AIP | Journal of Mathematical Physics

Continuity statements and counterintuitive examples in connection with Weyl quantization

Ingrid Daubechies

Citation: J. Math. Phys. **24**, 1453 (1983); doi: 10.1063/1.525882
View online: http://dx.doi.org/10.1063/1.525882
View Table of Contents: http://jmp.aip.org/resource/1/JMAPAQ/v24/i6
Published by the American Institute of Physics.

Continuity statements and counterintuitive examples in connection with Weyl quantization

Ingrid Daubechies[a)]
Physics Department, Princeton University, Princeton, New Jersey 08544

(Received 11 March 1982; accepted for publication 27 August 1982)

We use the properties of an integral transform relating a classical function f with the matrix elements between coherent states of its quantal counterpart Qf, to derive continuity properties of the Weyl transform from classes of distributions to classes of quadratic forms. We also give examples of pathological behavior of the Weyl transform with respect to other topologies (e.g., bounded functions leading to unbounded operators).

PACS numbers: 03.65.Ca, 02.30. + g

I. INTRODUCTION

The Weyl correspondence or Weyl transform defines a map from the functions on the classical phase space to the operators on the quantum mechanical Hilbert space.[1]

The classical phase space is here a 2ν-dimensional real vector space E, equipped with a nondegenerate symplectic form σ. It is customary to consider E as the direct sum of position and momentum spaces:

$$E \ni v = (q, p), \quad q, p \in \mathbb{R}^\nu ,$$

with σ of the form

$$\sigma((q,p),(q',p')) = \tfrac{1}{2}(p\cdot q' - q\cdot p') .$$

The Hilbert space $\mathscr{H}$ is a complex Hilbert space carrying an irreducible representation of the canonical commutation relations. Explicitly, $\exists\ W(v)$, unitary operators on $\mathscr{H}$, labeled by the points v of E, such that

$$\operatorname*{s-lim}_{v\to 0} W(v) = \mathbb{1}_{\mathscr{H}} ,$$

$$W(v_1)W(v_2) = e^{i\sigma(v_1,v_2)}W(v_1+v_2) .$$

One usually writes the $W(v)$ as

$$W(p,q) = \exp[i(p\cdot Q - q\cdot P)] = \exp(-\tfrac{1}{2}ipq)\exp(ip\cdot Q)\exp(-iq\cdot P) ,$$

where Q_j (P_j) are the generators of the ν-parameter group $W(0,p)$ $(W(-q,0))$; Q_j and P_j are called the position and momentum operators, respectively, and satisfy the usual commutation relations

$$[Q_j, P_k] = i\delta_{jk}\mathbb{1}$$

on a common core.

The Weyl transform of a function f on E is then an operator Qf defined by (formally)

$$Qf = 2^{-\nu}\int_E dv\ \tilde{f}(v)W(-v/2) , \tag{1}$$

where $\tilde{f}$ is the symplectic Fourier transform of f:

$$\tilde{f}(v) = 2^{-\nu}\int dv'\ e^{i\sigma(v,v')}f(v') .$$

One can check[1] that Eq. (1) defines an unambiguous extension ("symmetric ordering") of the usual correspondence

$$f_j(q,p) = q_j \Rightarrow Qf_j = Q_j ,$$

$$g_j(q,p) = p_j \Rightarrow Q\,g_j = P_j .$$

To give a precise sense to Eq. (1), one has to specify in what sense the integral converges. It can easily be shown that this integral is well defined in the usual weak sense for $\tilde{f}\in L^1 + L^2$; however, one can give a meaning to Eq. (1) for much larger classes of f (see below).

It is possible to write the map $f \mapsto Qf$ without involving Fourier transforms[2]

$$Qf = 2^\nu \int dv\, f(v)W(2v)\Pi . \tag{2}$$

Here Π is the parity operator, i.e., an involutive, unitary operator satisfying

$$\Pi W(v) = W(-v)\Pi ;$$

the operators $\Pi(v) = W(2v)\Pi$ used in Eq. (2) are called Wigner operators.[3] Like the Weyl operators, they satisfy specific multiplication rules; moreover, they are self-adjoint.

