Journal of Mathematical Physics

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Ingrid Daubechies

Citation: J. Math. Phys. **21**, 1377 (1980); doi: 10.1063/1.524562
View online: http://dx.doi.org/10.1063/1.524562
View Table of Contents: http://jmp.aip.org/resource/1/JMAPAQ/v21/i6
Published by the American Institute of Physics.

Coherent states and projective representation of the linear canonical transformations [a)]

Ingrid Daubechies [b)]
Theoretische Natuurkunde, Vrije Universiteit Brussel, Pleinlaan 2, B 1050 Brussels, Belgium

(Received 26 November 1979; accepted for publication 25 January 1980)

Using a family of coherent state representations we obtain in a natural and coordinate-independent way an explicit realization of a projective unitary representation of the symplectic group. Dequantization of these operators gives us the corresponding classical functions.

1. INTRODUCTION

Canonical transformations and their relations to quantum mechanics have been studied extensively and in many different settings.[1-10] See, for instance Refs. 2 and 3 for a representation in terms of coherent states, Ref. 4 for applications of this treatment of the homogeneous linear canonical transformations, Ref. 5 for an application of the inhomogeneous linear canonical transformations, and Ref. 6 for a relation with Bogoliubov transformations and quasi-free states on the CCR algebra. In Ref. 7 it was advocated that the most natural way to study canonical transformations (we are only concerned with the linear ones here, even if we don't specify so further on) is (1) to work in a phase space realization, and (2) to consider a suitable family of closed subspaces of $L^2(E;dv)$, the square integrable functions on phase space, instead of only one Hilbert space as the basic setting. We follow this point of view here, and use it to derive a simple and natural expression for the operators of the symplectic group, the so-called metaplectic representation. This metaplectic representation was constructed already some ten years ago by Bargmann[2] and Itzykson[3] independently, who both used a holomorphic representation of the canonical commutation relations. Another approach can be found in Ref. 4. In this latter treatment, however, a certain class of linear transformations cannot be treated by the direct formula, and can only be recovered by taking products of linear transformations outside this class; this is not the case in either Refs. 2, 3, or the present paper. Our treatment differs from the ones given in Refs. 2 and 3 in that we obtain the representation almost automatically from the structure of the family of closed subspaces of $L^2(E;dv)$ mentioned above. In fact, for any state ψ with wave function ϕ_ψ in the coherent state representation, we obtain the image $W_S\phi_\psi$ of ϕ_ψ under a canonical transformation S simply by a substitution $(U_S\phi_\psi)(v) = \phi_\psi(S^{-1}v)$, followed by a projection. This projection has to be introduced because the naive substitution above does not always leave invariant the Hilbert space of coherent states. It turns out that this succession of two simple operations (a naive substitution, and a projection back onto the right space when things threaten to go wrong because the substitution has taken us out of it) is, up to some constant factor, a unitary operator. The family of these operators gives us our projective representation. We work with intrinsic and coordinate-free notations differing from the notations used in Refs. 2, 3, or 4. At the end of the paper we rewrite some of the results in the more familiar x–p notations.

Following the prescription given in Ref. 11 for the dequantization of these operators, we proceed then to compute the classical functions corresponding to the symplectic transformations. This calculation of classical functions for symplectic transformations has been done for one-parameter subgroups of the symplectic group.[8,9] One then only catches a small part of the symplectic group at a time; moreover, since the group is not exponential, not every symplectic transformation can be considered as an element of such a one-parameter subgroup. In Ref. 10 a general formula for the classical functions corresponding to symplectic transformations is given, valid whenever the symplectic transformation S is nonexceptional, i.e., whenever $\det(\mathbb{1} + S) \neq 0$. The case of an exceptional S is also tackled in Ref. 10 but in an indirect way. In this paper we derive an explicit expression (7.1) or (8.1) which holds for all cases, whether S is exceptional or not. Of course, if we assume S to be nonexceptional, our result simplifies, and we fall back on Huguenin's result [see Eq. (7.2)].

The paper is organized as follows: In Sec. 2 we introduce some definitions and notations, which are essentially those used in Refs. 7 and 11. We also state our results at the end of this section. In Secs. 3–6 we construct a unitary projective representation of the symplectic group using the family of Hilbert spaces mentioned above. In Sec. 7 we dequantize these operators to obtain the corresponding classical functions. Up to Sec. 7 everything is written in intrinsic and coordinate-free notations. In Sec. 8 we show in which way the results can be rewritten in the usual x–p notations. Section 9 contains some applications: calculation of the classical functions for some one-parameter subgroups of the symplectic group; a method for calculating any matrix element of the evolution operator associated to a quadratic Hamiltonian. We end with some remarks.

2. DEFINITIONS AND NOTATIONS

Note: Following A. Grossmann, we are borrowing most of the following notations from D. Kastler, who introduced them in a slightly different setting.[12]

[a)] Part of this work was done at the CNRS Marseille.
[b)] Research fellow at the Interuniversitair Instituut voor Kernwetenschappen (Research Project 21 EN).

 0022-2488/80/061377-13$1.00

We denote by E a real vector space of even dimension $2n < \infty$. On this vector space a symplectic form σ (i.e., a bilinear, antisymmetric map from $E \times E$ to $\mathbb{R}$) is defined, which we assume to be nondegenerate [i.e., $\sigma(u,v) = 0$, $\forall u \in E \Rightarrow v = 0$]. Using this symplectic form we can define an affine function φ on $E \times E \times E$ [10,13]:

$$\varphi(u,v,w) = 4(\sigma(u,w) + \sigma(w,v) + \sigma(v,u)),$$

which can be interpreted as the surface of the oriented triangle with vertices u,v,w, and which plays a role in the so-called twisted product (see for instance Ref. 11).

We normalize the invariant measure dv on E by requiring $F^2 = \mathbb{1}$, where F is the symplectic Fourier transform

$$(Ff)(v) = 2^{-n} \int dw\, e^{i\sigma(v,w)} f(w).$$

Let $\mathscr{H}$ be the Hilbert space $L^2(E;dv)$.

On $\mathscr{H}$ we define a family of unitary operators $\{W(a); a \in E\}$ by

$$(W(a)\psi)(v) = e^{i\sigma(a,v)}\psi(v-a).$$

These operators $W(a)$ satisfy the relation

$$W(a)\, W(b) = e^{i\sigma(a,b)} W(a+b);$$

hence, they form a representation of the Weyl commutation relations. This representation is not irreducible, but we can build a family of irreducible subrepresentations by introducing complex structures.

A linear map $J: E \to E$ is said to be a σ-allowed complex structure if

$$J^2 = -\mathbb{1},$$

$$\sigma(Jv,Jw) = \sigma(v,w), \quad \forall v,w, \in E,$$

$$\sigma(v,Jv) > 0, \quad \text{if } v \neq 0.$$

For any such σ-allowed complex structure we define the function

$$\Omega_J(v) = \exp\left[-\tfrac{1}{2}\sigma(v,Jv)\right].$$

These Ω_J are elements of $\mathscr{H}$. We define now the following subspaces of $\mathscr{H}$:

$$\mathscr{H}_J = \{\psi \cdot \Omega_J | \psi \text{ is holomorphic w.r.t. } J$$

$$\text{(i.e., } \nabla^{Ja}\psi = i\nabla^a\psi, \forall a \in E) \text{ and } \psi \cdot \Omega_J \in \mathscr{H}\}.$$

These $\mathscr{H}_J$ are closed subspaces of $\mathscr{H}$,[14] which are left invariant by the $W(v)$. Furthermore, the restrictions $W_J(v) = W(v)|_{\mathscr{H}_J}$ of the $W(v)$ to the spaces $\mathscr{H}_J$ form irreducible representations of the Weyl commutation relations.[14] (The notations used in Ref. 14 are different from the ones used here. The reader who would want to compare should make the obvious unitary transformation.)

In each of the $\mathscr{H}_J$ we can consider the elements

$$\Omega_J^a = W(a)\Omega_J;$$

they are in fact the coherent states with respect to the choice of complex structure (or equivalently of complex polarization) J. The closed span of the Ω_J^a is the Hilbert space $\mathscr{H}_J$; the Ω_J^a have moreover the following useful "reproducing property"[14,15]:

$$\forall \Psi \in \mathscr{H}_J : \Psi(a) = (\Omega_J^a, \Psi). \tag{2.1}$$

As a result of this any operator A_J on $\mathscr{H}_J$ can be represented by its matrix elements $A_J(a,b) = (\Omega_J^a, A_J\Omega_J^b)$:

$$\Psi \in \mathscr{H}_J \Rightarrow (A_J\Psi)(a) = \int db A_J(a,b)\Psi(b).$$

Because of this property we also call $A(\cdot,\cdot)$ the kernel of the operator A.

Whenever a function f on phase space is given, we can compute its quantal counterpart on the Hilbert space $\mathscr{H}_J$:

$$Q_J(f) = 2^{-n} \int dv (Ff)(v) W_J(-v/2); \tag{2.2}$$

this is the usual Weyl quantization procedure when an irreducible representation of the Weyl commutation relations is given. We can rewrite this expression as[15]

$$Q_J(f) = 2^n \int dv\, f(v) \Pi_J(v), \tag{2.3}$$

where $\Pi_J(v) = W_J(2v)\Pi$, and $(\Pi\,\Psi)(v) = \Psi(-v)$ for any Ψ in $\mathscr{H}$.

Note that both expressions (2.2) and (2.3) can be used to define $Q(f)$ as an operator on the big space $\mathscr{H}$ (at least for reasonable f) which, when restricted to the different $\mathscr{H}_J$, yields $Q_J(f)$ again.[7] The correspondence $f \to Q_J(f)$ can be inverted, i.e., an operator A_J on $\mathscr{H}_J$ can be "dequantized" as follows[11]:

$$f_A(v) = \iint da\, db A_J(a,b) \{b,a|v\}_J, \tag{2.4}$$

with

$$\{b,a|v\}_J = 2^n(\Omega_J^b, \Pi_J(v)\Omega_J^a). \tag{2.5}$$

It is easy to check that the dequantized function of $Q_J(f)$ is always f, regardless of the chosen J.

In these notations our results can be stated as follows: For any symplectic transformation S (i.e., any linear map on E leaving σ invariant; see Sec.4) we have a classical function w_S given by

$$w_{S,J}(v) = (\det[(\mathbb{1} - iJ) + S(\mathbb{1} + iJ)])^{1/2} \times \int db \Omega_J(b + Sb - 2v) e^{i\varphi[(b/2),v,(Sb/2)]}$$

[see Eq. (7.1); we have chosen one fixed complex structure J]. Here one can choose either of the two square roots of the determinant. If there is no good reason to do otherwise, we choose the one with argument in $]-\pi/2, \pi/2]$. If $\det(\mathbb{1} + S) \neq 0$, this simplifies to give [see Eq. (7.2)]

$$w_{S,J}(v) = \frac{2^n}{\sqrt{\det(\mathbb{1} + S)}} e^{4i\sigma(v,(\mathbb{1}+S)^{-1}v)},$$

which is the result obtained in Ref. 10.

