > 0$; we always assume $\gamma > 0$, and $\beta > \frac{1}{2}(\gamma - 1)$. The resolution of the identity becomes then

$$\frac{2\beta + 1 - \gamma}{2\pi\gamma} \sum_{\varepsilon = + \text{ or } -} \int_0^\infty \mathrm{d}a\, a^{-2} \int_{-\infty}^\infty \mathrm{d}b\, h^\varepsilon_{z;\beta,\gamma} \langle h^\varepsilon_{z;\beta,\gamma}, f\rangle = f.$$

The phase-space localisation formulae (2.10) have to be replaced by

$$\langle x\rangle_{h^{\xi}_{\Sigma;\beta,\gamma}}=\beta\gamma^{-1}2^{1/\gamma}\Gamma(2\beta/\gamma)\{\Gamma[(2\beta+1)/\gamma]\}^{-1}ba^{-1+1/\gamma}\equiv C_2 ba^{-1+1/\gamma}$$

$$\langle p\rangle_{h^{\xi}_{\Sigma;\beta,\gamma}}=\varepsilon 2^{-1/\gamma}\Gamma[(2\beta+2)/\gamma]\{\Gamma[(2\beta+1)/\gamma]\}^{-1}a^{-1/\gamma}\equiv\varepsilon C_1 a^{-1/\gamma}.$$

Consequently we define

$$\Phi^{\beta,\gamma}_{p,q}(x)=\begin{cases}\Phi_{b_{\beta,\gamma}(p,q)+\mathrm{i}a_{\beta,\gamma}(p,q)} & \text{if } p>0\\ 0 & \text{if } p=0\\ \Phi_{b_{\beta,\gamma}(p,q)-\mathrm{i}a_{\beta,\gamma}(p,q)} & \text{if } p<0\end{cases}$$

where

$$a_{\beta,\gamma}(p,q)=(C_1)^{\gamma}p^{-\gamma}$$

$$b_{\beta,\gamma}(p,q)=(C_1)^{\gamma-1}C_2^{-1}qp^{1-\gamma}$$

with $C_1$, $C_2$ as above. This results in

$$(2\beta+1-\gamma)\Gamma[(2\beta+1)/\gamma]^2\{2\pi\beta\Gamma(2\beta/\gamma)\Gamma[(2\beta+2)/\gamma]\}^{-1}\int_{-\infty}^{\infty}\mathrm{d}p\int_{-\infty}^{\infty}\mathrm{d}q\,\Phi^{\beta,\gamma}_{p,q}\langle\Phi^{\beta,\gamma}_{p,q},f\rangle=f,$$

and

$$|\langle\Phi^{\beta,\gamma}_{p,q},\Phi^{\beta,\gamma}_{p_2,q_2}\rangle|$$

$$=\begin{cases}0 & \text{if } \operatorname{sgn} p_1\neq\operatorname{sgn} p_2\\ 2^{(2\beta+1)/\gamma}(p_1p_2)^{-(\beta+1/2)}\left((p_1^{-\gamma}+p_2^{-\gamma})^2+\dfrac{1}{C_1^2C_2^2}(p_2^{1-\gamma}q_2-p_1^{1-\gamma}q_1)^2\right)^{-(2\beta+1)/2\gamma} & \text{if } \operatorname{sgn} p_1=\operatorname{sgn} p_2.\end{cases}$$

The special sets for which the phase-space localisation operator can be completely characterised are given by

$$|z-\mathrm{i}C|^2\leqslant C^2-1$$

which corresponds, in the variables $p$, $q$, to

$$S^{\beta,\gamma}_C=\{(p,q)\in\mathbb{R}^2;\,(q/C_2)^2(C_1/|p|)^{2\gamma-2}+(C_1/|p|)^{2\gamma}+1\leqslant 2C(C_1/|p|)^{\gamma}\}.$$

In figure 6 we show a few examples of sets $S^{\beta,\gamma}_C$, for different values of $\beta$, $\gamma$.

The second-order differential operator of interest to us becomes (analogous to (3.4))

$$T_{\beta,\gamma}=-\gamma^{-2}k^{2-\gamma}\frac{\mathrm{d}^2}{\mathrm{d}k^2}-\gamma^{-1}k^{1-\gamma}\frac{\mathrm{d}}{\mathrm{d}k}+\beta^2\gamma^{-2}k^{-\gamma}+k^{\gamma}.$$

The proof that $T_{\beta,\gamma}$ commutes with the localisation operators $P_C^{\beta;\gamma}$

$$P_C^{\beta;\gamma}=(2\beta+1-\gamma)\Gamma[(2\beta+1)/\gamma]^2\{\beta\Gamma(2\beta/\gamma)\Gamma[(2\beta+2)/\gamma]\}^{-1}\int\mathrm{d}p\int_{(p,q)\in S_{C,\gamma}^{\beta}}\mathrm{d}q\;\Phi_{p,q}^{\beta,\gamma}\langle\Phi_{p,q}^{\beta,\gamma},\cdot\rangle$$

runs along the same lines as in §3.