The inverse map, from the operators on $\mathscr{H}$ to the functions on E, is the Wigner transform.[4] Formally this transform can be written as

$$f(v) = 2^\nu\,\mathrm{Tr}[Qf\Pi(v)] . \tag{3}$$

One sees immediately that Eq. (3) is well defined for Qf trace-class; again, we shall see below how this correspondence can be extended to more general classes of operators.

The Weyl correspondence and its inverse have been used in several different contexts. One obvious field of applications has been the study of classical limits.[5,6] In Ref. 7 the Weyl correspondence and its properties are used to find the equations of motion for a particle with spin $\frac{1}{2}$ in an electromagnetic field in a semirelativistic approximation, in particular to derive the correct magnetodynamic effect. An approach of quantum mechanics related to the Weyl transform can be found in Ref. 8, where quantum effects are studied using only functions on phase space (no Hilbert space picture), with a noncommutative product, which is usually called the twisted product, and which is the transposition, through the Weyl correspondence, of the noncommutative operator product. See also Ref. 3 for several beautiful applications, and discussions of quantum phenomena by means of the Wigner transform.

We shall be concerned here with the "topological" properties of the Weyl correspondence. We have many topolo-

[a)] On leave from Dienst voor Theoretische Natuurkunde, Vrije Universiteit Brussel, Belgium, and from Interuniversitair Instituut voor Kernwetenschappen, Belgium.

 0022-2488/83/061453-09$02.50

gies at our disposal, both on the functions on the classical phase space, and on the operators on the Hilbert space; it is a natural question to ask for the continuity properties of the Weyl correspondence with respect to these topologies. Some answers to this question were given in Ref. 9, showing that the Weyl correspondence maps L^2 unitarily onto τ_2, the space of Hilbert–Schmidt operators, and in Ref. 10, where it was proved that all Schwartz functions yield trace-class operators, and all L^1 functions compact operators. There also exists an extensive literature on the properties of the Weyl transform and its inverse when attention is restricted to the pseudodifferential operators (see Refs. 5, 11, 12, and the references quoted therein).

It is our purpose here to prove some new continuity statements ("positive results") and show the existence of counterexamples illustrating the breakdown of continuity if other topologies are chosen ("negative results"). To derive these results, we use extensively the properties of the harmonic oscillator coherent states[13,14] and of an integral transform[15] relating the function f with the matrix elements of Qf between these coherent states. Basically this integral transform maps functions to *analytic* functions; it is well known that such integral transforms have very special properties, which we shall exploit in our proofs, using ideas going back to Refs. 16, 17, and 18. A first application of our integral transform can be found in Ref. 19, where we exhibit a larger class of functions than $\mathscr{S}$, yielding trace-class operators, and, by duality, put some restrictions on the distributions corresponding to bounded operators. The mathematical properties of this integral transform were studied in some more detail in Ref. 20; using the results obtained there, we shall see that we can sharpen the results of Ref. 19, and derive some new ones. These results constitute the first part of this article.

While the first part contains mostly continuity results, which can be considered to be "positive" results, the second part contains essentially "negative results," i.e., counterexamples and no-go theorems showing which kind of continuity cannot be expected. For instance,

$$\exists\ f\in L^\infty \text{ such that } Qf \text{ is unbounded,}$$

$$\exists\ A \text{ trace-class for which } Q^{-1}A \notin L^1(E).$$

II. POSITIVE RESULTS: CONTINUITY STATEMENTS

We shall derive here some continuity properties of the Weyl transform and its inverse, using an integral transform introduced in Ref. 15, connecting the function f with the matrix element of Qf between coherent states. We therefore start by giving a short review of the definition and properties of the coherent states and of the integral transform in question; for more details the reader is referred to Refs. 15 and 20.