The operators $W_J(S)$ in $\mathscr{H}_J$ which are quantizations of these functions are given by

$$W_J(S) = 2^{-n}(\det[(\mathbb{1} - iJ) + S(\mathbb{1} + iJ)])^{1/2} \times \int db\, |\Omega_J^{Sb})(\Omega_J^b|$$

[see Eq. (7.4)]. Another form of this operator can be found in Sec. 6. These operators form a unitary projective representation of the symplectic group:

$$W_J(S_1)W_J(S_2) = \rho_J(S_1,S_2)W_J(S_1,S_2).$$

The multiplier ρ takes only the values ± 1; given S_1,S_2 it is possible to determine the sign of $\rho(S_1,S_2)$ once one has fixed one's choice of the square roots of the corresponding determinants $\det[(1-iJ)+S(1+iJ)]$ (see Sec. 6).

Moreover, the operators are the representation on the quantum level of the linear canonical transformations on phase space. We have indeed for any symplectic transformation S (the symplectic transformations are in fact just the linear canonical transformations) and for any function f on E:

$$W_J(S)Q_J(f)W_J^*(S) = Q_J(\mathbf{S}f) \quad \text{with } \mathbf{S}f(v) = f(S^{-1}v)$$

[see Eq. (6.4)]. Analogous relations hold for the $w_{S,J}$:

$$w_{S_1,J} \circ w_{S_2,J} = \rho_J(S_1,S_2)w_{S_1,S_2,J},$$

$$w_{S,J} \circ f \circ w_{S,J}^* = \mathbf{S}f,$$

where $\circ$ denotes the twisted product (see, for instance, Ref. 11). These formulas depend on the choice of J. The relation between the $w_{S,J}$ and the $w_{S,J'}$ are given in Sec. 9.

3. PROJECTION OPERATORS ON THE $\mathscr{H}_J$

Since the $\mathscr{H}_J$ are closed subspaces of $\mathscr{H}$, there exist orthogonal projection operators P_J mapping $\mathscr{H}$ to $\mathscr{H}_J$. With the help of the Ω_J^a, these projection operators can be explicitly constructed.

Indeed, since the Ω_J^a span the subspace $\mathscr{H}_J$, we have

$$P_J\Psi = 0 \Longleftrightarrow (\Omega_J^a,\Psi) = 0.$$

On the other hand, we have also Eq. (2.1):

$$P_J\Psi = \Psi \Longleftrightarrow \Psi(a) = (\Omega_J^a,\Psi).$$

It is now obvious that P_J is given by

$$(P_J\Psi)(a) = (\Omega_J^a,\Psi). \tag{3.1}$$

Written more explicitly this means that the projection $P_J\psi$ of any square integrable function ψ on $\mathscr{H}_J$ is given by

$$(P_J\psi)(a) = \int dv\, \overline{\Omega^a(v)}\, \psi(v).$$

This function $P_J\psi$ has automatically the right holomorphy properties.

This can also be written as (in Dirac's bra–ket notation)

$$P_J = \int |\Omega_J^a)da(\Omega_J^a|.$$

Since on the other hand the $\mathscr{H}_J$ are invariant under the $W(v)$, we have

$$P_J W(v) = W_J(v)P_J, \quad \forall v \in E. \tag{3.2}$$

4. THE SYMPLECTIC GROUP AND ITS NATURAL REPRESENTATION IN $L^2(E;dv)$

The symplectic group $\mathrm{Sp}(E,\sigma)$ is defined as the set of real linear maps from E to E which leave σ invariant:

$$S \in \mathrm{Sp}(E,\sigma) \Longleftrightarrow \sigma(Sv,Sw) = \sigma(v,w), \quad \forall v,w \in E.$$

Note that for any given complex structure J, and for any $S \in \mathrm{Sp}(E,\sigma)$, the map SJS^{-1} is again a complex structure. The converse is also true: Whenever two complex structures J, J' are given, there exists a symplectic transformation S in $\mathrm{Sp}(E,\sigma)$ such that $J' = SJS^{-1}$ [For any J one can construct a J-symplectic basis of E, i.e., a basis $\{e_1,...,e_n,f_1,...,f_n\}$ in E such that $\sigma(e_i,e_j) = 0 = \sigma(f_i,f_j)$, $\sigma(e_i,f_j) = \delta_{ij}$ and $f_j = Je_j$. The map S mapping a J-symplectic basis to a J'-symplectic basis is in $\mathrm{Sp}(E,\sigma)$,[16] and satisfies $SJS^{-1} = J'$.]

A symplectic transformation always has determinant 1.[16] Since any complex structure J is obviously in $\mathrm{Sp}(E,\sigma)$, we have in particular $\det J = 1$. This will be used in calculations later on.

As in the case of the Galilei group or the Poincaré group we can define the inhomogeneous symplectic group $\mathrm{ISp}(E,\sigma)$ by taking the semidirect product of $\mathrm{Sp}(E,\sigma)$ with the translation group on E: the elements of $\mathrm{ISp}(E,\sigma)$ are pairs (S,a) with $S \in \mathrm{Sp}(E,\sigma)$, $a \in E$; the product of two such pairs is defined as

$$(S,a)(S',a') = (SS',Sa'+a).$$

The natural representation of $\mathrm{Sp}(E,\sigma)$ on $L^2(E;dv)$ is given by

$$(U_S\Psi)(v) = \Psi(S^{-1}v).$$

This is obviously a unitary representation of $\mathrm{Sp}(E,\sigma)$. Note that the $\mathscr{H}_J$ are not invariant under U_S unless $SJS^{-1} = J$. An easy calculation yields

$$U_S\Omega_J^a = \Omega_{SJS^{-1}}^{Sa}. \tag{4.1}$$

Taking into account the definition (3.1) of the orthogonal projection operators P_J, we see that this implies

$$\begin{aligned}(U_SP_{J'}\Omega_J^b)(a) &= (\Omega_{J'}^{S^{-1}a},\Omega_J^b)\\ &= (\Omega_{SJ'S^{-1}}^a,\Omega_{SJS^{-1}}^{Sb})\\ &= (P_{SJ'S^{-1}}U_S\Omega_J^b)(a);\end{aligned}$$

hence,

$$U_S \circ P_{J'}\big|_{\mathscr{H}_J} = P_{SJ'S^{-1}} \circ U_S\big|_{\mathscr{H}_J}. \tag{4.2}$$

It is easy to see that

$$U_S W(v) = W(Sv)U_S. \tag{4.3}$$

Hence, we have also a unitary representation of $\mathrm{ISp}(E,\sigma)$ on $L^2(E;dv)$ given by

$$U_{S,a} = W(a)U_S.$$

5. INTERTWINING OPERATORS BETWEEN THE $\mathscr{H}_J$ (SEE ALSO REF. 7, AND IN A SOMEWHAT DIFFERENT CONTEXT REF. 17)

We will use the natural representation of $\mathrm{Sp}(E,\sigma)$ on $L^2(E;dv)$ to define a projective representation on each $\mathscr{H}_J$. Since the U_S map each $\mathscr{H}_J$ to $\mathscr{H}_{SJS^{-1}}$, we will need some device to map everything back from $\mathscr{H}_{SJS^{-1}}$ to $\mathscr{H}_J$. This device will be given by the maps intertwining the $W_J(v)$: moreover, we will be able to construct these intertwining maps explicitly.

Let any two J, J' be given. Since the $W_J(v)$ form an irreducible representation of the Weyl commutation relations on $\mathscr{H}_J$, and the same is true for the $W_{J'}(v)$ on $\mathscr{H}_{J'}$, von Neumann's theorem tells us there exists a unitary map $T_{J',J}$ from $\mathscr{H}_J$ to $\mathscr{H}_{J'}$ intertwining the $W_J(v)$ and $W_{J'}(v)$. Hence,

$$T_{J',J}W_J(v) = W_{J'}(v)T_{J',J}. \tag{5.1}$$

We proceed now to compute these $T_{J',J}$.

Combining $T_{J',J}^{-1}$ [Eq. (5.1)] $T_{J',J}^{-1}$ with Eq. (3.2), we see that

$$P_{J'}\,|_{\mathscr{H}_J} T_{J',J}^{-1} W_{J'}(v) = P_{J'}\,|_{\mathscr{H}_J} W_J(v) T_{J',J}^{-1} = W_{J'}(v) P_{J'}\,|_{\mathscr{H}_J} T_{J',J}^{-1}.$$

Hence, the operator $P_{J'}\,|_{\mathscr{H}_J} T_{J',J}^{-1} \in \mathscr{B}(\mathscr{H}_{J'})$ commutes with all the $W_{J'}(v)$, which implies that it is a multiple of $\mathbb{1}_{\mathscr{H}_{J'}}$, or

$$P_{J'}\,|_{\mathscr{H}_J} = \gamma_{J',J} T_{J',J}. \tag{5.2}$$

The constant $\gamma_{J',J}$ is always different from zero: If it were zero, we would have $\mathscr{H}_{J'} \perp \mathscr{H}_J$; hence, $(\Omega_{J'},\Omega_J) = 0$, which is impossible since this inner product is the integral of a strictly positive function. On the other hand, if $|\gamma_{J',J}| = 1$, then $\|P_{J'}\Psi\| = \|\Psi\|$; hence, $P_{J'}\Psi = \Psi$ for any Ψ in $\mathscr{H}_J$, or $\mathscr{H}_{J'} = \mathscr{H}_J$. From Eq. (2.1) we see that this implies that the $\mathscr{H}_J$ are all different ($J' \neq J \Rightarrow \mathscr{H}_{J'} \neq \mathscr{H}_J$), have trivial intersection (this is essentially Schur's lemma), but that no nontrivial vector in $\mathscr{H}_J$ can be orthogonal to all vectors $\mathscr{H}_{J'}$.

Note also that Eq. (5.2) implies that, up to some constant, $P_{J'}P_J$ is a partial isometry in $\mathscr{H}$ with initial subspace $\mathscr{H}_J$ and final subspace $\mathscr{H}_{J'}$ which, as a map from $\mathscr{H}_J$ to $\mathscr{H}_{J'}$, intertwines W_J with $W_{J'}$. From Eq. (5.2) we see that

$$|\gamma_{J',J}|^2 = \|P_{J'}\Omega_J\|^2 = (\Omega_J,P_{J'}\Omega_J) = \int da\,|(\Omega^a_{J'},\Omega_J)|^2. \tag{5.3}$$

For the time being, we choose $\gamma_{J',J} = |\gamma_{J',J}|$. This amounts to fixing the up to now undetermined phase factor in $T_{J',J}$.

