AIP Journal of Mathematical Physics

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Citation: J. Math. Phys. **21**, 2080 (1980); doi: 10.1063/1.524702
View online: http://dx.doi.org/10.1063/1.524702
View Table of Contents: http://jmp.aip.org/resource/1/JMAPAQ/v21/i8
Published by the American Institute of Physics.

An integral transform related to quantization

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(Received 18 December 1978; accepted for publication 39 August 1979)

We study in some detail the correspondence between a function f on phase space and the matrix elements $(Q_f)(a, b)$ of its quantized Q_f between the coherent states $|a\rangle$ and $|b\rangle$. It is an integral transform: $Q_f(a, b) = \int \{a, b\,|v\} f(v)\, dv$ which resembles in many ways the integral transform of Bargmann. We obtain the matrix elements of Q_f between harmonic oscillator states as the Fourier coefficients of f with respect to an explicit orthonormal system.

I. INTRODUCTION

Quantization is a word which should be used with caution, since it means many things to many people. We understand it here in the sense first sketched by Weyl,[1] where it describes a "harmonic analysis" procedure. It consists in Fourier analyzing a (fairly arbitrary) function on phase space, and then replacing the "elementary building blocks" (i.e., exponentials on phase space) by appropriate operators (which have since been known as Weyl operators, and are exponentials of linear combinations of the operators X and P).

A satisfactory and intrinsic description of the procedure became possible when von Neumann[2] proved the uniqueness theorem (Steps towards the theorem can be found in Weyl's book.[1]) which states that for a given (finite) number of degrees of freedom there exists—up to unitary equivalence—essentially only one *irreducible* family of Weyl operators in Hilbert space. This theorem is a cornerstone of quantum mechanics for a finite number of degrees of freedom. It seems however to have appeared too late to be fully incorporated in the mainstream of texbooks on the subject.

The intrinsic and symplectic formulation of quantum mechanics, made possible by von Neumann's theorem, was developed by Segal[3,4] and Kastler,[5] largely as a by-product of work aimed at systems with infinitely many degrees of freedom (The MIT thesis of R. Lavine[6] is devoted to finite numbers of degrees of freedom). The ingredients are as follows:

(1) A phase space E which is defined (without any "*a priori*" decomposition into position and momentum) as an even-dimensional vector space ($\dim E = 2\nu$) with an antisymmetric nondegenerate bilinear form σ.

(2) A Weyl system W which is a family of unitary operators, labeled by points in phase space, acting irreducibly on a Hilbert space $\mathscr{H}$ and satisfying

$$W(v_1)W(v_2) = e^{i\sigma(v_1,v_2)}W(v_1 + v_2). \qquad (1.1)$$

Given E and σ, von Neumann guarantees the existence and uniqueness (up to unitary equivalence) of W, but does not commit us to any concrete realization of W. The Weyl quantization procedure is then a two-step affair: (a) Fourier analysis: $f(v)$ is written as

$$f(v) = 2^{-\nu}\int e^{i\sigma(v,v')}\tilde{f}(v')dv'; \qquad (1.2)$$

(b) substitution of $W(-v/2)$ for $e^{i\sigma(v,\cdot)}$, giving

$$Q(f) = 2^{-\nu}\int W\left(\frac{-v}{2}\right)\tilde{f}(v)\,dv \qquad (1.3)$$

as the definition of the "quantized" of f.

It was shown in Ref. 3 that the correspondence $\tilde{f}\rightarrow Q(f)$ is inverted by

$$\tilde{f}(v) = 2^{-\nu}\mathrm{tr}\left(W\left(\frac{1}{2}v\right)Q(f)\right) = 2^{-\nu}\left(\left(W\left(\frac{-v}{2}\right),\ Q(f)\right)\right)_{\mathrm{HS}}, \qquad (1.4)$$

where $((,))_{\mathrm{HS}}$ is the inner product in the Hilbert space $\mathscr{L}_{\mathrm{HS}}$ of Hilbert–Schmidt operators in $\mathscr{H}$, and that the map $f\rightarrow Q(f)$ is unitary from $L^2(E)$ onto $\mathscr{L}_{\mathrm{HS}}$. Consequently,

$$(f_1, f_2)_{L^2(E)} = ((Q(f_1), Q(f_2)))_{\mathrm{HS}}. \qquad (1.5)$$

If e^a denotes the function $e^a(v) = e^{i\sigma(a,v)}$ we have $Q(e^a) = W(-a/2)$, and so, by extension of Eq. (1.5)

$$((Q(e^a), Q(e^b)))_{HS} = \int e^{-a}(v)e^b(v)dv = 2^{2\nu}\,\delta(a-b),$$

in a sense to be made precise (see, e.g., Ref. 7). This map is discussed in more detail by Pool.[8]

In Ref. 9, one of us made the remark that step (a) of the quantization procedure (Fourier analysis) can be avoided at the price of replacing the Weyl operators $W(v)$ by *Wigner operators* $\Pi(v)$ which are simply Weyl operators multiplied by parity, i.e., if $\Pi(0)$ is the parity operator (which can be defined intrinsically up to a sign in any Hilbert space that carries a Weyl system), and if $\Pi(v)$ is defined by

$$\Pi(v) = W(2v)\,\Pi(0) = W(v)\Pi W(-v),$$

then $Q(f)$ can be written directly as

$$Q(f) = 2^{\nu}\int f(v)\Pi(v)\,dv \qquad (1.6)$$

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 0022-2488/80/082080-11$1.00

and we do not have to consider the Fourier transform $\tilde{f}$ of the function f. The reason for calling $\Pi(v)$ a Wigner operator is that the Wigner quasiprobability density $\rho_\psi(v)$ corresponding to a pure state ψ is just the expectation value of $\Pi(v)$:

$$\rho_\psi(v) = 2^\nu(\psi,\Pi(v)\psi)$$

(see Ref. 7).

Equation (1.6) expresses $Q(f)$ as a superposition of Wigner operators, which are in some ways simpler than Weyl operators, namely, (i) Every Wigner operator $\Pi(v)$, in addition to being unitary $[\Pi^*(v) = (\Pi(v))^{-1}]$ is also selfadjoint $[\Pi^*(v) = \Pi(v)]$. Consequently, $\Pi(v)$ is involutive $[(\Pi(v))^2 = 1]$ and its spectrum consists of the numbers $+1$ and -1. (ii) The relationship (1.1) for Weyl operators is replaced by

$$\Pi(v_1)\Pi(v_2)\Pi(v_3) = e^{i\varphi(v_1,v_2,v_3)}\Pi(v_1 - v_2 - v_3) \quad (1.7)$$

(see Huguenin[10]), where $\varphi(v_1,v_2,v_3) = 4[\sigma(v_1,v_3) + \sigma(v_3,v_2) + \sigma(v_2,v_1)]$ is the oriented area of the triangle spanned by v_1, v_2, v_3; thus, Eq. (1.7) is affine (i.e., independent of the choice of origin in phase space) while Eq. (1.1) is vectorial (i.e., dependent on the choice of origin).

We can again invert formula (1.6) to obtain an expression analogous to Eq. (1.4), but giving now a direct correspondence between f and Q_f:

$$f(v) = 2^\nu \mathrm{tr}\,(Q_f\Pi(v)). \quad (1.8)$$

Wigner operators (without the name) were already present in Ref. 11. They can also be found in Ref. 12, where a decomposition of operators with respect to Wigner operators is given, analogous to Eq. (1.6), and relation (1.8) is derived for the function used in this decomposition. The Wigner operators were however not discussed in Ref. 12 as a means to do Weyl quantization without having to pass through the Fourier analysis step. That the $\Pi(v)$ formulas (1.6) and (1.8) may be more convenient then the $W(v)$ formulas (1.3) and (1.4) was also implicitly recognized in Ref. 13, where indeed some nondiagonal matrix elements involving a parity operation were used rather than the diagonal ones to compute classical function, which amounts exactly to preferring Eq. (1.8) as a direct formula to the indirect version (1.4) containing still a Fourier transform.