This can be used, as in §4, to determine the eigenfunctions and eigenvalues of $P_C^{\beta;\gamma}$. The results are

$$P_C^{\beta;\gamma}\psi_{n;\beta,\gamma}^{\mathrm{c\ or\ o}}=\lambda_n^{\beta,\gamma}(C)\psi_{n;\beta,\gamma}^{\mathrm{c\ or\ o}}$$

with

$$\psi_{n;\beta,\gamma}^{\mathrm{c}}(x)=N_{n;\beta,\gamma}\int_0^\infty \mathrm{d}k\,k^{\beta}\exp(-k^{\gamma})L_n^{-1+(2\beta+1)/\gamma}(2k^{\gamma})\cos kx$$

$$\psi_{n;\beta,\gamma}^{\mathrm{o}}(x)=N_{n;\beta,\gamma}\int_0^\infty \mathrm{d}k\,k^{\beta}\exp(-k^{\gamma})L_n^{-1+(2\beta+1)/\gamma}(2k^{\gamma})\sin kx$$

with

$$N_{n;\beta,\gamma}=(\gamma 2^{(2\beta+1)/\gamma}n!)^{1/2}\{\pi\Gamma[n+(2\beta+1)/\gamma]\}^{-1/2},$$

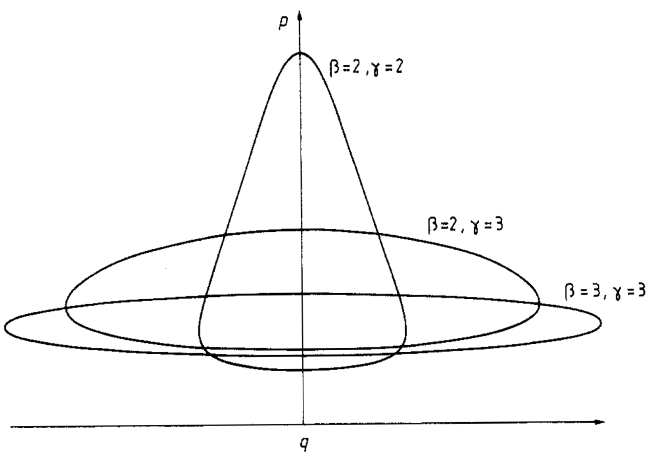


**Figure 6.** A few examples of the positive frequency half of $S_C^{\beta;\gamma}$, for one fixed value of $C(C=1.5)$, and different values of $\beta$, $\gamma$.

and

$$\lambda_n^{\beta,\gamma}(C) = \Gamma[2+\gamma+(2\beta+1)/\gamma]\{\Gamma(1+\gamma)\Gamma[1+(2\beta+1)/\gamma]\}^{-1}\int_0^{(C-1)/(C+1)} t^{\gamma}(1-t)^{(2\beta+1)/\gamma}\mathrm{d}t.$$

## References


[1] Daubechies I 1987 Time–frequency localisation operators—a geometric phase space approach *IEEE Trans. Information Theory* to be published

[2] Grossmann A, Morlet J and Paul Th 1985 Transforms associated to square integrable group representations: I *J. Math. Phys.* **26** 2473–9; 1986 Transforms associated to square integrable group representations: II *Ann. Inst. Henri Poincaré* **45** 293–309

[3] Aslaksen E W and Klauder J R 1968 Unitary representations of the affine group *J. Math. Phys.* **9** 206–11; 1969 Continuous representation theory using the affine group *J. Math. Phys.* **10** 2267–75

[4] Paul Th 1987 Affine coherent states and the radial Schrödinger equation *Ann. Inst. Henri Poincaré* submitted

[5] Slepian D and Pollak H O 1961 Prolate spheroidal wavefunctions, Fourier analysis and uncertainty: I. *Bell Syst. Tech. J.* **40** 43–64

Landau H J and Pollak H O 1961 Prolate spheroidal wavefunctions, Fourier analysis and uncertainty: II *Bell Syst. Tech. J.* **40** 65–84; 1962 Prolate spheroidal wavefunctions, Fourier analysis and uncertainty: III *Bell Syst. Tech. J.* **41** 1295–336

Slepian D 1976 On bandwidth *Proc. IEEE* **64** 292–300 (review)

[6] Daubechies I 1987 The wavelet transform, time frequency localisation and signal analysis. Submitted for publication