Putting now $\beta_{J',J} = \gamma_{J',J}^{-1}$ (which we are allowed to do, since $\gamma_{J',J} \neq 0$) we have

$$T_{J',J} = \beta_{J',J} P_{J'}\,|_{\mathscr{H}_J}. \tag{5.4}$$

We can use Eq. (5.3) to compute $\beta_{J',J}$; after some calculation [see Eq. (A16)] we get

$$\beta_{J',J} = 2^{-n/2}[\det(J+J')]^{1/4}. \tag{5.5}$$

It is obvious from Eq. (5.5) that

$$\beta_{J',J} = \beta_{J',J},$$

$$\beta_{SJ'S^{-1},SJS^{-1}} = \beta_{J',J}, \quad \forall S\in \mathrm{Sp}(E,\sigma).$$

Moreover, if we consider three subspaces $\mathscr{H}_J$, $\mathscr{H}_{J'}$, $\mathscr{H}_{J''}$, then the map $T_{J'',J'} \circ T_{J',J}$ is a unitary map intertwining the $W_J(v)$ and the $W_{J''}(v)$. Owing to the irreducibility of the $W_J(v)$, this implies the existence of a phase factor $\alpha(J'',J',J)$ such that

$$T_{J'',J'} \circ T_{J',J} = \alpha(J'',J',J) T_{J'',J}. \tag{5.6}$$

With our choice for $\beta_{J',J}$, this α is given by

$$\alpha(J'',J',J) = \|P_{J'}\Omega_J\|^{-1}\|P_J\Omega_{J''}\|^{-1}P_{J''}\Omega_{J'}\|^{-1} \times (P_{J''}\Omega_J,P_{J'}\Omega_J) = \frac{(P_{J''}\Omega_J,P_{J'}\Omega_J)}{|(P_{J''}\Omega_J,P_{J'}\Omega_J)|}. \tag{5.7}$$

Since

$$(P_{J''}\Omega_J,P_{J'}\Omega_J) = \int da(\Omega_J,\Omega^a_{J''})(\Omega^a_{J'},\Omega_J) = \int da(\Omega_J^{-a},\Omega_{J''})(\Omega_{J'},\Omega_J^{-a}) = (\Omega_{J'},P_J\Omega_{J''}),$$

we can also write α as

$$\alpha(J'',J',J) = \frac{(\Omega_{J'},P_J\Omega_{J''})}{|(\Omega_{J'},P_J\Omega_{J''})|}. \tag{5.7'}$$

In particular, $\alpha(J'',J,J) = \alpha(J,J',J) = \alpha(J',J',J) = 1$.

Note, incidentally, that as a by-product of our reasoning above we have proved that

$$|(\Omega_{J'},P_J\Omega_{J''})| = |(P_{J''}\Omega_J,P_{J'}\Omega_J)| = \|P_{J'}\Omega_J\|\,\|P_J\Omega_{J''}\|\,\|P_{J''}\Omega_{J'}\|.$$

Since $\alpha(J,J',J) = 1$, we have

$$T^*_{J',J} = T^{-1}_{J',J} = T_{J,J'}. \tag{5.8}$$

Inverting Eq. (5.6) and using Eq. (5.8), we get

$$\alpha(J,J',J'') = \alpha^{-1}(J'',J',J) = \alpha^*(J'',J',J).$$

Combining this with Eq. (5.7') one can easily show that

$$\alpha(J'',J',J) = \alpha(J',J,J'') = \alpha(J,J'',J').$$

From Eq. (5.7) or (5.7') one sees again that

$$\alpha(SJ''S^{-1},SJ'S^{-1},SJS^{-1}) = \alpha(J'',J',J), \quad \forall S\in \mathrm{Sp}(E,\sigma).$$

We have of course also

$$\alpha(J''',J'',J')\,\alpha(J''',J',J) = \alpha(J''',J'',J)\,\alpha(J'',J',J).$$

We can calculate α explicitly from Eq. (5.7') (see Appendix A). The result is

$$\alpha(J'',J',J) = \lim_{\xi\to 1}\left(\exp(i\arg\sqrt{\det(2J+J'+J''-i\xi\mathbb{1}-i\xi J'J'')}\,)\right) \tag{5.9}$$

Here the argument of the square root of the determinant is determined by the requirement that it be continuous in ξ and equal to zero for $\xi = 0$ (see Appendix A).

6. A PROJECTIVE REPRESENTATION OF THE SYMPLECTIC GROUP ON THE $\mathscr{H}_J$

We have now a device to map from a $\mathscr{H}_J$ to a $\mathscr{H}_{J'}$: It is given by the orthogonal projection operator onto $\mathscr{H}_{J'}$, which, when restricted to $\mathscr{H}_J$, is a unitary map up to some constant we can compute. This device will now be used to define a family of maps $\{V_J(S);\, S\in \mathrm{Sp}(E,\sigma)\}$ which will be unitary maps from $\mathscr{H}_J$ to itself:

$$V_J(S) = T_{J,SJS^{-1}} \circ U_S\,|_{\mathscr{H}_J} = \beta_{J,SJS^{-1}} P_J \circ U_S\,|_{\mathscr{H}_J}. \tag{6.1}$$

Here β is given by Eq. (5.5):

$$\beta_{J,SJS^{-1}} = 2^{-n/2}[\det(SJ+JS)]^{1/4}.$$

Since both the U_S and the $T_{J',J}$ are unitary, and since $U_S\mathscr{H}_J = \mathscr{H}_{SJS^{-1}}$, the $V_J(S)$ are obviously unitary maps. In some sense they are even the most natural unitary maps in $\mathscr{B}(\mathscr{H}_J)$ representing the symplectic transformations: For any S, we simply apply U_S; since U_S does not leave $\mathscr{H}_J$ invariant in general, we project back onto $\mathscr{H}_J$, and we normalize.

The $V_J(S)$ form a projective representation of $\mathrm{Sp}(E,\sigma)$, and we can even give an expression for the multiplier. Indeed, we have

$$U_S \circ T_{J',J} = \beta_{J',J} U_S \circ P_{J'} |_{\mathscr{H}_J} = \beta_{SJ'S^{-1},SJS^{-1}} P_{SJ'S^{-1}} \circ U_S |_{\mathscr{H}_J} = T_{SJ'S^{-1},SJS^{-1}} \circ U_S |_{\mathscr{H}_J} .$$

Hence,

$$\begin{aligned} V_J(S_1) \circ V_J(S_2) &= T_{J,S_1JS_1^{-1}} \circ U_{S_1} \circ T_{J,S_2JS_2^{-1}} \circ U_{S_2} |_{\mathscr{H}_J} = T_{J,S_1JS_1^{-1}} \circ T_{S_1JS_1^{-1},S_1S_2JS_2^{-1}S_1^{-1}} \circ U_{S_1} \circ U_{S_2} |_{\mathscr{H}_J} \\ &= \alpha(J,S_1JS_1^{-1},S_1S_2JS_2^{-1}S_1^{-1}) T_{J,S_1S_2J(S_1S_2)^{-1}} \circ U_{S_1S_2} |_{\mathscr{H}_J} \\ &= \alpha(S_1^{-1}JS_1,J,S_2JS_2^{-1}) V_J(S_1S_2) . \end{aligned}$$

So we have indeed a projective representation of $\mathrm{Sp}(E,\sigma)$, with multiplier $\tilde{\alpha}(S_1,S_2) = \alpha(S_1^{-1}JS_1,J,S_2JS_2^{-1})$, where the right-hand side is given by Eqs. (5.7′) and (5.9):

$$\begin{aligned} \tilde{\alpha}(S_1,S_2) = \alpha(S_1^{-1}JS_1,J,S_2JS_2^{-1}) &= \exp\big(i \arg(\Omega_{S_1^{-1}JS_1}, P_J \Omega_{S_2JS_2^{-1}})\big) \\ &= \exp\Big(i \arg\Big(\int da(\Omega_J, U_{S_1} \Omega_J^a)(\Omega_J^a, U_{S_2} \Omega_J)\Big)\Big) \\ &= \lim_{\xi \to 1} \big(\exp(i \arg \sqrt{\det(2J + S_2JS_2^{-1} + S_1^{-1}JS_1 - i\xi \mathbb{1} - i\xi S_1^{-1}JS_1S_2JS_2^{-1})}\,)\big) . \end{aligned}$$

Here the argument of the square root of the determinant is determined by the same continuity requirement as at the end of the preceding section.

These $\tilde{\alpha}$ have the usual multiplier property

$$\tilde{\alpha}(S_1,S_2S_3)\tilde{\alpha}(S_2,S_3) = \tilde{\alpha}(S_1,S_2)\tilde{\alpha}(S_1S_2,S_3) .$$

The properties of the $\alpha(J'',J',J)$ listed at the end of the preceding section imply

$$\tilde{\alpha}(S,\mathbb{1}) = 1 \tag{6.2a}$$

or even

$$S_1JS_1^{-1} = J \Rightarrow \tilde{\alpha}(S_1,S_2) = 1 . \tag{6.2b}$$

Also

$$\tilde{\alpha}(S_1^{-1},S_2^{-1}) = \tilde{\alpha}^*(S_2,S_1) , \tag{6.2c}$$

$$\tilde{\alpha}(S,S^{-1}) = 1. \tag{6.2d}$$

The operators $V_J(S)$ thus form a projective representation of $\mathrm{Sp}(E,\sigma)$ which is, however, not the metaplectic representation. In this latter representation one deals in fact with a true representation R of a two-sheeted covering of $\mathrm{Sp}(E,\sigma)$ in which the representation images of the two lifts Σ_1, Σ_2 of the same symplectic operator S differ only by a sign: $R(\Sigma_1) = -R(\Sigma_2)$. This implies that the multiplier of the projective representation of $\mathrm{Sp}(E,\sigma)$ induced by the metaplectic representation takes only the values ± 1, which is not the case for our multiplier $\tilde{\alpha}$. We can, however, reduce our representation above to the metaplectic one. To do this, one should define

$$W_J(S) = \zeta_{J,S} V_J(S) ,$$

where $\zeta_{J,S}$ is a phase factor ($|\zeta_{J,S}| = 1$). These $W_J(S)$ form again a projective representation of $\mathrm{Sp}(E,\sigma)$ with a new multiplier:

$$\rho(S_1,S_2) = \zeta_{J,S_1} \zeta_{J,S_2} \zeta_{J,S_1S_2}^{-1} \tilde{\alpha}(S_1,S_2) .$$

We want this multiplier to take only the values ± 1; hence,

$$[\tilde{\alpha}(S_1,S_2)]^2 = \zeta_{J,S_1S_2}^2 \zeta_{J,S_1}^{*2} \zeta_{J,S_2}^{*2} .$$

So any decomposition of $\tilde{\alpha}^2$ in this form will give us a possibility to reduce our representation to the metaplectic one. However [see Eq. (B5)], one has

$$\begin{aligned} [\tilde{\alpha}(S_1,S_2)]^2 = \exp(i \arg(&\det(\mathbb{1} - iJ) + S_1S_2(\mathbb{1} + iJ) \\ &\cdot\det[(\mathbb{1} + iJ) + S_1(\mathbb{1} - iJ)] \\ &\cdot\det[(\mathbb{1} + iJ) + S_2(\mathbb{1} - iJ)] . \end{aligned}$$

This decomposition has exactly the right form. Moreover,

$$\begin{aligned} |\det[(\mathbb{1} - iJ) + S(\mathbb{1} + iJ)]| &= 2^n [\det(SJ + JS)]^{1/2} \\ &= 2^{2n} \beta_{J,SJS^{-1}}^2 \end{aligned}$$

[see Eq. (B6)].