In this paper we will exploit Eq. (1.6) to study directly the relationship between the function f and matrix elements of the corresponding operator Q_f. We are particularly interested in the matrix elements of Q_f between coherent states. So the coherent state formalism will be the second main input in this paper.

The coherent state formalism has "a long and proud history in quantum theory."[14] Coherent states can be considered as eigenstates of a displaced harmonic oscillator (it is in this form they historically made their first appearance; see Ref. 15), as wave packets satisfying the minimum-uncertainty conditions, or as the eigenfunctions of the annihilation operator associated to the harmonic oscillator. For more details concerning these different points of view, see Ref. 14 and the references quoted there. We will consider the coherent states as displacements of the harmonic oscillator vacuum Ω:

$$|a\rangle = \Omega^a = W(a)\,\Omega, \quad (1.9)$$

where the operator representing a shift in phase space by the vector a is the Weyl operator $W(a)$.

The coherent states have many interesting properties and have therefore been widely used in quantum physics. One property which has been frequently made use of is the resolution of the identity operator[16]:

$$1 = \int dv\, |\Omega^v)(\Omega^v|. \quad (1.10)$$

One can use this property to represent every vector ψ in $\mathscr{H}$ by the wavefunction ϕ_ψ defined as

$$\phi_\psi(v) = (\Omega^v,\psi). \quad (1.11)$$

The set of wavefunctions constructed in this way forms a Hilbert space when equipped with the L^2 norm, and one has

$$\forall\ \varphi,\, \psi\in\mathscr{H}: \quad (\varphi,\psi) = (\phi_\varphi,\phi_\psi) = \int dv\ \overline{\phi_\varphi(v)}\phi_\psi(v) \quad (1.12)$$

[this is again Eq. (1.10)] and

$$\phi_\psi(a) = (\phi_{\Omega^a},\phi_\psi). \quad (1.13)$$

In fact, the mapping $\psi\mapsto\phi_\psi$ is, up to a Gaussian factor, the unitary map from our abstract Hilbert space $\mathscr{H}$ to the Bargmann–Segal Hilbert space of analytic functions $\mathscr{H}_{\rm BS}$[17,18] intertwining the operators $W(a)$ of the irreducible representation of the Weyl commutation relations on $\mathscr{H}$ with the usual Weyl operators on $\mathscr{H}_{\rm BS}$. For the special choice $\mathscr{H} = L^2(R^\nu)$ with the Schrödinger realization of the commutation relations this unitary map was constructed explicitly in Ref. 17.

With the help of the unitary map $\psi\mapsto\phi_\psi$ one can easily transport the Weyl operators $W(a)$ on $\mathscr{H}$ to the Hilbert space $\{\phi_\psi;\,\psi\in\mathscr{H}\}$. The irreducible representation of the Weyl commutation relations obtained in this way is the coherent state representation of the canonical commutation relations. Note that to define the coherent state representation we have used the harmonic oscillator vacuum Ω. To define this Ω in an unambiguous way some additional structure on E, σ is needed. Therefore, one introduces a complex structure J on E, compatible with σ (see Sec. 2). This is analogous to but weaker than the usual decomposition of E into a direct sum of x space and p space.

In what follows we will define an integral transform which relates f to the matrix elements of Q_f between coherent states:

$$\langle a|Q_f|b\rangle = (\Omega^a,Q_f\Omega^b) = \int dv\ \{a,b\,|v\}f(v). \quad (1.14)$$

Such a direct relationship between f and the matrix elements of Q_f enables us to study some aspects of the quantization procedure by means of a correspondence between function or distribution spaces instead of as a map from functions on phase space to operators on a Hilbert space. Of course, we could achieve this by using directly the classical functions, equipped with the twisted product rather than the usual product—this is the point of view of deformation theory; see for instance Ref. 19— or the functions occurring in the so-called diagonal or P respresentation.[16,20-23] In both these approaches, however, functions corresponding to very nice operators may have quite singular features: the classical function corresponding to the parity operator is a delta function, and it is a well-known fact that the P-representation function of a trace class operator may have such big growth at infinity that it is not even a tempered distribution. Working with the correspondence $f \rightarrow \langle a|Q_f|b\rangle$ we gain in smoothness with respect to these two approaches; the price we pay for this is an increase of the numbers of variables used (4ν variables instead of 2ν).

Coherent states can be defined in any Hilbert space carrying an irreducible Weyl system, which means that the matrix elements (1.14) can be computed in any representation, and are representation independent. We will use this to choose one specific representation, namely, the coherent state representation (written in an intrinsic, i.e., coordinateless way), which is particularly well suited for calculations with coherent states; the matrix elements we compute will however be independent of this particular choice of representation. The kernel $\{a, b\,|v\}$ (which was briefly discussed in Ref. 24) is studied in Sec. 3; the notations are explained in Sec. 2. We consider in particular a bilinear expansion for the kernel with respect to a basis formed by the matrix elements $h_{mn}(v)$ of the Wigner operators $\Pi(v)$ between harmonic oscillator states. These functions h_{mn} are given explicitly by Eq. (3.28). The Fourier coefficients of an arbitrary function f with respect to the basis h_{mn} are the matrix elements of Q_f between harmonic oscillator states.

The integral transform (1.4) which is discussed in Sec. 4 is analogous in many ways to the transform of Bargmann.[17] This analogy and the differences are discussed in Sec. 6.

The discussion of the integral transform given here is not at all exhaustive: a deeper study will be carried out in a forthcoming paper; we will study in particular the correspondence between some classes of distributions and the corresponding operators. A first application of Eq. (1.14) can be found in the computation of the classical functions corresponding to linear canonical transformations in Ref. 25.

2. THE GEOMETRICAL SETTING

We find it convenient to work in phase space without coordinates whenever possible. We shall however also rewrite some of the main formulas in a notation with coordinates which may be more familiar to most readers.

A. Affine phase space (symplectic geometry)

We denote by E a set which has the structure of an affine space (i.e., which can be identified to a real vector space after the choice of an origin). Assume that E is even dimensional and that we have associated an "oriented area" $\varphi(a, b, c)$ to every triangle with vertices a, b, c (taken in a given order). We assume the following:

(i) φ does not change if all its arguments are shifted by the same vector.

(ii) If a point o is chosen as the origin, then $\sigma(a, b)$ defined by

$$\sigma(a,b) = \tfrac{1}{4}\varphi(a,o,b)$$

is symplectic (i.e., bilinear, antisymmetric, and nondegenerate).

The function φ can now be expressed in terms of σ:

$$\varphi(a, b, c) = 4(\sigma(b, a) + \sigma(a, c) + \sigma(c, b)).$$

We see that it is totally antisymmetric: it changes sign if any two arguments are interchanged.

B. Phase space with a symplectic and a Euclidean geometry

Consider in E a reference frame, i.e., a family of vectors $a_1, b_1, \ldots, a_\nu, b_\nu$ that span E and such that $\sigma(a_i, a_j) = \sigma(b_i, b_j) = 0$, and $\sigma(a_i, b_j) = \delta_{ij}$. For our purposes (the building of a representation space for the Heisenberg commutation relations) all the relevant information is contained in the map J defined by

$$Ja_i = b_i, \quad J\,a_i = b_i, \quad J\,b_i = -a_i \qquad (i = 1,\ldots,\nu).$$

Notice that J has the following properties:

$$J^2 = -\mathbb{1} \tag{2.1}$$

(which is expressed by saying that J is a complex structure),

$$\sigma(Ja, Jb) = \sigma(a, b), \tag{2.2}$$

and

$$\sigma(a, Ja) > 0, \text{ if } a \neq 0. \tag{2.3}$$

It follows that the bilinear form

$$s(a, b) = \sigma(a, Jb) \tag{2.4}$$

defines a Euclidean geometry on E. A Triangle a, b, c has now not only an oriented area $\varphi(a, b, c)$ but also side lengths $(s(a-b, a-b))^{1/2}, \cdots$, which however depend on the choice of J.

We shall also use the complex combination

$$h(a, b) = s(a, b) + i\sigma(a, b), \tag{2.5}$$

which makes E into a ν-dimensional Hilbert space.

Examples: (1) Take $E = \mathbb{C}^\nu$ with

$$\sigma(a, b) = \operatorname{Im}(\bar{a}.b), \tag{2.6}$$

$$Ja = ia. \tag{2.7}$$

Then

$$s(a, b) = \operatorname{Re}(\bar{a}.b), \tag{2.8}$$

$$h(a, b) = \bar{a}.b, \tag{2.9}$$

and all the conditions above are satisfied. (2) Take $E = \mathbb{R}^\nu \oplus \mathbb{R}^\nu$. Any a in E is written as (x_a, p_a). Define

$$\sigma(a,b) = \tfrac{1}{2}(p_a.x_b - p_b.x_a), \tag{2.10}$$

$$J(x_a, p_a) = (p_a, -x_a). \tag{2.11}$$

Then

$$s(a,b) = \tfrac{1}{2}(x_a.x_b + p_a.p_b), \tag{2.12}$$

and all our conditions are fulfilled again. We can now use this structure to build a representation of the cannonical commutation relations.

C. A representation space for canonical commutation relations

On E, we consider the space of holomorphic functions

$$\mathscr{F}(E) = \{\varphi : E \rightarrow \mathbb{C} \mid \nabla^{Ja}\varphi = i\nabla^a\varphi \text{ for any } a \text{ in } E\}, \quad (2.13)$$

where

$$(\nabla^a\varphi)(v) = \lim_{\lambda\to 0} \frac{1}{\lambda}\varphi(v + \lambda a).$$

On the other hand, we define the Gaussian Ω by

$$\Omega(v) = \exp[-\tfrac{1}{2}s(v,v)]. \quad (2.14)$$

We shall say that a function ϕ on E is *modified holomorphic* if it is the product of a $\varphi \in \mathscr{F}$ and of the Gaussian: $\varphi(v) = \phi(v)\Omega(v)$.

This combination of both $\mathscr{F}$ and Ω gives us Z, the space of modified holomorphic functions:

$$Z(E) = \{\varphi\Omega \mid \varphi \in \mathscr{F}\}. \quad (2.15)$$

An alternative way of defining Z is

$$Z(E) = \{\phi : E \rightarrow \mathbb{C} \mid D^{Ja}\phi = iD^a\phi\}, \quad (2.15')$$

with $(D^a\phi)(v) = (\nabla^a\phi)(v) + s(a,v)\phi(v)$.

The Hilbert space we will use in the sequel whenever we want to consider a concrete representation space is now given by

$$\mathscr{L}_0 = Z(E) \cap L^2(E;dv), \quad (2.16)$$

where dv is the Lebesgue measure, translationally invariant, normalized by the requirement

$$\int \Omega^2(v)\,dv = 1. \quad (2.17)$$

On this Hilbert space, we define a set of unitary operators $W(a)$ by

$$(W(a)\phi)(v) = \exp[i\sigma(a,v)]\phi(v-a), \text{ for any } a \text{ in } E. \quad (2.18)$$

These $W(a)$ are called Weyl operators. It is easy to check that

$$W(a)W(b) = \exp[i\sigma(a,b)]W(a+b), \quad (2.19)$$

which implies we have a representation of the canonical commutation relations. One can easily see this in the example given above:

$$E = \mathbb{R}^\nu \oplus \mathbb{R}^\nu,$$

$$W((x_a,0))W(0,p_b))$$

$$= \exp\left(-\frac{i}{2}x_a \cdot p_b\right)W((x_a,p_b))$$

$$= \exp(-ix_a \cdot p_b)W((0,p_b))W((x_a,0)).$$

This is exactly what one would have expected from

$$W((x_a,p_b)) = \exp(i(x_a.X + p_b.P)),$$

with

$$[X_j,P_k] = i\delta_{jk}.$$

The representation given by Eq. (2.18) in the space (2.16) is irreducible.

Owing to von Neumann's uniqueness theorem for representations of the canonical commutation relations for a finite number of degrees of freedom, any result we will obtain in our particular representation on $\mathscr{L}_0$ can be transcribed to any irreducible representation.

Some particular functions in $\mathscr{L}_0$ will play a special role in the sequel: They are called the coherent states and are defined as

$$\Omega^a(v) = (W(a)\Omega)(v) = \exp(i\sigma(a,v))\Omega(v-a). \quad (2.20)$$

These coherent states have the following "reproducing" property[16,17]:

$$(\Omega^a,\phi) = \int \overline{\Omega^a(v)}\,\phi(v)\,dv = \phi(a) \text{ for any } \phi \in \mathscr{L}_0. \quad (2.21)$$

Writing this otherwise, we have

$$(\phi,\psi) = \int \overline{\phi(v)}\,\psi(v)\,dv$$

$$= \int (\phi,\Omega^v)(\Omega^v,\psi)dv;$$

hence

$$\int |\Omega^v\rangle\langle\Omega^v|\,dv = 1. \quad (2.22)$$

It is now easy to see that the Ω^a are normalized elements of $\mathscr{L}_0$:

$$(\Omega^a,\Omega^a) = \Omega^a(a) = \Omega(0) = 1. \quad (2.23)$$

As we already mentioned in the Introduction it is often useful to introduce Wigner operators, i.e., products of Weyl operators with the parity operator. We define

$$\pi : \psi \rightarrow \check{\psi}, \quad (2.24)$$

with $\check{\psi}(v) = \psi(-v)$. This operator conserves the modified holomorphy properties of and is thus an involutive unitary operator from $\mathscr{L}_0$ to itself. Moreover, one easily sees that

$$\Pi W(v) = W(-v)\Pi$$

or

$$\Pi W(v)\Pi = W(-v).$$

Hence, Π represents the parity $v \rightarrow -v$ on phase space. The Wigner operators $\Pi(a)$ are now defined as

$$\Pi(a) = W(2a)\Pi \quad (2.25)$$

i.e.,

$$(\Pi(a)\psi)(v) = e^{i\sigma(2a,v)}\psi(2a - v), \text{ for any } \psi \text{ in } \mathscr{L}_0. \quad (2.26)$$

It is easy to check that

$$\Pi(a)\Pi(b)\Pi(c) = \exp[i\varphi(a,b,c)]\Pi(a-b+c). \quad (2.27)$$

3. THE FUNCTIONS $\{a,b|v\} = \{\xi/v\}$

Definition: Let $\mathscr{H}$ be a Hilbert space carrying an irreducible representation of the Weyl commutation relations for ν degrees of freedom. Denote by Ω^a the coherent state centered at $a \in E$, and by $\Pi(v)$ the parity operator around v (Wigner operator). Given a, b, $v \in E$, we define

$$\{a,b\,|v\} = \kappa^{-1}(\Omega^a,\Pi(v)\Omega^b), \quad (3.1)$$

with

$$\kappa = 2^{-\nu}. \quad (3.1')$$

The numbers $\{a,b\,|v\}$ can be easily calculated and have simple properties.

A. Explicit form, symmetries, and special values

One has

$$\{a, b\,|v\} = 2^{\nu}\exp\left[i\varphi\left(\frac{a}{2},v,\frac{b}{2}\right)\right]\Omega(a+b-2v), \tag{3.2}$$

i.e., the phase of $\{a, b\,|v\}$ is the oriented area of the triangle with vertices $\frac{1}{2}a, v, \frac{1}{2}b$. The number $\{a, b\,|v\}$ is real if and only if the three points $\frac{1}{2}a, v, \frac{1}{2}b$ are collinear.

The absolute value of $\kappa\{a, b\,|v\}$ is the exponential of the negative of half the squared Euclidean distance from v to the midpoint $(a+b)/2$ of the segment (a, b). It takes its maximal value (which is 1) when v is the midpoint of (a, b).