Hence, we can define

$$\eta_{J,S} = 2^{-n} (\det[(\mathbb{1} - iJ) + S(\mathbb{1} + iJ)])^{1/2} \tag{6.3a}$$

(since $|\eta_{J,S}| = \beta_{J,SJS^{-1}}$, this is always different from zero) and

$$\begin{aligned} W_J(S) &= \exp(i \arg \eta_{J,S}) V_J(S) \\ &= \eta_{J,S} P_J \circ U_S |_{\mathscr{H}_J} . \end{aligned} \tag{6.3b}$$

In the definition of $\eta_{J,S}$ we choose the square root with argument in $]-\pi/2,\pi/2]$. [A continuity procedure to determine the phase of this square root would not be unambiguous for all S: There do exist S for which $\det(\mathbb{1} + S) = 0$.] In fact, there is absolutely no reason to prefer the root with positive real part to the one with negative real part. It is just a topological fact of life that it is impossible to choose the signs of the $\eta_{J,S}$ in such a way that the projective representation of $\mathrm{Sp}(E,\sigma)$ becomes a true one. Changing the sign of $\eta_{J,S}$ for a subfamily of $\mathrm{Sp}(E,\sigma)$ means only changing some signs of multipliers where elements of this subfamily occur. We will use this freedom in the choice of the sign of $\eta_{J,S}$ in the treatment of nonexceptional S later on.

Note that our constant $\eta_{J,S}$ leads to the same matrix elements as Bargmann's constant $\nu_g{}^2$ (see Appendix B).

By construction the $W_J(S)$ form a projective representation of $\mathrm{Sp}(E,\sigma)$ with a multiplier which takes only the values ± 1:

$$W_J(S_1) W_J(S_2) = \rho_J(S_1,S_2) W_J(S_1S_2) ,$$

$$\rho J(S_1,S_2) = \tilde{\alpha}(S_1,S_2) \exp[i \arg(\eta_{J,S_1} \eta_{J,S_2} \eta_{J,S_1S_2}^{-1})] = \pm 1 .$$

Note that the constant $\eta_{J,S}$ depends explicitly on S and not only on SJS^{-1}. Indeed it may happen that $SJS^{-1} = J$, and hence $\beta_{J,SJS^{-1}} = 1$, yet $\eta_{J,S} \neq 1$. As a consequence of this ρ does not inherit $\tilde{\alpha}$'s nice property (6.2b). Properties (6.2c) and (6.2d) also fail to hold in general for ρ: One can find S such that $\eta_{J,S} = i$, and hence $\eta_{J,S^{-1}} = i$, which implies $\rho(S,S^{-1}) = -1$. So the only property of $\tilde{\alpha}$ which passes on to ρ is Eq. (6.2a).

Using Eqs. (4.3) and (5.1) we see that for any $S\in \mathrm{Sp}(E,\sigma)$,

$$W_J(S)W_J(v) = \eta_{J,S}\,\beta^{-1}_{J,SJS^{-1}}\,T_{J,SJS^{-1}}\,W_{SJS^{-1}}(Sv)U_S\,|_{\mathscr{H}_J}$$
$$= W_J(Sv)W_J(S)$$

or

$$W_J(S)W_J(v)W_J(S)^{-1} = W_J(Sv)\,. \tag{6.4}$$

Combining this with Eq. (2.2) or (2.3), we see that

$$W_J(S)Q_J(f)W_J(S)^{-1} = Q_J(\mathbf{S}f), \tag{6.5}$$

where $\mathbf{S}f$ is the function defined by $(\mathbf{S}f)(v) = f(S^{-1}v)$.

Of course, we can extend all this to the inhomogeneous group $\mathrm{ISp}(E,\sigma)$. We have

$$W_J(S,a) = W_J(a)W_J(S)\,,$$

with

$$W_J(S_1,a_1)W_J(S_2,a_2)$$
$$= e^{i\sigma(a_1,S_1a_2)}W_J(a_1+S_1a_2)\,\rho(S_1,S_2)W_J(S_1S_2)$$
$$= e^{i\sigma(a_1,S_1a_2)}\rho(S_1,S_2)W_J((S_1,a_1)(S_2,a_2))\,.$$

Generalizing Eq. (6.4), we get

$$W_J(S,a)W_J(v)W_J(S,a)^{-1} = e^{2i\sigma(a,Sv)}W_J(Sv)$$

or

$$W_J(S,a)\Pi_J(v)W_J(S,a)^{-1} = \Pi_J(Sv+a);$$

hence,

$$W_J(S,a)Q_J(f)W_J(S,a)^{-1} = Q_J((\mathbf{S},\mathbf{a})f), \tag{6.6}$$

with

$$((\mathbf{S},\mathbf{a})f)(v) = f(S^{-1}v - S^{-1}a).$$

Note that, for n even, the operators $W(\pm\mathbb{1},a)$ are the $W(\pm;a)$ introduced in Ref. 13, and that, as was to be expected, this representation $\mathrm{ISp}(E,\sigma)$ is thus an extension of the Wigner–Weyl system as defined in Ref. 13. [For n odd a phase factor has to be introduced: in this case we have indeed $W_J = (-\mathbb{1},0) = i\Pi_J = i\,W(-;0)$.]

From Eqs. (6.4) and (6.5) we see that our operators $W_J(S)$ are exactly the quantal counterparts of the functions w in Ref. 10, up to some phase factor. Hence, we can apply the dequantization procedure given in Ref. 11 to calculate these functions. This will be done in the next section.

7. DEQUANTIZATION OF THE OPERATORS $W_J(S)$ AND $W_J(S,a)$

To apply the dequantization procedure sketched in Eqs. (2.4) and (2.5), we have to compute first the matrix elements of the operators $W_J(S)$ with respect to the coherent states:

$$W_J(S)(a,b) = (\Omega^a_J, W_J(S)\Omega^b_J) = \eta_{J,S}(\Omega^a_J,\Omega^{Sb}_{SJS^{-1}})\,.$$

We calculate now the corresponding function w_S:

$$w_S(v) = 2^n\eta_{J,S}\iint da\,db\,(\Omega^b_J,\Pi(v)\Omega^a_J)(\Omega^a_J,\Omega^{Sb}_{SJS^{-1}})$$
$$= 2^n\eta_{J,S}\int db\,(\Omega^b_J,\Pi(v)\Omega^{Sb}_{SJS^{-1}})\,.$$

A straightforward calculation (Appendix C), using $F\Omega_J = \Omega_J$ and formula (6.3) for η, yields

$$w_S(v) = \{\det[(\mathbb{1}-iJ) + S(\mathbb{1}+iJ)]\}^{1/2}$$
$$\times\int db\Omega_J(b+Sb-2v)e^{i\varphi(b/2,v,Sb/2)}\,, \tag{7.1}$$

where φ is defined in Sec. 2.

Formula (7.1) is valid for any S in $\mathrm{Sp}(E,\sigma)$. If S is exceptional, i.e., if $\mathbb{1}+S$ is singular, we see that for some directions in E the Ω_J factor in the integrand of Eq. (7.1) plays no role, which leaves us with an integral of the phase factor $e^{i\varphi}$, and hence gives us δ functions in the final result. If, however, $\mathbb{1}+S$ is regular, we can always find $u = (\mathbb{1}+S)^{-1}v$ such that $v = (\mathbb{1}+S)u$; hence,

$$w_S(v)$$
$$= 2^n\eta_{J,S}\int db\Omega_J[(\mathbb{1}+S)(b-2u)]e^{i\varphi(b/2,u+Su,Sb/2)}$$
$$= \left(2^n\eta_{J,S}\int db\Omega_J[(\mathbb{1}+S)b]e^{i\sigma(b,Sb)}\right)e^{4i\sigma(Su,u)}$$
$$= K_S\exp[4i\sigma(v,(\mathbb{1}+S)^{-1}v)]$$
$$= K_S\exp\left[2i\sigma\left(v,\frac{\mathbb{1}-S}{\mathbb{1}+S}v\right)\right]. \tag{7.2}$$

Since S is nonexceptional, we can use our freedom in the choice of a sign for $\eta_{J,S}$ to redefine η as

$$\eta_{J,S} = 2^{-n}$$
$$\times\lim_{\xi\to 1}\left(\exp\{i\arg\sqrt{\det[\mathbb{1}+S-i\xi J(\mathbb{1}-S)]}\}\right),$$

with again the assumptions that the root of the determinant is continuous in ξ and positive for $\xi = 0$. With this choice for the sign of η, we have

$$K_S = 2^n/\sqrt{\det(\mathbb{1}+S)}\,. \tag{7.3}$$

The calculation is given in Appendix A.

Note that the result (7.2) and (7.3) is exacly what was obtained in Ref. 10 for the classical functions corresponding to nonexceptional S.

When S is exceptional, but $J\ker(\mathbb{1}+S)$, we can again simplify formula (7.1) to obtain something anaologous to Eq. (7.2). Indeed, in this case we can decouple the degrees of freedom associated with $\ker(\mathbb{1}+S)$, i.e., we can write E as a direct sum $E = E'\oplus E''$ $[E'' = \ker(\mathbb{1}+S)]$, such that $\sigma(E',E'') = 0$, $JE' = E'$, $JE'' = E''$; S can then be considered as a sum $S = S' + S''$, where S' is a non exceptional element of $\mathrm{Sp}(E',\sigma_{E'\times E'})$, and $S'' = -\mathbb{1}_{E''}$. Formula (7.1) can then be simplified to give ($v = v' + v''$, with $v'\in E'$, $v''\in E''$)

$$w_S(v) = K_S\delta(v'')\exp[4i\sigma(v',(\mathbb{1}+S)^{-1}v')]\,,$$

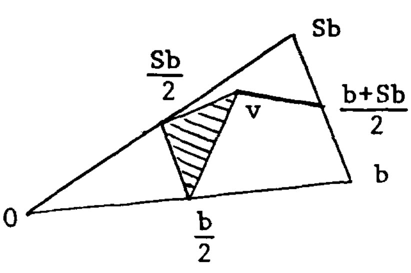

FIG. 1

with

$$K_S = \frac{2^{n'} 2^{-n''}}{\sqrt{\det_{E'}(\mathbb{1} + S)}} \left(\frac{1 - (-1)^{n''}}{2} + i \, \frac{1 + (-1)^{n''}}{2} \right).$$

Here $n' = \frac{1}{2}\dim E'$, $n'' = \frac{1}{2}\dim E''$.

The extra factor gives a coefficient 1 if n'' is even, and i if n'' is odd. In particular, we have

$$w_{-1}(v) = 2^{-n}\delta(v)\left(\frac{1 - (-1)^{n}}{2} + i \, \frac{1 + (-1)^{n}}{2} \right).$$

There exist, however, exceptional S for which no J can be found such that $J \ker(\mathbb{1} + S) = \ker(\mathbb{1} + S)$. For these S, we have to apply directly formula (7.1).