If we denote by ζ the pair $\{a, b\}$ we have

$$\{-\zeta\,|-v\} = \{-a,-b\,|-v\} = \{a, b\,|v\} = \{\zeta\,|v\}. \tag{3.3}$$

Denote by $\bar{\zeta}$ the pair $\{b, a\}$. (This will be justified below.) Then

$$\{\bar{\zeta}\,|v\} = \{b, a|v\} = \overline{\{a, b\,|v\}} = \overline{\{\zeta\,|v\}}. \tag{3.4}$$

If the arguments of $\{a, b\,|v\}$ are shifted, we have

$$\{a+c, b+c|v+c\} = \exp[i\sigma(c, a-b)]\{a, b\,|v\}. \tag{3.5}$$

One has

$$\{a, a|v\} = |\{a, a|v\}| = \kappa^{-1}\Omega(2v-2a) \tag{3.6}$$

and

$$\left\{a, b\,\middle|\,\frac{a+b}{2}\right\} = \kappa^{-1}. \tag{3.7}$$

B. Expression of $\{a, b|v\}$ in coordinates

(1) Identify E with $\mathbb{C}^{\nu}$. Then $\sigma(a, b) = \mathrm{Im}(\bar{a}.b)$, $Ja = ia$, and $s(a, b) = \mathrm{Re}(\bar{a}.b)$. So

$$\{a, b\,|v\} = 2^{\nu}\exp(-\tfrac{1}{2}|a|^2 - \tfrac{1}{2}|b|^2 - 2|v|^2 - a.\bar{b} + 2\bar{b}.v + 2\bar{v}.a). \tag{3.8}$$

(2) Identify E with $\mathbb{R}^{\nu}\oplus\mathbb{R}^{\nu}$. So a is the pair $a = \{x_a, p_a\}$: $\sigma(a, b) = \frac{1}{2}(p_a x_b - p_b x_a)$, $J(x_a, p_a) = (p_a, -x_a)$, and $s(a, b) = \frac{1}{2}(p_a p_b + x_a x_b)$.

Then

$$\{a, b\,|v\} = 2^{\nu}\exp[-\tfrac{1}{4}|x_a + x_b - 2x_v|^2 - \tfrac{1}{4}|p_a + p_b - 2p_v|^2 + i(x_v(p_b - p_a) - p_v(x_b - x_a) + \tfrac{1}{2}p_a x_b - \tfrac{1}{2}p_b x_a)]. \tag{3.9}$$

C. Analyticity and regularity properties

The expression (3.2) can be rewritten as

$$\{a, b\,|v\} = 2^{\nu}\exp[2h(b,v) + 2h(v,a) - h(b,a)] \times\Omega(a)\,\Omega(b)\,\Omega(2v), \tag{3.10}$$

where $h(a, v)$ is defined by Eq. (2.5). In coordinates, Eq. (3.10) is just Eq. (3.8).

Since Eq. (3.10) can be rewritten as

$$\begin{aligned}\{a, b\,|v\} &= 2^{\nu}\exp[2i\sigma(b,v)]\Omega^{2v-b}(a)\\ &= 2^{\nu}\exp[2i\sigma(v,a)]\,\overline{\Omega^{2v-b}(b)}\\ &= 2^{\nu}\exp[i\sigma(a,b)]\Omega^{-1}(a-b)\\ &\quad\times\Omega^{\sqrt{2}b}(\sqrt{2}v)\overline{\Omega^{\sqrt{2}a}(\sqrt{2}v)},\end{aligned}$$

we see that $\{a, b\,|v\}$ is modified holomorphic in a, modified antiholomorphic in b, and a product of a modified holomorphic function with a modified antiholomorphic one in v. In each of these variables it is infinitely differentiable and of Gaussian decrease at infinity.

D. Fourier transforms and integrals

The (symplectic) Fourier transform F can be defined by

$$(Fg)(v) = \kappa\int\exp[i\sigma(v,v')]g(v')\,dv'. \tag{3.11}$$

Then

$$F^2 = \mathbb{1},\quad F\Omega = \Omega,\quad F\Omega^{a} = \Omega^{-a}. \tag{3.12}$$

If a function ϕ is modified holomorphic [see Eq. (2.15')], it satisfies $(F\phi)(v) = \phi(-v)$. If it is modified antiholomorphic, it satisfies $(F\phi)(v) = \phi(v)$. So

$$\kappa\int\exp[i\sigma(a,a')]\{a', b\,|v\}\,da' = \{-a, b\,|v\}. \tag{3.13}$$

In particular,

$$\kappa\int\{a', b\,|v\}\,da' = \kappa^{-1}\exp[i\sigma(b, 2v)]\,\Omega(2v-b). \tag{3.14}$$

Similarly,

$$\kappa\int\exp[i\sigma(b,b')]\{a, b'|v\}\,db' = \{a, b\,|v\}. \tag{3.15}$$

In particular,

$$\kappa\int\{a, b'|v\}\,db' = \kappa^{-1}\exp[i\sigma(2v,a)]\,\Omega(2v-a). \tag{3.16}$$

The Fourier transform in the variable v can be computed directly. It is

$$\kappa\int\exp[i\sigma(v,v')]\{a, b\,|v'\}\,dv' = \kappa^2\{a, -b\,|\tfrac{1}{4}v\}. \tag{3.17}$$

In particular,

$$\kappa\int\{a, b\,|v'\}\,dv' = \kappa\,\Omega^{-b}(a). \tag{3.18}$$

One has also

$$\int\{a, a|v\}\,da = 1. \tag{3.19}$$

We now consider integrals that are bilinear in the symbols $\{a, b\,|v\}$. Particularly important is the relationship

$$\int\int\{a, b\,|v\}\{b, a|v'\}\,da\,db = \delta(v-v'), \tag{3.20}$$

which can also be written as

$$\int\{\zeta\,|v\}\{\bar{\zeta}\,|v'\}\,d\zeta = \int\{\zeta\,|v\}\overline{\{\zeta\,|v'\}}\,d\zeta = \delta(v-v'). \tag{3.21}$$

We shall derive it here, in order to show how simple the calculations are:

$$\int\int\{a, b\,|v\}\{b, a|v'\}\,da\,db$$

$$= \kappa^{-2} \int \int (\Omega^a, \Pi(v) \Omega^b)(\Omega^b, \Pi(v') \Omega^a)\, da\, db$$
$$= \kappa^{-2} \int (\Omega^a, \Pi(v) \Pi(v') \Omega^a)\, da$$
$$= \kappa^{-2} \int \exp(i\varphi(v, v', \frac{a}{2}))\Omega^{2(v-v')+a}(a)\, da$$
$$= \kappa^{-2} \exp(4i\sigma(v', v))\, \Omega(2v - 2v')$$
$$\times \int \exp[4i\sigma(a, v - v')]\, da$$
$$= \exp[4i\sigma(v', v)]\, \Omega(2v' - 2v)\kappa^{-4}\delta(4v' - 4v)$$
$$= \delta(v' - v).$$

Another useful relation is

$$\int \{a, b\,|v\}\{c, d\,|v\}\, dv = \Omega^d(a)\Omega^b(c). \tag{3.22}$$

We give finally two integrals of triple products:

$$\int\int \{a, b\,|v\}\{b, c|v'\}\{c, d\,|v''\}\, db\, dc$$
$$= \kappa^{-2}\exp[i\varphi(v, v', v'')]\{a, d\,|v - v' + v''\} \tag{3.23}$$

and

$$\int\int\int \{a, b\,|v\}\{b, c|v'\}\{c, a|v''\}\, da\, db\, dc$$
$$= \kappa^{-2}\exp[i\varphi(v, v', v'')]. \tag{3.24}$$

E. Bilinear expansions of $\{\zeta|v\}$; The orthonormal family $h_{mn}(v)$

Since $\{\zeta\,|v\}$ will be used as the kernel of an integral transform, it is natural to expand it into a sum of products of functions of v and of functions of ζ. Such expansions can be found immediately since the $\{a, b\,|v\}$'s are matrix elements of irreducible families of operators.

1. Generalities

For any orthonormal base e_n in $\mathscr{H}$, we define the functions e_{mn} by

$$e_{mn}(v) = \kappa^{-1}(e_m, \Pi(v)e_n). \tag{3.25}$$

These e_{mn} form an orthonormal base in $L^2(E)$: orthogonality is a consequence of Eq. (3.22), and completeness follws from the fact that the family $\{\Pi(a)\}$ is an irreducible family of operators.

It is now obvious that

$$\{a, b\,|v\} = \sum_{m,n} (\Omega^a, e_m)(e_n, \Omega^b)e_{mn}(v). \tag{3.26}$$

2. In the representation on $\mathscr{L}_0$

Let $\{e_1, \ldots, e_\nu\}$ be a symplectic base on E, σ, i.e., a set of ν vectors satisfying

$$\sigma(e_j, e_k) = 0, \qquad \sigma(e_j, Je_k) = \delta_{jk}.$$

We define the normalized monomials $h^{[n]}$ on E by

$$h^{[n]}(v) = \frac{1}{\sqrt{[n]!}} \prod_{j=1}^{\nu} (h(e_j, v))^{n_j}. \tag{3.27}$$

Here $[n]$ is a multi-index, with $[n]! = \Pi_{j=1}^{\nu}(n_j!)$.

The functions $h^{[n]}\Omega$ form an orthonormal base in $\mathscr{L}_0$. In fact, they are the eigenfunctions of the harmonic oscillator (see also Sec. 5). In the following we shall drop the square brackets in the notation $h^{[n]}$. In this particular case we have now

$$h_{mn}(v) = \kappa^{-1}(h^m\Omega, \Pi(v)h^n\Omega)$$
$$= \kappa^{-1}\Omega(2v)\sum_{s=0}^{\min(m,n)} (-1)^s \left(\binom{n}{s}\binom{m}{s}\right)^{1/2}$$
$$\times h^{n-s}(2v)\, \overline{h^{m-s}(2v)} \tag{3.28}$$

and

$$\{a, b\,|v\} = \Omega(a)\,\Omega(b) \sum_{m,n} h^m(a)\, \overline{h^n(b)}\, h_{mn}(v). \tag{3.29}$$

Formula (3.29) implies we can also write the h_{mn} as

$$h_{mn}(v) = \frac{1}{m!\, n!}[(D^a)^m(D^b)^n\{a, b\,|v\}]_{a=b=0}$$
$$= \frac{1}{m!\, n!}[(\nabla^a)^m(\nabla^b)^n(\{a, b\,|v\}\Omega^{-1}(a)\Omega^{-1}(b))]_{a=b=0},$$

where D^a and ∇^a are defined in Sec. 2c.

If one identifies E with $\mathbb{C}^\nu$, the h^n are simply

$$h^n(z) = \frac{1}{\sqrt{n!}} z^n,$$

and the h_{mn} are given by

$$h_{mn}(z) = 2^\nu 2^{m+n} e^{-2|z|^2} \sum_{s=0}^{\min(m,n)} \left[-\frac{1}{4}\right]^s$$
$$\times \frac{\sqrt{n!\, m!}}{s!\,(n-s)!\,(m-s)!} z^{n-s}\bar{z}^{m-s}.$$

Identifying E with $\mathbb{R}^{2\nu}$, one gets

$$h^n(x,p) = \frac{1}{\sqrt{n!}} 2^{-n/2}(p + ix)^n$$

and

$$h_{mn}(x,p) = 2^\nu e^{-(x^2+p^2)} \sum_{s=0}^{\min(m,n)} (-2)^{-s}$$
$$\times \frac{\sqrt{n!\, m!}\, 2^{(n+m)/2}}{s!\,(n-s)!\,(m-s)!}$$
$$\times (p + ix)^{n-s}(p - ix)^{m-s}. \tag{3.30}$$

In particular, we have

$$h_{nn}(x,p) = (-1)^n 2^\nu e^{-(x^2+p^2)}$$
$$\times \sum_{s=0}^{n} (-2)^s \frac{n!}{s!\, s!\,(m-s)!}(x^2 + p^2)^s$$
$$= (-1)^n 2^\nu \exp(-x^2 - p^2)L_n(2x^2 + 2p^2), \tag{3.31}$$

where L_n is the Laguerre polynomial with multi-index $[n]$ in the variables $x_j^2 + p_j^2$ $(j = 1, \cdots, \nu)$.

As a consequence of Eq. (3.17), we see that

$$(Fh_{mn})(v) = \kappa\left(h^m\Omega, \Pi\left(\frac{v}{4}\right)\Pi h^n\Omega\right)$$
$$= (-1)^n\kappa^2 h_{mn}\left(\frac{v}{4}\right). \tag{3.32}$$

4. THE INTEGRAL TRANSFORM: MAIN PROPERTIES

A. The map from $f(v)$ to $Q_f(\zeta)$

Let $f \in S'(E)$ be a tempered distribution on E. Let $\tilde{f}$ be its Fourier transform, which we write freely as

$$\tilde{f}(v) = \kappa \int \exp[i\sigma(v,v')]f(v')\,dv'. \quad (4.1)$$

Let

$$f(v) = \sum_{m,n} f_{mn} h_{mn}(v) \quad (4.2)$$

be the Fourier series expansion of f in the orthonormal system (3.28). By Eq. (3.30), the expansion of $\tilde{f}$ is

$$\tilde{f}(v) = \kappa^2 \sum (-1)^n f_{mn} h_{mn}\left(\frac{v}{4}\right).$$

Definition: The Q transform of f is the function Q_f on $E \times E$, defined by

$$Q_f(\zeta) = Q_f(a,b) = \int \{a,b\,|v\} f(v)\,dv, \quad (4.3)$$

to be interpreted, if necessary, as the evaluation of the functional f on the testing function $\{a, b\,|.\}$.

Remark: The above definition is more restrictive than necessary since the testing functions $\{a, b\,|.\}$ can handle more general distributions "of type S".[26,27] We shall not try here to study in detail the functional analysis associated with Eq. (4.3).

By Eq. (3.17), Q_f and $Q_{\tilde{f}}$ are related through

$$Q_{\tilde{f}}(a,b) = \kappa^{-2} Q_{\check{f}_4}(a, -b), \quad \text{with} \quad \check{f}_4(v) = f(-4v). \quad (4.4)$$

In fact, this relationship between the matrix elements of $Q_{\tilde{f}}$ and $Q_{\check{f}_4}$ is just a consequence of the equivalence of formulas (1.3) and (1.6). Indeed, we have from Eqs. (1.3) and (1.6)

$$\begin{aligned} Q_{\tilde{f}}(a,b) &= \kappa^{-1} \int dv \tilde{f}(v)(\Omega^a, \Pi(v)\Omega^b) \\ &= \kappa \int dv\, f(v)\left(\Omega^a, W\left(-\frac{v}{2}\right)\Omega^b\right) \\ &= \kappa \int dv\, f(-v)\left(\Omega^a, \Pi\left(\frac{v}{4}\right)\Pi\Omega^b\right) \\ &= \kappa^{-3} \int dv\, f(-4v)(\Omega^a, \Pi(v)\Omega^{-b}) \\ &= \kappa^{-2} Q_{\check{f}_4}(a, -b). \end{aligned} \quad (4.5)$$

Furthermore, by Eq. (3.29) and (4.3), the function $Q_f(a,b)$ can be expressed in terms of the Fourier coefficients f_{mn}:

$$Q_f(a,b) = \Omega(a)\Omega(b) \sum_{m,n} h^n(a)\overline{h^m(b)} f_{mn},$$

where the functions $h^n(a)$ and $h^m(b)$ are defined by Eq. (3.27).

An examination of either Eq. (4.3) or (4.5) shows that $Q_f(a,b)$ is modified holomorphic in a and modified antiholomorphic in b, i.e., it is holomorphic in ζ with respect to the complex structure $(J, -J)$.

B. Inverting the map $f \to Q_f$

Given $Q_f(\zeta)$, we can reconstruct f through

$$f(v) = \int \{\zeta\,|v\} Q_f(\zeta)\,d\zeta = \int\int \{a,b\,|v\} Q_f(b,a)\,da\,db, \quad (4.6)$$

provided the integrals converge. This is an immediate consequence of (3.21).

In other words, the same kernel is used for (4.3) and its inverse just as in Fourier transforms and the integral transforms[17] of Bargmann.