Note that the integrand in the general formula (7.1) has the following nice geometric interpretation: Take the triangle with vertices $0,b,Sb$. The midpoints of the sides of this triangle are $b/2, Sb/2$, and $b + Sb/2$. Then $\varphi(b/2,v,Sb/2)$ is exactly the surface of the oriented triangle $(b/2,v,Sb/2)$, while $\log \Omega_J(b - 2v + Sb)$ is $-2\times$ the distance of v to the third midpoint $(b + Sb)/2$ ["distance" being defined with respect to the Euclidean form $s(u,w) = \sigma(u,Jw)$] (see Fig. 1).

We can of course also calculate the functions corresponding to the $W(S,a)$ for the inhomogeneous group; this gives

$$\begin{aligned} w_{S,a}(v) &= 2^n \eta_{J,S} e^{2i\sigma(a,v)} \int db \, \Omega_J(b - 2v + a + Sb) e^{i\varphi(b/2, v - a/2, Sb/2)} \\ &= e^{2i\sigma(a,v)} w_S(v - a/2) . \end{aligned}$$

As a special case we have the well-known result

$$w_a(v) = w_{1,a}(v) = e^{2i\sigma(a,v)} .$$

Requantization of the functions (7.1) along the procedure sketched in Eq. (2.3) yields (for the detailed calculation, see Appendix C)

$$\begin{aligned} W_J(S) &= 2^n \int dv \, w_S(v) \Pi_J(v) \\ &= 2^{2n} \eta_{J,S} \int dv \int db \Omega_J(b + Sb - 2v) e^{i\varphi(b/2,v,Sb/2)} \\ &\quad \times \Pi_J(v) \\ &= \eta_{J,S} \int db \, | \Omega_J^{Sb})(\Omega_J^{Sb} | \\ &= 2^{-n} \{ \det[(\mathbb{1} - iJ) + S(\mathbb{1} + iJ)] \}^{1/2} \\ &\quad \times \int db \, | \Omega_J^{Sb})(\Omega_J^{b} | . \end{aligned} \tag{7.4}$$

This is of course again the same operator as given by Eq. (6.1), as one can easily check by comparing the kernels corresponding to Eqs. (6.1) and (7.4).

8. THE TRANSLATION TO x–p NOTATIONS

The translation of our intrinsic notation system to any particular more explicit notation system is completely determined once one has given explicit expressions for E, σ, and J.

Writing everything in coordinate notations amounts to taking

$E = \mathbb{R}^n \oplus \mathbb{R}^n$ (with usually $n = 3N$, N being the number of particles),

$$\begin{aligned} &E \ni v = (x,p), \\ &\sigma((x,p),(x',p')) = \tfrac{1}{2}(p \cdot x' - x \cdot p'), \\ &J((x,p)) = (p, -x). \end{aligned}$$

Hence, $\Omega_J(v) = \exp[-\frac{1}{4}(x^2 + p^2)]$ and

$$dv = \frac{1}{(2\pi)^n} d^n x \, d^n p .$$

A symplectic transformation can be represented by a matrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$, where A, B, C, D, are real $n\times n$ matrices such that

$$S((x,p)) = (Ax + Bp, Cx + Dp).$$

The fact that S is symplectic is equivalent to

$$\begin{cases} A\,{}^tC - C\,{}^tA = 0, \\ B\,{}^tD - D\,{}^tB = 0, \\ C\,{}^tB - D\,{}^tA = \mathbb{1}. \end{cases}$$

Another explicit but less frequently used notation system is Bargmann's. Here one takes

$$\begin{aligned} &E = \mathbb{C}^n, \\ &E \ni v = z, \\ &\sigma(z,z') = \mathrm{Im}(\bar{z}\cdot z') = \frac{1}{2i}(\bar{z}\cdot z' - z\cdot\bar{z}') , \\ &J(z) = iz. \end{aligned}$$

Hence, $\Omega_J(z) = \exp[-\frac{1}{2}(z)^2]$ and

$$dv = (1/\pi^n) d(\mathrm{Re}\, z) d(\mathrm{Im}\, z) .$$

9. APPLICATIONS

We have computed the operators $W_J(S)$ of the metaplectic representation on one hand, and on the other hand the corresponding classical functions. Both these results can be used for applications.

A. Applications of the classical function formula

We give here some explicit calculations of the classical function corresponding to a given symplectic transformation. In the first three cases the symplectic transformations form a one-parameter subgroup of $\mathrm{Sp}(E,\sigma)$ which is defined as the classical evolution group for a quadratic Hamiltonian. Since for any quadratic Hamiltonian h the quantum mechanical evolution operator $\exp(iQ_h t)$ is exactly given by $W(S_t)$, where S_t is the one-parameter symplectic transformation group associated to h, one sees that the calculated functions are, at least formally, the twisted exponentials of h (see also Refs. 8 and 18). It is to be noted that one can show, using some recent results,[19] that these functions really are the twisted exponentials (not only formally), i.e., that the series of the twisted exponential makes sense in $\mathscr{S}'$, and does converge (again in $\mathscr{S}'$) to w_{S_t}. This means that the quite complicated proofs (see, for example, Ref. 18) for this convergence in particular cases are no longer necessary.

We give our different results in the x–p notation. Since we are here on the level of the classical functions, the results are independent of the particular representation of the Weyl commutation relations we used:

(1) The harmonic oscillator ($n = 1$): $H = \frac{1}{2}(x^2 + p^2)$ gives rise to the evolution

$$\begin{cases} x_t = x_0 \cos t + p_0 \sin t, \\ p_t = -x_0 \sin t + p_0 \cos t; \end{cases}$$

hence, $(x_t, p_t) = S_t(x_0, p_0)$, with

$$S_t = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}.$$

Calculating the classical function corresponding to this, we find [we can apply Eq. (7.2) since S_t is nonexceptional whenever $t \neq (2k+1)\pi$; for the special values $t = (2k+1)\pi$ we have $S_t = -\mathbb{1}$ and $w_{S_t} = \frac{1}{2}\delta(x)\delta(p)$.]

$$w_{S_t}(x,p) = (\cos[t/2])^{-1}\exp(-i(x^2+p^2)\tan[t/2]).$$

This is the result found in Refs. 9 and 18.

(2) The same for $H = \frac{1}{2}(p^2 - x^2)$ gives

$$S_t = \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}$$

and

$$w_{S_t}(x,p) = \left(\cosh\frac{t}{2}\right)^{-1} e^{-2i(p^2 - x^2)\tanh(t/2)}.$$

(3) The same for $H = \frac{1}{2}p^2 + x$ gives $(x_t, p_t) = S_t(x_0, p_0) + a_t$, with

$$S_t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

and $a_t = (-\frac{1}{2}t^2, -t)$. We have $w_{S_t}(x,p) = e^{-itp^2}$; hence,

$$w_{S_t, a_t}(x,p) = e^{-2i(p^2 t/2) + tx + t^3/8)}.$$

Again these are the same expressions as in Ref. 9.

In our last calculation we treat a "general" exception S. It is general in the sense that no J can be found such that $J\ker(\mathbb{1}+S) = \ker(\mathbb{1}+S)$, which compels us to use the nonsimplified formula (7.1).

(4) Take $\left(\begin{smallmatrix} -1 & \alpha \\ 0 & -1 \end{smallmatrix}\right)$, with $\alpha > 0$. We have $\eta_{J,S} = (i/\sqrt{2})\sqrt{2 + i\alpha}$ and

$$\int db\, \Omega_J(b + Sb - 2v) e^{i\sigma(b,Sb) + 2i\sigma(Sb,v) + 2i\sigma(v,b)}$$

$$= \text{(after some calculation)}\ \frac{\sqrt{\pi}}{2}\frac{\sqrt{2}}{\sqrt{\alpha}}\frac{1}{\sqrt{\alpha - 2i}} \times \delta(p) e^{-4ix^2/\alpha}.$$

Hence,

$$w_S(x,p) = \tfrac{1}{2}\sqrt{\pi/\alpha}\, i\sqrt{i}\,\delta(p) e^{-4ix^2\alpha^{-1}}.$$

B. Applications of the expression for $W_J(S)$

We have [see Eq. (6.3b)] $W_J(S) = \eta_{J,S} P_J \circ U_S|_{\mathscr{H}_J}$. Hence, for any φ, ψ in $\mathscr{H}_J$,

$$(\varphi, W_J(S)\psi) = \eta_{J,S}\int dv\, \overline{\varphi(v)}\,\psi(S^{-1}v). \tag{9.1}$$

Suppose we are interested in the time evolution operator e^{itH} associated with a quadratic Hamiltonian H. Dequantizing H we get a quadratic function h on phase space, for which the corresponding classical time evolution on phase space is given by a symplectic one-parameter group $(S_h)_t$. It is easy to check that $e^{itH} = W(S_h)_t$. Hence, the matrix elements of the time evolution operator e^{itH} for an at most quadratic Hamiltonian are given by

$$(\varphi, e^{itH}\psi) = \eta_{J,S_{h,t}}\int dv\, \overline{\varphi(v)}\,\psi(S_{h,-t}v). \tag{9.2}$$

This formula is of course only true if the chosen representation of the Weyl commutation relations is a $\mathscr{H}_J$ representation. However, we can use an extension for arbitrary representation spaces.

Indeed, let $\mathscr{H}$ be any Hilbert space carrying an irreducible representation of the Weyl commutation relations [usually one chooses $\mathscr{H} = L^2(\mathbb{R}^n)$ with the Schrödinger representation]. Choose a nice complex structure J on E, σ, and let $\tilde{\Omega}_J \in \mathscr{H}$ be the ground eigenstate of the harmonic oscillator Hamiltonian corresponding to $h_J(v) = s_J(v,v) = \sigma(v, Jv)$. [Usually one takes $J(x,p) = (p, -x)$; hence, $h(v) = \frac{1}{2}(x^2 + p^2)$; $\tilde{\Omega}_J$ is then—in the Schrödinger representation—the well-known Hermite function $\pi^{-n/2} \times \exp(-\frac{1}{2}x^2)$.] We define the coherent states $\tilde{\Omega}_J^a$ to be the translated [by $W(a)$] of $\tilde{\Omega}_J$: $\tilde{\Omega}_J^a = W(a)\tilde{\Omega}_J$. For any vector ψ in $\mathscr{H}$ we define the function ϕ_ψ by

$$\phi_{J,\psi}(a) = (\Omega_J^a, \psi)_{\mathscr{H}}.$$

One can easily check that, as a function of a, these $\phi_{J,\psi}$ are elements of $\mathscr{H}_J$. The converse is also true: To any function in $\mathscr{H}_J$ corresponds a unique vector in $\mathscr{H}$ for which the relation above holds. The matrix elements of the evolution operator e^{itH} for any quadratic Hamiltonian $H = Q_h$ are then given by

$$(\varphi, e^{itH}\psi) = \eta_{J,S_{h,t}}\int_E da\, \overline{\phi_{J,\varphi}(a)}\,\phi_{J,\psi}(S_{h,-t}a). \tag{9.3}$$

So once the classical solutions of the Hamiltonian equations for the Hamiltonian h are known, we can compute any matrix element of the quantum evolution operator for the corresponding Hamiltonian $H = Q_h$. This Hamiltonian H, though at most quadratic, may be quite nontrivial, e.g., a system of N particles, in a homogeneous electromagnetic field (with arbitrary strength), with harmonic oscillator pair potentials, is described by a Hamiltonian falling into this class.