C. Physical interpretation

So far, we have only defined some integral transform $f(v) \to Q_f(\zeta)$ by means of the kernel $\{\zeta\,|v\}$. Of course, we always had in mind the physical interpretation of all this when we defined our map from one function space to another. This physical interpretation follows immediately from formula (1.6) and definition (3.1) of the kernel $\{a, b\,|v\}$. One has

$$Q_f(a,b) = \kappa^{-1} \int dv\, f(v)(\Omega^a, \Pi(v)\Omega^b) = (\Omega^a, Q_f\,\Omega^b).$$

So for any a, b in E, $Q_f(a, b)$ is the matrix element between the coherent states Ω^a and Ω^b of the quantum mechanical operator Q_f corresponding to the "classical observable" f.

In an analogous way, the Fourier coefficient

$$f_{mn} = \int \overline{h_{mn}(v)} f(v)\,dv, \quad (4.7)$$

with

$$h_{mn}(v) = \kappa^{-1}(h^m\Omega, \Pi(v)h^n\Omega),$$

the h's being normalized monomials [see Eq. (3.27)], is equal to the matrix element of Q_f between harmonic oscillator states:

$$f_{mn} = \langle n|Q_f|m\rangle.$$

D. Action of Q_f in $\mathscr{L}_0(E)$

The action of the operator Q_f on $\mathscr{L}_0(E)$ can be written explicitly with the help of the function $Q_f(\zeta)$ [see Eq. (2.21)]:

$$\forall \psi \in \mathscr{L}_0(E) : (Q_f\psi)(a) = \int db\, Q_f(a,b)\psi(b). \quad (4.8)$$

E. Unitarity of the correspondence $f \to Q_f$

The function $Q_f(\zeta)$ is an element of $Z(E \times E; (J, -J))$, i.e., modified holomorphic in its first variable and modified antiholomorphic in its second variable.

Define $\mathscr{L}(E \times E) = L^2(E \times E; dv \otimes dv) \cap Z(E \times E; (J, -J))$. Equipped with the L^2 norm, this is a Hilbert space. Suppose f is square integrable. Then

$$\begin{aligned} &\int d\zeta\, |Q_f(\zeta)|^2 \\ &\quad = \int\int da\,db\, \overline{Q_f(a,b)}\, Q_f(a,b) \\ &\quad = \int\int\int\int da\,db\,dv\,dv' f^*(v)\{b,a|v\}\, f(v')\{a,b\,|v'\} \\ &\quad = \int\int dv\,dv' f^*(v) f(v')\delta(v-v') \\ &\quad = \int dv\, |f(v)|^2. \end{aligned}$$

Hence the map $f(.) \to Q_f(.)$ is a unitary map from $L^2(E)$ to $\mathscr{L}_0(E \times E)$; its inverse is defined by Eq. (4.6).

In fact, this unitarity is nothing else than the well-known unitarity of the correspondence between square integrable functions and Hilbert–Schmidt operators.[8] Indeed, one can check that the operator Q_f is Hilbert–Schmidt if the function $Q_f(.)$ is square integrable, and one has the equality

$$\|Q_f\|^2_{\mathrm{HS}} = \mathrm{Tr}(Q_f^* Q_f) = \int d\zeta\, |Q_f(\zeta)|^2.$$

F. Products

Let A and B be operators on $\mathscr{L}_0$. Then

$$(A, B)(a,b) = (\Omega^a, AB\Omega^b)$$
$$= \int (\Omega^a, A\Omega^c)(\Omega^c, B\Omega^b)\, dc$$
$$= \int A(a,c)B(c,b)\, dc.$$

Hence,

$$(Q_f \cdot Q_g)(a,b) = \int Q_f(a,c)Q_g(c,b)\, dc.$$

We define the twisted product $f \circ g$ by

$$Q_{f\circ g} = Q_f \cdot Q_g. \tag{4.9}$$

Hence,

$$(f\circ g)(v)$$
$$= \int \{\zeta\,|v\} Q_{f\circ g}(\bar{\zeta})\, d\zeta$$
$$= \int\int\int \{a,b\,|v\} Q_f(b,c)Q_g(c,a) da\, db\, dc$$
$$= \int\int\int\int\int f(v')g(v'')$$
$$\times \{a,b\,|v\}\{b,c|v'\}\{c,a|v''\} da\, db\, dc\, dv'\, dv''$$
$$= \kappa^{-2}\int\int f(v'')g(v'')\exp[i\varphi(v,v',v'')]dv'\, dv'', \tag{4.10}$$

which is a well-known expression.[10,28]

G. Bounds on $Q_f(a,b)$

Define the following two *regularized functions* associated to f:

$$f_R(v) = \int f(v')\Omega(2(v'-v))\, dv', \tag{4.11}$$

$$f_r(v) = \int f(v')\,\Omega\left(\frac{v'-v}{2}\right) dv'. \tag{4.12}$$

They can be obtained by choosing f as the the initial value of a diffusion (heat) equation and waiting the appropriate time.

Assume now that f is a (locally integrable) function so that $|f|$ (the absolute value of f) is well defined. Denote by $|f|_R$ the regularized Eq. (4.11) of $|f|$. Then Eq. (4.3) gives

$$|Q_f(a,b)| \leqslant \kappa^{-1}|f|_R\left(\frac{a+b}{2}\right). \tag{4.13}$$

On the other hand, if the Fourier transform $\tilde{f}$ of f is a function (here one should not think of Q_f as, say, a Hamiltonian but for example a resolvent), and if $|\tilde{f}|_r$ is the regularized Eq. (4.12) of $|\tilde{f}|$, we obtain from Eq. (4.4)

$$|Q_f(a,b)| = \kappa^{-2}|Q_{\tilde{f}_4}(a,-b)|$$
$$\leqslant \kappa^{-3}|\tilde{f}_4|_R\left(\frac{a-b}{2}\right)$$
$$= \kappa|\tilde{f}|_r(2(a-b)). \tag{4.14}$$

If both f and $\tilde{f}$ are functions we obtain

$$|Q_f(a,b)|^2 \leqslant |f|_R\left(\frac{a+b}{2}\right)|\tilde{f}|_r(2(a-b)). \tag{4.15}$$

For the diagonal matrix elements one obtains[24] an equality, which does not require any special assumption on f,

$$Q_f(a,a) = \kappa^{-1}f_R(a). \tag{4.16}$$

More generally, f can be assumed to be a measure.

H. Positivity of Q_f

Suppose that Q_f is positive, i.e.,

$$\forall\psi\in\mathscr{L}_0: (\psi, Q_f\psi) = \int\int \overline{\psi(a)}\psi(b)Q_f(a,b)\, da\, db \geqslant 0. \tag{4.17}$$

Since

$$\{a,b\,|v\} = \kappa^{-1}\exp[2i\sigma(v,a)]\,\overline{\Omega^{2v-a}(b)}$$

(see Sec. 3c), this is equivalent to

$$\forall\psi\in\mathscr{L}_0: \int\int \overline{\psi(a)}\psi(2v-a)\exp[2i\sigma(v,a)]f(v)\, da\, dv \geqslant 0, \tag{4.18}$$

provided we are allowed to change the order of the integrations, which is certainly true, for example, for $f\in L^2(E;dv)$.

We can rewrite condition (4.18) as

$$\forall\psi\in\mathscr{L}_0: \int\int \overline{\psi(a)}\psi(b)\exp[i\sigma(b,a)]f\left(\frac{a+b}{2}\right) da\, db \geqslant 0. \tag{4.19}$$

For $f\in L^2(E;dv)$, Eq. (4.19) is a necessary and sufficient condition for Q_f to be positive.

If, moreover, we suppose f is essentially bounded ($f\in L_\infty$) and integrable ($f\in L^1$), then Eq. (4.19) is implied by

$$\forall n\in\mathbb{N},\ \forall\, a_1,\ldots,a_n\in E: \text{the matrix}$$
$$\times\left[\exp[i\sigma(a_j,a_k)]f\left(\frac{1}{2}(a_j+a_k)\right)\right]_{j,k} \text{ is positive.} \tag{4.20}$$

So, for $f\in L_\infty \cap L^1$, Eq. (4.20) is a sufficient condition for Q_f to be positive.

A similar result, though in a different context, can be found in Ref. 29; the fact that the matrices

$$[\exp[i\sigma(a_j,a_k)]f(a_k-a_j)]_{jk}$$

are considered in Ref. 29 and not

$$\left[\exp[i\sigma(a_j,a_k)]f\left(\frac{a_j+a_k}{2}\right)\right]_{jk}$$

as here, is due to their studying the correspondence $f\rightarrow\int dv\, f(v)\, W(v)$ and not $f\rightarrow\int dv\, f(v)\Pi(v)$.

Examples: Using Eq. (4.20), one can easily check that the following functions yield positive operators:

$$f(v) = \Omega^{(1-\alpha)}(2v), \quad \text{with } \alpha < 1.$$

In particular,

$$f(v) = \Omega(2v) \text{ and } f(v) = \Omega(v),$$
$$f(v) = \exp[2s(c,v)]\,\Omega(2v),$$
$$f(v) = \exp[2\sigma(c,v)]\,\Omega(2v).$$

5. EXAMPLES

A. Operators corresponding to elementary functions

For some functions we shall use Eq. (4) to compute both Q_f and $Q_{\tilde{f}}$:

(1) $f(v) = 1$. Then $Q_1(a,b) = \Omega^b(a)$ or $Q_1 = \mathbb{1}$. $\tilde{1}(v) = \kappa^{-1}\delta(v)$. Hence, $Q_\delta(a,b) = \kappa^{-1}Q_1(a,-b) = \kappa^{-1}\Omega^{-b}(a)$. This implies $Q_\delta = \kappa^{-1}\Pi$, which is, of course, implicit in Eq. (4.3).

(2) $f(v) = \exp[i\sigma(c,v)]$. This gives $Q_f(a,b) = \exp[\frac{1}{2}i\sigma(c,b)]\,\Omega^{b+c/2}(a)$; hence $Q_f = \Pi(c/4)\Pi$. Applying again Eq. (4.4), we see that

$$[F e^{i\sigma(c,.)}](v) = \kappa^{-1}\delta(v-c) = \kappa^{-1}\delta_c(v)$$

and

$$Q_{\delta_c} = \kappa^{-1}\Pi(c).$$

(3) $f(v) = \Omega(\alpha v)$ for $\alpha \in \mathbb{R}$.

Then

$$Q_f(a,b) = \left[\frac{4}{4+\alpha^2}\right]^{\nu} \exp[i\sigma(a,b)]^{(\alpha^2-4)/(4+\alpha^2)} \times [\Omega(a+b)]^{(\alpha^2)/(4+\alpha^2)}[\Omega(a-b)]^{4/(4+\alpha^2)}.$$

As a consequence of the fact

$$[F\Omega(\alpha .)](v) = \alpha^{-2\nu}\Omega(\alpha^{-1}v),$$

we see that

$$Q_{\Omega(\alpha^{-1}.)}(a,b) = (2\alpha)^{2\nu} Q_{\Omega(4\alpha .)}(a,-b).$$

(4) As a special case of (3), we have

$$Q_{\Omega(2.)}(a,b) = \kappa\,\Omega(a)\,\Omega(b);$$

hence,

$$Q_{\Omega(2.)} = \kappa\,|\Omega)(\Omega|.$$

(5) $f(v) = s(v,v)$. This is the Hamiltonian of the harmonic oscillator. We have

$$Q_f(a,b) = \left[\frac{\nu}{2} + h(b,a)\right]\Omega^b(a),$$

or

$$Q_f = \frac{\nu}{2} + N,$$

with

$$N: \Omega^b \mapsto h(b,.)\Omega^b.$$

We see here the expected vacuum energy term $\nu/2$; moreover, one can easily check that for $u_n = h^n\Omega$, one has

$$N u_n = n\, u_n,$$

which is in accordance with the well-known fact that the u_n are the harmonic oscillator eigenfunctions.[17]

(6) $f(v) = \sigma(c,v)$. This gives $Q_f(a,b) = (i/2)\Omega^b(a) \times [h(b,c) - h(c,a)]$. Define $H_c: \psi \mapsto h(c,.)\psi$. Then $H_c^*: \psi \to D_c\psi$ with $D_c = \nabla_c + s(c,.)$ or $\varphi.\Omega \mapsto (\nabla_c\varphi)\Omega$ or $\Omega^b \to h(b,c)\Omega^b$, and $Q_{\sigma(c,.)} = (i/2)(H_c^* - H_c)$. Analogously, $Q_{s(c,.)} = \frac{1}{2}(H_c^* + H_c)$ and $Q_{h(c,.)} = H_c$.

(7) $f(v) = \sigma(c_1,v)\,\sigma(c_2,v)$. Then

$$Q_{\sigma(c_1,.)\sigma(c_2,.)} = \tfrac{1}{4}[s(c_1,c_2) + H_{c_1}H_{c_2}^* + H_{c_2}H_{c_1}^* - H_{c_1}H_{c_2} - H_{c_1}^*H_{c_2}^*].$$

Analogously,

$$Q_{h(c_1,.)h(c_2,.)} = H_{c_1}H_{c_2}.$$

B. Functions corresponding to elementary operators

1. Dyadics

Take A to be the dyadic $A = |\phi)(\psi|$. Then

$$f_A(v) = \int\int \{a,b\,|v\}\,\phi(b)\overline{\psi(a)}\,da\,db = (\psi, Q_{\delta_v}\phi).$$

In particular,

$$f_{|\Omega^d)(\Omega^c|}(v) = \{c,d\,|v\},$$

$$f_{|\Omega^c)(\Omega^c|}(v) = \kappa^{-1}\,\Omega(2v-2c),$$

$$f_{|\Omega)(\Omega|}(v) = \kappa^{-1}\,\Omega(2v),$$

$$f_{|u_n)(u_m|}(v) = h_{mn}(v).$$

From this last example we see that h_{mn} [given for instance by Eq. (3.31′)] is the classical function corresponding to the projection onto the subspace $\mathbb{C}u_n$. For the special case $\nu = 1$, this is the projection onto the nth eigenspace of the harmonic oscillator (a similar expression, obtained in a different way, can be found in Ref. 19).

2. Multiplication operators by holomorphic functions

Consider $A: \psi \to F.\psi$, where F is some (holomorphic) function such that $F.\Omega^a \in \mathscr{L}_0$ for any a.

Then

$$A(a,b) = F(a)\Omega^b(a),$$

and

$$f_A(v) = \kappa^{-1}\int \Omega(2v-2b)\,F(b)\,db;$$

hence,

$$f_A = \kappa^{-1}(\Omega_2 * F), \quad \Omega_2(v) = \Omega(2v).$$

3. Permutation operators

Suppose $E = E_1 \oplus \dots \oplus E_n$, with $JE_j \subset E_j, \forall j, \sigma(E_j,E_k) = 0$ for $j \neq k$: this is the phase space for the simultaneous description of n particles ($\dim E_j = 2\nu'$ for any j; $n\nu' = \nu$).