The procedure given above for applying our formula for $W_J(S)$ even if the representation chosen is not a $\mathscr{H}_J$ representation can of course also be applied if one is not interested in one-parameter subgroups but in the whole symplectic group: We can define a projective representation of $\mathrm{Sp}(E,\sigma)$ on any Hilbert space $\mathscr{H}$ carrying an irreducible representation of the Weyl commutation relations

$$(\varphi, W(S)\psi) = \eta_{J,S}\int da\, \overline{\phi_{J,\varphi}(a)}\,\phi_{J,\psi}(S^{-1}a). \tag{9.4}$$

In the case where $\mathscr{H} = L^2(\mathbb{R}^n)$, with the Schrödinger representation

$$(W(x_a, p_a)\psi)(x) = \exp\left(-\frac{i}{2}x_a p_a\right)e^{ip_a x}\psi(x - x_a),$$

one can check that this yields

$$(\varphi, W(S)\psi) = \iint dx\, dx'\, \overline{\varphi(x)}\, U_S(x,x')\psi(x'),$$

where $U_S(x,x')$ is given, up to a phase factor, by expression (3.27) in Ref. 4 for the cases considered there. The phase factor occurs because we really have a (projective) representation of the whole group $\mathrm{Sp}(E,\sigma)$ while in Ref. 4 only individual symplectic transformations were studied.

10. REMARKS

(1) In the preceding section we showed how one can reconstruct, using our expression in $\mathscr{H}_J$, the metaplectic representation on any Hilbert space carrying an irreducible representation $W(v)$ of the Weyl commutation relation. To do this, we introduced the coherent states (with respect to some J) in $\mathscr{H}$. We can avoid these coherent states in the reconstruction if we use the classical functions w_S: Let Π be the representation on $\mathscr{H}$ of phase space parity ($v \to -v$). Then define $W(S)$ on $\mathscr{H}$ as

$$W(S) = 2^n \int dv\, w_S(v) W(2v)\, \Pi.$$

(2) We have given explicit expression (6.3b) and (7.4) for the operator $W_J(S)$. [In fact, Eq. (9.4) shows us that expression (7.4) is also valid in other representation spaces than $\mathscr{H}_J$.] We can use these expressions to calculate the matrix elements of $W_J(S)$ between coherent states:

$$\begin{aligned} W_J(S)(a,b) &= (\Omega_J^a, W_J(S)\Omega_J^b) \\ &= e^{i\sigma(Sb,a)}(\Omega_J^{a-Sb}, W_J(S)\Omega_J) \\ &= \eta_{J,S} e^{i\sigma(Sb,a)} \int dv\, \overline{\Omega_J^{a-Sb}(v)}\, \Omega_{SJS^{-1}}(v). \end{aligned}$$

Using Eq. (A5) this gives

$$\begin{aligned} W_J(S)(a,b) = (\eta^*_{J,S})^{-1} \exp[i\sigma(Sb,a) - i\sigma(a-Sb,\ & J\hat{Z}(a-Sb)) - \sigma(a-Sb, \hat{Z}(a-Sb))], \end{aligned}$$

with

$$\hat{Z} = -(J + SJS^{-1})^{-1}.$$

It is easy to check that this is in fact the same expression as in Bargmann.[2]

(3) Formula (7.1) for w_S depends on the choice of J. So let us denote for the time being this function by $w_{S,J}$. For two J, J' there exists of course a relation between $w_{S,J}$ and the $w_{S,J'}$. Since one sees easily from Eq. (6.3) that $\eta_{J,S'^{-1}SS'} = \eta_{S'JS'^{-1},S}$, a simple substitution in the integration in Eq. (7.1) gives us the following relation between $w_{S,J'}$ and $w_{S,J}$ (we put $J' = S'JS'^{-1}$):

$$w_{S,J'}(v) = w_{S,S'JS'^{-1}}(v) = w_{S'^{-1}SS',J}(S'^{-1}v). \tag{10.1}$$

On the other hand, we know that for any function f on phase space

$$f(S'^{-1}v) = S'f(v) = (w_{S',J} \circ f \circ w^*_{S',J})(v),$$

where $\circ$ denotes the twisted product (see for instance Refs. 11 and 13). Substituting $w_{S'^{-1}SS',J}$ for f, and introducing the multipliers ρ, we get

$$w_{S'^{-1}SS',J}(S'^{-1}v) = \rho_J(S', S'^{-1}SS')\rho_J(SS', S'^{-1}) w_{S,J}(v).$$

Combining this with Eq. (10.1), we see that

$$w_{S,J'}(v) = \rho_J(S', S'^{-1}SS')\rho_J(SS', S'^{-1}) w_{S,J}(v).$$

So, up to a sign depending on J, J', and S, $w_{S,J'}$ is equal to $w_{S,J}$. If we choose to consider our representation as a double valued representation of $\mathrm{Sp}(E,\sigma)$ instead of as a projective representation, this implies that the double valued representation $S \nrightarrow \pm w_{S,J}$ is independent of J.

(4) Formula (9.4) is only valid for linear canonical transformations. In fact, once the canonical transformation T is nonlinear, there does not exist any more a bounded operator V_T satisfying $\forall f: Q_f V_T = V_T Q_{f\circ T^{-1}}$. (This can easily be seen if one realizes that up to a constant this V_T would have to be unitary. One can then use an argument found in Ref. 10 to show that T cannot be linear.) One can of course try to find V_T satisfying the relation above for just n independent functions f_j (see, for example, Ref. 20). The operator constructed in this way is however dependent on the choice of the f_j.

ACKNOWLEDGMENTS

I would like to thank Professor P. Huguenin for several useful remarks. It is a pleasure to thank Professor A. Grossmann for many helpful discussions and for his constant interest and encouragement. Thanks are also due for the hospitality of the CPT2, CNRS Marseille, where part of this work was done.

APPENDIX A

All the calculations in this Appendix are based on the following general principle:

Let B be a real linear map $E \to E$ such that

$$\sigma(u,Bv) = \sigma(v,Bu), \quad \forall u,v \in E, \tag{A1}$$

$$\sigma(u,Bu) > 0, \quad \text{if } u \neq 0, \tag{A2}$$

then the function $\Omega_B(v) = \exp[-\frac{1}{2}\sigma(v,Bv)]$ is integrable, and

$$\int dv \Omega_B(v) = 2^n (\det B)^{-1/2}. \tag{A3}$$

Here we choose the positive square root of det B.

By a simple analyticity argument one can extend (A3) to all complex combinations $B + iC$ of real linear maps from E to E, where B is chosen as above [B satisfies both Eqs. (A1) and (A2)] and C is symmetric [i.e., it satisfies Eq. (A1)]. For any such complex combinations we have again

$$\int dv\, e^{-\sigma(v,Bv)/2} e^{-i\sigma(v,Cv)/2} = 2^n [\det(B + iC)]_c^{-1/2}. \tag{A3'}$$

Here we have introduced the notation $[\det(B+iC)]_c^{\pm 1/2}$ in the following meaning: let $f_\pm : [0,1] \to \mathbb{C}$ be a continuous function with $f_\pm(0) \in \mathbb{R}_+$ and $f_\pm(\xi) = [\det(B + i\xi C)]^{\pm 1/2}$. The continuity of f and its initial value in $\mathbb{R}_+$ select without ambiguity one of the two possible roots of $[\det(B + i\xi C)]^{\pm 1}$ as the value of $f_\pm(\xi)$ at any ξ. Then we define

$$[\det(B + iC)]_c^{\pm 1/2} = f_\pm(1).$$

As usual in Gaussian integrals, the integration variable in Eq. (A3) can be shifted by a complex vector:

$$\int dv\, \Omega_B(v+a+ib) = 2^n(\det B)^{-1/2}\,, \qquad \text{(A3'')}$$

where we define $\sigma(u+iu',v+iv')$ to be the obvious complex linear extension:

$$\sigma(u+iu',v+iv') = \sigma(u,v) - \sigma(u',v') + i\sigma(u',v) + i\sigma(u,v').$$

For any real linear map B satisfying both Eqs. (A1) and (A2), we can construct $\hat{B} = -B^{-1}$ [B is regular because of Eq. (A2)]. It is easy to check that Eqs. (A1) and (A2) are again satisfied by $\hat{B}$. As a corollary of Eq. (A3'') we have now

$$\begin{aligned}\int dv\, e^{i\sigma(a,v)}\Omega_B(v) &= \int dv\, e^{i\sigma(\hat{B}a,Bv)}\, e^{-\sigma(v,Bv)/2}\\ &= \int dv\, e^{-\sigma(v-i\hat{B}a,B(v-i\hat{B}a))/2}\, e^{-\sigma(a,\hat{B}a)/2}\\ &= 2^n(\det B)^{-1/2}\Omega_{\hat{B}}(a)\,.\end{aligned}$$

Finally, note that the family of real linear maps satisfying Eqs. (A1) and (A2) is a convex cone containing the σ-allowed complex structures.

We can now start with our calculations. We begin with $\beta_{J',J}$. Equation (5.3) tells us that $\beta_{J',J}$ is given by

$$\beta_{J',J} = \left(\int da\, |(\Omega^a_{J'},\Omega_J)|^2\right)^{-1/2}.$$

So we start by calculating $(\Omega^a_{J'},\Omega_J)$. Put $Z = J+J'$. Then

$$\begin{aligned}(\Omega^a_{J'},\Omega_J) &= \int dv\, e^{-i\sigma(a,v)}\Omega_{J'}(a)e^{\sigma(a,J'v)}\Omega_{J'}(v)\Omega_J(v)\\ &= \Omega_{J'}(a)\int dv\, e^{i\sigma(iJ'a-a,v)}\Omega_Z(v)\\ &= 2^n(\det Z)^{-1/2}\Omega_{J'}(a)\Omega_{\hat{Z}}(a-iJ'a)\\ &= 2^n(\det Z)^{-1/2}\, e^{-i\sigma(a,J'\hat{Z}a)}\\ &\quad\times e^{-\sigma(a,(J'+\hat{Z}+J'\hat{Z}J')a)/2}\,.\end{aligned}$$

Since, however, $Z = J+J'$, and $J^2 = J'^2 = -1$, we have $J'ZJ = -Z$; hence, $J\hat{Z}J' = -\hat{Z}$, or

$$J'\hat{Z}J' = Z\hat{Z}J' - J\hat{Z}J' = -J' + \hat{Z}. \qquad \text{(A4)}$$

This implies

$$(\Omega^a_{J'},\Omega_J) = 2^n(\det Z)^{-1/2}\, e^{-i\sigma(a,J'\hat{Z}a)}\Omega^2_{\hat{Z}}(a)\,. \qquad \text{(A5)}$$

Hence,

$$\begin{aligned}\int da\, |(\Omega^a_{J'},\Omega_J)|^2 &= 2^{2n}(\det Z)^{-1}\int da\, \Omega^4_{\hat{Z}}(a)\\ &= 2^n(\det Z)^{-1}(\det \hat{Z})^{-1/2}\\ &= 2^n(\det Z)^{-1/2}\,.\end{aligned}$$