Let $\{e_1^j,\dots,e_{2\nu'}^j\}$ be a symplectic base in each E_j. For any $\pi \in P(1,\dots,n)$[$P(1,\dots,n)$ is the set of all permutations of $(1,\dots,n)$], we define

$$P_\pi: E \to E,$$

$$\sum_{j=1}^{n}\sum_{k=1}^{2\nu'} v_k^j e_k^j \to \sum_{j=1}^{n}\sum_{k=1}^{2\nu'} v_k^{\pi j} e_k^j;$$

$$Q_\pi: \mathscr{L}_0 \mapsto \mathscr{L}_0,$$

$$\phi(v) \to \phi(P_\pi(v)).$$

Clearly,

$$Q_\pi(a,b) = \Omega^b(P_\pi(a)).$$

To compute the classical function corresponding to Q_π, we split up π into a product of independent cyclic permutation operators. The classical function splits up in a product of independent functions, corresponding to these cyclic permutations. For the cyclic permutation $\pi = (1,\dots,m)$ (this permutation maps 1 to 2, 2 to 3,..., $m-1$ to m, m to 1), we get

$$f_\pi(v_1,\dots,v_m) = 2^{\nu'(m-1)}\prod_{j=1}^{(1/2)(m-1)} \times \exp[i\varphi(V_{2j-1}, P_\pi^{-1}(v_{2j}), P_\pi^{-2}(v_{2j+1}))] \quad \text{for } m \text{ odd},$$

and

$$f_\pi(v_1,\dots,v_m) = 2^{\nu'(m-1)}\prod_{j=1}^{(m/2)-1} \times \exp[i\varphi(V_{2j-1}, P_\pi^{-1}(v_{2j}), P_\pi^{-2}(v_{2j+1}))]\delta(V_m),$$

for m even, with

$$V_k = \sum_{j=1}^{k} (-1)^{j-k} (P_\pi)^{j-k} (v_j).$$

In particular, if we describe two particles, and we want to consider the operator Q_π for $\pi 1 = 2$, $\pi 2 = 1$, we have

$$f_{(12)}(v_1,v_2) = \delta(v_1 - P_\pi(v_2)).$$

For three particles, we see that

$$f_{(123)} = 2^{2\nu} \exp[i\varphi(v_1, P_\pi^{-1}(v_2), P_\pi^{-2}(v_3))]$$

and

$$f_{(12)(3)} = \delta(v_1 - P_\pi(v_2)).$$

These different expressions can be considered as special cases of the classical functions corresponding to general symplectic transformations computed in Refs. 28 and 25.

6. A COMPARISON WITH BARGMANN'S INTEGRAL TRANSFORM REF. 17

In Ref. 17 some explicit expressions are given for the unitary operator intertwining the Schrödinger representation with the coherent state representation of the Weyl commutation relations. We rewrite here this result in our notations.

Identify E with $\mathbb{R}^{2\nu} = \mathbb{R}^\nu \oplus \mathbb{R}^\nu = x$ space $\oplus\, p$ space. Let us denote the x space by E_1. In what is usually called the Schrödinger representation the Hilbert space used is $L^2(E_1)$, i.e., the space of square integrable functions on E_1 with respect to a Lebesgue measure on E_1.

Bargmann's integral transform is a unitary map A from $L^2(E_1)$ to $\mathscr{L}_0(E, J)$ which can be represented by a kernel:

$$A : L^2(E_1) \to \mathscr{L}_0(E,J),$$

$$\forall \psi \in L^2(E_1) : (A\psi)(v) = \int dx\, A(v,x)\, \psi(x). \tag{6.1}$$

The kernel $A(v, x)$ has many interesting properties. For fixed x, it is an element of Z, and for fixed v it is square integrable on E_1. This is analogous with our kernel $\{\zeta | v\}$ which for fixed ζ is square integrable on E, and for fixed v an element of $Z\{(E \oplus E; (J, -J)\}$. Moreover, we know (see Sec. 4.E) that our integral transform Q is unitary from $L^2(E)$ on $\mathscr{L}_0(E \oplus E, (J, -J))$. So it would seem that our integral transform is just a double Bargmann transform:

$$A : L^2(E_1) \to \mathscr{L}_0(E, J),$$

$$Q: L^2(E) \simeq L^2(E_1) \otimes L^{2*}(E_1) \to \mathscr{L}_0(E, J) \otimes \mathscr{L}_0(E, J)^* \simeq \mathscr{L}_0(E \oplus E, (J, -J)).$$

We donote here by $\mathscr{H}^*$ the dual of $\mathscr{H}$; the isomorphy $\mathscr{L}_0(E, J) \otimes \mathscr{L}_0(E,J)^* \simeq \mathscr{L}_0(E \oplus E, (J, -J))$ follows from the fact that $\mathscr{L}_0(E \oplus E, (J, -J))$ is isomorphic to the Hilbert space of Hilbert–Schmidt operators on $\mathscr{L}_0(E, J)$ (see Sec. 4.E). It is however not altogether true that Q is just twice A. Indeed, on has

$$A(v,x) = \sum_{[m]} u_{[m]}(v) \phi_{[m]}(x), \tag{6.2}$$

where $u_{[m]} = h^{[m]} \Omega$ and $\phi_{[m]}$ are the eigenfunctions of the harmonic oscillator, respectively, in $\mathscr{L}_0(E, J)$ and $L^2(E_1)$[17]; on the other hand [see Eq. (3.29)],

$$\{a,b | (x_v, p_v)\} = \sum_{[m][n]} u_{[m]}(a)\, \overline{u_{[n]}(b)}\, h_{[m][n]}(x_v, p_v), \tag{6.3}$$

where $h_{[m][n]}(v)$ is given by Eq. (3.29) and is definitely different from $\phi_{[m]}(2^{1/2} x_v) \phi_{[n]}(2^{1/2} p_v)$ (the factor $\surd 2$ has to be introduced because of a difference in normalization in the measures on E_1 and E). This can readily be checked in an example. Take $\nu = 1$, $m = n = 1$. Then

$$\phi_1(2^{1/2} x_v) (\overline{\phi_1(\sqrt{2} p_v)} = x_v p_v e^{-(x_v^2 + p_v^2)}$$

and

$$h_{11}(x_v, p_v) = 2\, e^{-(x_v^2 + p_v^2)(-x_v^2 - p_v^2 + 1)}$$
$$\neq \phi_1(\sqrt{2} x_v) \overline{\phi_1(\sqrt{2} p_v)}.$$

Another way of seeing that the integral transform Q is not merely a double Bargmann transform is to look at the explicit expressions for the kernels $A(v, x)$ and $\{\zeta | v\}$. We have[17]

$$A(v,x) = \Pi^{-\nu/4} e^{(i/2) x_v p_v} e^{-i p_v x} e^{-(1/2)(x - x_v)^2}$$
$$= \Omega^{(x_v, p_v)}(x), \tag{6.4}$$

where $\Omega^{(x_v, p_v)}(x)$ is the coherent state centered round (x_v, p_v), written in the x representation, while a direct calculation from Eq. (3.9) gives

$$\{a,b | (x_v, p_v)\} = \Omega^{(x_a + x_b)/\sqrt{2}, (p_b - p_a)/\sqrt{2}} (\sqrt{2} x_v)$$
$$\times \overline{\Omega^{(p_a + p_b)/\sqrt{2}, (x_b - x_a)/\sqrt{2}} (\sqrt{2} p_v)}, \tag{6.5}$$

which is again very different from the expected

$$A(a, \sqrt{2} x_v) \overline{A(b, \sqrt{2} p_v)}$$
$$= \Omega^{(x_a, p_a)}(\sqrt{2} x_v) \overline{\Omega^{(x_b, p_b)}(\sqrt{2} p_v)}. \tag{6.6}$$

In a certain sense these differences are due to the fact that the integral transform Q has to do with quantization, while A is just a unitary map from one quantum mechanical realization to another. Indeed, if we look at Eqs. (6.2) and (6.3), we see that on the $\mathscr{L}_0$ side everything is all right: Eq. (6.3) contains one straight copy $u_{[m]}(a)$ and one complex conjugate copy $\overline{u_{[n]}(b)}$ of the $\mathscr{L}_0$ function $u_{[m]}(v)$ in Eq. (6.2); but things go wrong with the L^2 function. This is precisely because $L^2(E)$, the initial space of Q, has to be considered as a space of classical functions, while the initial space of A is a quantum Hilbert space.

Another way of seeing this is the following: By taking a double Bargmann transform one treats x_v and p_v as two equivalent but independent ("commuting") variables: in Eq. (6.6) x_v is only linked with x_a, p_a, and p_v only with x_b, p_b. However, this is not what happens in a quantization procedure; there x_v and p_v are linked with the x_a, p_a as well as with x_b, p_b: some mixing has taken place.

ACKNOWLEDGMENT

One of the authors (I.D.) would like to thank the CPT, CNRS Marseille for hospitality.

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