So finally

$$\begin{aligned}\beta_{J',J} &= \left(\int da\, |(\Omega^a_{J'},\Omega_J)|^2\right)^{-1/2} = 2^{-n/2}(\det Z)^{1/4}\\ &= 2^{-n/2}[\det(J+J')]^{1/4}\,. \qquad \text{(A6)}\end{aligned}$$

We now compute $\alpha(J'',J',J)$. From Eq. (5.7') we see that

$$\alpha(J'',J',J) = \exp\{i\arg[(\Omega_{J'},P_J\Omega_{J''})]\}\,.$$

Put $Z_1 = J+J'$, $Z_2 = J+J''$. Using Eq. (A5) we can now calculate $\arg(\Omega_{J'},P_J\Omega_{J''})$:

$$\begin{aligned}\arg((\Omega_{J'},P_J\Omega_{J''})) &= \arg\left[\int da(\Omega_{J'},\Omega^a_J)(\Omega^a_J,\Omega_{J''})\right]\\ &= \arg\bigg(\int da\, \exp[-\sigma(a,(\hat{Z}_1+\hat{Z}_2)a)]\\ &\qquad - i\sigma(a,J(\hat{Z}_2-\hat{Z}_1)a)\bigg)\,.\end{aligned}$$

From Eq. (A4) we see that $J\hat{Z}_i = -\mathbb{1} - \hat{Z}_iJ$; hence, $J(\hat{Z}_2-\hat{Z}_1) = -(\hat{Z}_2-\hat{Z}_1)J$. Combining this with the fact that J, $\hat{Z}_i$ satisfy Eq. (A1), we see now that both $\hat{Z}_1+\hat{Z}_2$ and $J(\hat{Z}_2-\hat{Z}_1)$ fulfill condition (A1), while $\hat{Z}_1+\hat{Z}_2$ obviously satisfies Eq. (A2). Hence,

$$\begin{aligned}\arg((\Omega_{J'},P_J\Omega_{J''})) = \arg([\det(\hat{Z}_1+\hat{Z}_2\\ - iJ(\hat{Z}_1-\hat{Z}_2)]_c^{-1/2})\,. \qquad \text{(A7)}\end{aligned}$$

The determinant in Eq. (A7) can be simplified. Indeed

$$\begin{aligned}&\hat{Z}_1+\hat{Z}_2 = \hat{Z}_1(-Z_2-Z_1)\hat{Z}_2\,,\\ &J(\hat{Z}_1-\hat{Z}_2)\\ &\quad = J\hat{Z}_1(-Z_2+Z_1)\hat{Z}_2\\ &\quad = -\hat{Z}_1J(-Z_2+Z_1)\hat{Z}_2 + \hat{Z}_1Z_1(-Z_2+Z_1)\hat{Z}_2\,.\end{aligned}$$

Hence,

$$\begin{aligned}&\det(\hat{Z}_1+\hat{Z}_2-iJ(\hat{Z}_1-\hat{Z}_2))\\ &= \det\hat{Z}_1\det(-Z_2-Z_1-iJZ_2+iJZ_1+iZ_1Z_2-iZ_1Z_1)\\ &\quad\times\det\hat{Z}_2\\ &= \det(\hat{Z}_1\hat{Z}_2)\det(2J+J'+J''+i\mathbb{1}+iJ'J'')\,.\end{aligned}$$

Finally,

$$\begin{aligned}\alpha(J'',J',J) = \exp(i\arg([\det(2J+J'+J''\\ - i\mathbb{1} - iJ'J'')]_c^{1/2}))\,. \qquad \text{(A8)}\end{aligned}$$

Our last calculation concerns the coefficient in Eq. (7.2).

We have to calculate

$$\begin{aligned}I &= \int db\Omega_J((\mathbb{1}+S)b)e^{i\sigma(b,Sb)}\\ &= [\det(\mathbb{1}+S)]^{-1}\int db\Omega_J(b)e^{-i\sigma(b,(\mathbb{1}+S)^{-1}b)}\\ &= [\det(\mathbb{1}+S)]^{-1}\int db\,\exp\left[-\tfrac{1}{2}\sigma\left(b,Jb+i\frac{\mathbb{1}-S}{\mathbb{1}+S}b\right)\right]\,.\end{aligned}$$

One can easily check that $(\mathbb{1}-S)(\mathbb{1}+S)^{-1}$ satisfies Eq. (A1) (see also Ref. 10). Hence,

$$\begin{aligned}I &= \frac{2^n}{\det(\mathbb{1}+S)}\left[\det\left(J+i\frac{\mathbb{1}-S}{\mathbb{1}+S}\right)\right]_c^{-1/2}\\ &= \frac{2^n}{\sqrt{\det(\mathbb{1}+S)}}(\det[J(\mathbb{1}+S)+i(\mathbb{1}-S)])_c^{-1/2}\\ &= \frac{2^n}{\sqrt{\det(\mathbb{1}+S)}}(\det[\mathbb{1}+S-iJ(\mathbb{1}-S)])_c^{-1/2}\,.\end{aligned}$$

Combining this with the other coefficient in $w_S(v)$, this yields the result stated in Eq. (7.3).

APPENDIX B

Our first calculation here will be the decomposition of

$\tilde{\alpha}^2$ (see Sec. 6). Before computing this, we derive some simple relations which will turn out to be very useful.

The first of these is

$$J(\mathbb{1} \pm iJ) = J \mp i\mathbb{1} = \mp(\mathbb{1} \pm iJ); \tag{B1}$$

hence,

$$(\mathbb{1} + iJ)(\mathbb{1} - iJ) = (\mathbb{1} - iJ)(\mathbb{1} + iJ) = 0, \tag{B2}$$

$$(\mathbb{1} \pm iJ)^2 = 2(\mathbb{1} \pm iJ). \tag{B3}$$

On the other hand, we already mentioned the existence for any complex structure J of J-symplectic bases, i.e., of bases e_j, f_j of E such that $f_j = Je_j$, $\sigma(e_j,e_k) = \sigma(f_j,f_k) = 0$, $\sigma(e_j,f_k) = \delta_{jk}$. With respect to such a basis J is represented by the matrix

$$M_J = \begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}.$$

Hence, there exists a complex unitary matrix U such that $UM_J U^{-1}$ has the form

$$i \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}.$$

Let now L be any linear map from E to itself, with matrix representation M_L w.r.t. a J-symplectic basis. We can write $UM_L U^{-1}$ as $\left(\begin{smallmatrix} X & Y \\ Z & W \end{smallmatrix}\right)$, where X, Y, Z, W are $n \times n$ matrices.

Now,

$$U((\mathbb{1} - iM_J) + M_L(\mathbb{1} + iM_J))U^{-1} = 2 \begin{pmatrix} \mathbb{1} & Y \\ 0 & W \end{pmatrix},$$

$$U((\mathbb{1} - iM_J) + (\mathbb{1} + iM_J)M_L)U^{-1} = 2 \begin{pmatrix} \mathbb{1} & 0 \\ Z & W \end{pmatrix},$$

$$U((\mathbb{1} - iM_J) + (\mathbb{1} + iM_J)M_L(\mathbb{1} + iM_J))U^{-1} = 2\begin{pmatrix} \mathbb{1} & 0 \\ 0 & 2W \end{pmatrix}.$$

This implies

$$\begin{aligned}\det[(\mathbb{1} - iJ) + L(\mathbb{1} + iJ)] &= \det[U(\mathbb{1} - iM_J) + M_L(\mathbb{1} + iM_J))U^{-1}] \\ &= 2^{2n}\det W\end{aligned}$$

and analogously

$$\det[(\mathbb{1} - iJ) + (\mathbb{1} + iJ)L] = 2^{2n}\det W,$$

$$\det[(\mathbb{1} - iJ) + (\mathbb{1} + iJ)L(\mathbb{1} + iJ)] = 2^{3n}\det W.$$

Hence,

$$\begin{aligned}\det[(\mathbb{1} - iJ) + L(\mathbb{1} + iJ)] &= 2^{-n}\det[(\mathbb{1} - iJ) + (\mathbb{1} + iJ)L(\mathbb{1} + iJ)] \\ &= \det[(\mathbb{1} - iJ) + (\mathbb{1} + iJ)L].\end{aligned} \tag{B4}$$

We can now proceed to compute the decomposition of $\tilde{\alpha}^2(S_1,S_2)$.

Let S_1, S_2 be any symplectic transformations. Define

$$J_1 = S_1^{-1}JS_1, \quad J_2 = S_2JS_2^{-1},$$

$$Z_1 = J + J_1, \quad Z_2 = J + J_2,$$

$$\hat{Z}_1 = -Z_1^{-1}, \quad \hat{Z}_2 = -Z_2^{-1}.$$

Then (see Sec. 6 and Appendix A)

$$\tilde{\alpha}_J^2(S_1,S_2) = \exp\{-i\arg(\det[(\mathbb{1} - iJ)\hat{Z}_1 + (\mathbb{1} + iJ)\hat{Z}_2])\}.$$

Since $J\hat{Z}_1 = -\hat{Z}_1J - \mathbb{1}$ (see Appendix A), we have

$$\begin{aligned}\det[(\mathbb{1} - iJ)\hat{Z}_1 + (\mathbb{1} + iJ)\hat{Z}_2] &= \det[\hat{Z}_1(\mathbb{1} + iJ) + i\mathbb{1} + (\mathbb{1} + iJ)\hat{Z}_2] \\ &= (\det Z_1 \det Z_2)^{-1}\det[-iZ_1Z_2 \\ &\quad + (\mathbb{1} + iJ)Z_2 + Z_1(\mathbb{1} + iJ)] \\ &= (\det Z_1Z_2)^{-1}\det[(\mathbb{1} - iJ_1)Z_2 + (\mathbb{1} + iJ_1)Z_1]\end{aligned}$$

(we have used $-J + Z_1 = J_1$ and $Z_1J = J_1Z_1$). However,

$$\begin{aligned}\det[(\mathbb{1} - iJ_1)Z_2 + (\mathbb{1} + iJ_1)Z_1](\det Z_1)^{-1} &= \det[(\mathbb{1} + iJ_1) + (\mathbb{1} - iJ_1)Z_2(-\hat{Z}_1)] \\ &= \det[(\mathbb{1} + iJ_1) - Z_2\hat{Z}_1(\mathbb{1} - iJ_1)] \qquad \text{[use Eq. (B4)]} \\ &= \det[(\mathbb{1} + iJ_1)Z_1 + Z_2(\mathbb{1} - iJ)](\det Z_1)^{-1} \\ &= \det[Z_1(\mathbb{1} + iJ) + Z_2(\mathbb{1} - iJ)](\det Z_1)^{-1}.\end{aligned}$$

Hence,

$$\tilde{\alpha}_J^2(S_1,S_2) = \exp\{-i\arg(\det[Z_1(\mathbb{1} + iJ) + Z_2(\mathbb{1} - iJ)])\}.$$

We have

$$\begin{aligned}[Z_1(\mathbb{1} + iJ) + Z_2(\mathbb{1} - iJ)]J &= -i(J + J_1)(\mathbb{1} + iJ) + i(J + J_2)(\mathbb{1} - iJ) \\ &= -i(-2i\mathbb{1} + J_1 - J_2 + iJ_1J + iJ_2J) \\ &= -(\mathbb{1} + iJ_1 + iJ - J_1J + \mathbb{1} - iJ_2 - iJ - J_2J) \\ &= -(\mathbb{1} + iJ_1)(\mathbb{1} + iJ) - (\mathbb{1} - iJ_2)(\mathbb{1} - iJ).\end{aligned}$$

Hence,

$$\det[Z_1(\mathbb{1} + iJ) + Z_2(\mathbb{1} - iJ)] = \det[(\mathbb{1} + iJ_1)(\mathbb{1} + iJ) + (\mathbb{1} - iJ_2)(\mathbb{1} - iJ)]$$

(remember that $\det J = \det S_1 = \det S_2 = 1$!, see Sec. 4) and

$$\begin{aligned}&= \det[S_1^{-1}(\mathbb{1} + iJ)S_1(\mathbb{1} + iJ) \\ &\quad + S_2(\mathbb{1} - iJ)S_2^{-1}(\mathbb{1} - iJ)] \\ &= 2^{-2n}\det\{[S_1^{-1}(\mathbb{1} + iJ) + S_2(\mathbb{1} - iJ)] \\ &\quad \times [(\mathbb{1} + iJ)S_1(\mathbb{1} + iJ) + (\mathbb{1} - iJ)S_1^{-1}(\mathbb{1} - iJ)]\} \\ &= 2^{-4n}\det[S_1^{-1}(\mathbb{1} + iJ) + S_2(\mathbb{1} - iJ)] \\ &\quad \times \det\{[(\mathbb{1} + iJ)S_1(\mathbb{1} + iJ) + (\mathbb{1} - iJ)] \\ &\quad \times [(\mathbb{1} + iJ) + (\mathbb{1} - iJ)S_2^{-1}(\mathbb{1} - iJ)]\} \\ &= 2^{-3n}\det[(\mathbb{1} + iJ) + S_1S_2(\mathbb{1} - iJ)] \\ &\quad \times \det[(\mathbb{1} - iJ) + S_1(\mathbb{1} + iJ)]\{\det[(\mathbb{1} - iJ) \\ &\quad + S_2^{-1}(\mathbb{1} + iJ)]\}^* \\ &= 2^{-2n}\det[(\mathbb{1} + iJ) + S_1S_2(\mathbb{1} - iJ)] \\ &\quad \times \{\det[(\mathbb{1} + iJ) + S_1(\mathbb{1} - iJ)]\}^* \\ &\quad \times \{\det[(\mathbb{1} + iJ) + S_2(\mathbb{1} - iJ)]\}^*.\end{aligned}$$

So finally

$$\begin{aligned}\tilde{\alpha}_J^2(S_1,S_2) = \exp\{i\arg(&\det[(\mathbb{1} - iJ) + S_1S_2(\mathbb{1} + iJ)] \\ &\times \det[(\mathbb{1} + iJ) + S_1(\mathbb{1} - iJ)] \\ &\times \det[(\mathbb{1} + iJ) + S_2(\mathbb{1} - iJ)])\}.\end{aligned} \tag{B5}$$

This is exactly the decomposition of $\tilde{\alpha}^2$ as used in Sec. 6.

Our next calculation is the computation of $|\det[(\mathbb{1}-iJ)+S(\mathbb{1}+iJ)]|$:

$$|\det[(\mathbb{1}-iJ)+S(\mathbb{1}+iJ)]|^2$$
$$=\det[(\mathbb{1}-iJ)+S(\mathbb{1}+iJ)]\det[(\mathbb{1}+iJ)+(\mathbb{1}-iJ)S]$$
$$=\det\{[(\mathbb{1}-iJ)+S(\mathbb{1}+iJ)]$$
$$\times J[(\mathbb{1}+iJ)+(\mathbb{1}-iJ)S]\}$$
$$=\det[2i(\mathbb{1}-iJ)S-2iS(\mathbb{1}+iJ)]$$
$$=2^{2n}\det(iS+JS-iS+SJ)$$
$$=2^{2n}\det(JS+SJ).$$

Hence,

$$|\det[(\mathbb{1}-iJ)+S(\mathbb{1}+iJ)]|$$
$$=2^n[\det(SJ+JS)]^{1/2}. \tag{B6}$$

Finally, we give here the connection with Bargmann's constant $(\det\lambda)^{-1/2}$.[2] We introduce the x–p notation (see also Sec. 8): $S((x,p))=(Ax+Bp,Cx+Dp)$. In Bargmann's notations one has $\lambda=\frac{1}{2}(D+A+iB-iC)$, and $\nu_g=(\det\lambda)^{-1/2}=2^{n/2}[\det(A+D+iB-iC)]^{-1/2}$. This constant ν_g is in fact the matrix element $(\Omega_J,W_J(S)\Omega_J)$ (see Ref. 2). We have

$$(\Omega_J,W_J(S)\Omega_J)=\eta_{J,S}(\Omega_J,\Omega_{SJS^{-1}})$$
$$=\eta_{J,S}\,\beta^{-2}_{J,SJS^{-1}}=(\eta^*_{J,S})^{-1}$$
$$=2^n\{\det[(\mathbb{1}+iJ)+S(\mathbb{1}-iJ)]\}^{-1/2}.$$

However,

$$S=\begin{pmatrix}A & B\\ C & D\end{pmatrix},\quad J=\begin{pmatrix}0 & \mathbb{1}\\ -\mathbb{1} & 0\end{pmatrix};$$

hence,

$$\det(\mathbb{1}+iJ)+S(\mathbb{1}-iJ)$$
$$=\det\begin{pmatrix}\mathbb{1}+A+iB & i\mathbb{1}+B-iA\\ -i\mathbb{1}+C+iD & \mathbb{1}+D-iC\end{pmatrix}$$
$$=\det\begin{pmatrix}A+D+i(B-C) & -i(A+D)+B-C\\ -i\mathbb{1}+C+iD & \mathbb{1}+D-iC\end{pmatrix}$$
$$=\det\begin{pmatrix}A+D+i(B-C) & 0\\ -i\mathbb{1}+C+iD & 2\mathbb{1}\end{pmatrix}$$
$$=2^n\det(A+D+iB-iC).$$

So

$$(\Omega_J,W_J(S)\Omega_J)=2^{n/2}(\det(A+D+iB-iC))^{-1/2}.$$

Comparing this result with Bargmann's (B7) we see that they coincide, as was to be expected.

APPENDIX C

We give here the details of the calculation leading to formula (7.1):

$$w_S(v)=2^n\eta_{J,S}\int db\,(\Omega^b_J,\Pi(v)\,\Omega^{Sb}_{SJS^{-1}}), \tag{C1}$$

with

$$\int db\,(\Omega^b_J,\Pi(v)\,\Omega^{Sb}_{SJS^{-1}})$$
$$=\int db\,(\Pi(v)\Omega^b_J,\Omega^{Sb}_{SJS^{-1}})$$
$$=\int db\int dc\,e^{2i\sigma(v,b)}\overline{\Omega^{2v-b}_J(c)}\Omega^{Sb}_{SJS^{-1}}(c)$$
$$=\int db\int dc\,e^{2i\sigma(v,b)}e^{-2i\sigma(v,c)+i\sigma(b,c)}$$
$$\times e^{i\sigma(Sb,c)}\Omega_J(2v-b-c)\Omega_J(b-S^{-1}c)$$
$$=\int db\int dc\,e^{2i\sigma(v,b)}$$
$$\times e^{-2i\sigma(v,Sc)+i\sigma(b,Sc)+i\sigma(b,c)}\Omega_J(2v-b-Sc)\Omega_J(b-c)$$
$$=\int db\int dc\,e^{i\sigma(2v-c-Sc,b)}$$
$$\times e^{-2i\sigma(v,Sc)}\Omega_J\left(\sqrt{2b}+\frac{Sc-c-2v}{\sqrt{2}}\right)$$
$$\times\Omega_J\left(\frac{Sc+c-2v}{\sqrt{2}}\right)$$
$$=2^{-n}\int dc\left(e^{-2i\sigma(v,Sc)}e^{-i\sigma(2v-c-Sc,Sc-c-2v)/2}\right.$$
$$\left.\times\Omega_J\left(\frac{Sc+c-2v}{\sqrt{2}}\right)\int db\,e^{i\sigma(2v-c-Sc,b)/\sqrt{2}}\Omega_J(b)\right)$$
$$=\int dc\,e^{-2i\sigma(v,Sc)}e^{i\sigma(2v-Sc,c)}\Omega^2_J\left(\frac{Sc+c-2v}{\sqrt{2}}\right)$$
$$=\int dc\,e^{i\varphi(c/2,v,Sc/2)}\Omega_J(Sc+c-2v).$$

Combining this result with Eq. (C1), we get formula (7.1).

For the requantization of $w_S(v)$ we have to calculate

$$2^n\int dv\,w_S(v)\Pi(v)$$
$$=2^{2n}\eta\int dv\int db\,\Omega_J(2v-c-Sc)e^{i\varphi(c/2,v,Sc/2)}\Pi(v). \tag{C2}$$

We give here the calculation of this integral:

$$I=\int dv\int db\,\Omega_J(2v-c-Sc)\,e^{i\varphi(b/2,v,Sb/2)}\Pi_J(v)$$
$$=\int dv\int db\,\Omega_J(-2v)\,e^{i\varphi(b,2v+b+Sb,Sb)/4}$$
$$\times W_J(2v+b+Sb)\Pi$$
$$=\int dv\int db\,\Omega_J(2v)\,e^{-i\sigma(b,Sb)+2i\sigma(Sb,v)+2i\sigma(v,b)}$$
$$\times W_J(b+Sb)W_J(2v)\Pi\,e^{2i\sigma(v,b+Sb)}$$
$$=\int db\,W_J(b+Sb)\,e^{-i\sigma(b,Sb)}\int dv\,\Omega_J(2v)\,e^{4i\sigma(v,b)}\Pi(v).$$

Using the notations of Ref. 11, we have

$$\Omega_J e^{4i\sigma(v,b)}=2^{-n}\{b,-b\,|\,v\};$$

hence (see Ref. 11, Sec. 5.B.1),

$$\int dv\,\Omega_J(2v)\,e^{4i\sigma(v,b)}\Pi_J(v)=2^{-n}\int dv\{b,-b\,|v\}\Pi_J(v)$$
$$=2^{-2n}Q_J(\{b,-b\,|\cdot\})$$
$$=2^{-2n}|\Omega_J^{-b})(\Omega^b_J|.$$

This implies

$$I = 2^{-2n} \int db\, W_J(Sb) W_J(b) |\Omega_J^{-b})(\Omega_J^{b}|$$
$$= 2^{-2n} \int db\, W_J(Sb) |\Omega_J)(\Omega_J^{b}|$$
$$= 2^{-2n} \int db\, |\Omega_J^{Sb})(\Omega_J^{b}| .$$

Combining this with Eq. (C2), we get Eq. (7.4).

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