A. The coherent states Ω^a; the integral transform $f\mapsto(\Omega^a, Q f\Omega^b)$

Let Ω be the ground state of the harmonic oscillator P^2+Q^2 [alternatively, one can define Ω as the vector for which $(P_j - iQ_j)\Omega = 0\ \ \forall\ j$]. We define then

$$\forall\ a\in E:\quad \Omega^a = W(a)\Omega .$$

It is well known[13] that the following resolution of the identity holds:

$$\int_E da|\Omega^a)(\Omega^a| = \mathbb{1}_{\mathscr{H}} , \tag{4}$$

with $da = [1/(2\pi)^\nu]\, d^\nu x_a d^\nu p_a$, and where the integral converges in the weak sense. Inserting Eq. (4) twice, one sees that for every (bounded) operator A

$$A = \int_E da \int_E db\, |\Omega^a)(\Omega^a, A\Omega^b)(\Omega^b| , \tag{5}$$

which means that every (bounded) operator can be reconstructed from its matrix elements between coherent states [alternatively, one can say that in the Bargmann representation of the canonical commutation relations every (bounded) operator is given by an integral kernel]. Actually, the reconstruction of A from its coherent state matrix elements $(\Omega^a, A\Omega^b)$ works for much larger classes than only the bounded operators (it works, e.g., for all closed operators for which the span of the coherent states is a core).

Applying Eq. (5) to Eq. (2), we see that (formally)

$$Qf = \int_E da \int_E db\, |\Omega^a) \int_E dv\, f(v) 2^\nu (\Omega^a, W(2v)\Pi\Omega^b)(\Omega^b| . \tag{6}$$

In Ref. 15 we used the notations

$$\begin{aligned}\{a,b\,|v\} &= 2^\nu(\Omega^a, W(2v)\Pi\Omega^b)\\ &= 2^\nu \exp[\tfrac{1}{2} i(p_a x_b - p_b x_a + 2p_v x_a\\ &\quad - 2p_a x_v + 2p_b x_b - 2p_v x_b)\\ &\quad - \tfrac{1}{4}(2x_v - x_b - x_a)^2 - \tfrac{1}{4}(2p_v - p_b - p_a)^2]\end{aligned}$$

and introduced the integral transform

$$(If)(a,b) = \int dv\, f(v)\{a,b\,|v\} . \tag{7}$$

What Eq. (6) is telling us then is that the integral transform I can be used as a tool to study the Weyl correspondence $f\leftrightarrow Qf$. This integral transform was studied in some detail in Ref. 20. We review here some of its properties. Since, for fixed a,b, the function $\{a,b\,|\cdot\}$ is C^∞, with Gaussian decrease, the integral transform I can be defined for all tempered distributions, and also for some classes of nontempered distributions. The images If have quite remarkable analyticity properties:

$$\begin{aligned}If(a,b) &= \exp[-\tfrac{1}{4}(x_a^2 + p_a^2 + x_b^2 + p_b^2)]\\ &\quad\times F(p_a + ix_a, p_b - ix_b) ,\end{aligned} \tag{8}$$

where F is an analytic function (depending on f, of course) on $\mathbb{C}^\nu\times\mathbb{C}^\nu$. The set of all functions which can be written in such a form we denote by $Z(E_2)$.

Two special sets of elements of $Z(E_2)$ are given by

$$\begin{aligned}u_{[m,n]}(a,b) &= \exp[-\tfrac{1}{4}(x_a^2 + p_a^2 + x_b^2 + p_b^2)]\\ &\quad\times\frac{1}{([m!][n!])^{1/2}}\left(\frac{p_a + ix_a}{\sqrt{2}}\right)^{[m]}\left(\frac{p_b - ix_b}{\sqrt{2}}\right)^{[n]},\\ \omega^{(c,d)}(a,b) &= \exp\{\tfrac{1}{2} i(p_c x_a - p_a x_c + p_b x_d - p_d x_b)\\ &\quad + \tfrac{1}{4}[(x_a - x_c)^2 + (p_a - p_c)^2 + (x_b - x_d)^2\\ &\quad + (p_b - p_d)^2]\} .\end{aligned} \tag{9}$$

For every element ϕ of $Z(E_2)$, one can write a Taylor series for its analytic part, which can be considered as an expansion of ϕ with respect to the $u_{[m,n]}$:

$$\phi \in Z(E_2) \Rightarrow \phi(a,b) = \sum_{[m],[n]} \phi_{[m,n]} u_{[m,n]}(a,b) \qquad (10)$$

with uniform and absolute convergence on compact sets. One can, moreover, show[20] that

$$\phi_{[m,n]} = \int da \int db\, \phi(a,b)\, \overline{u_{[m,n]}(a,b)} = \int da \int db\, \phi(a,b) u_{[n,m]}(a,b) \qquad (11)$$

for all ϕ in $Z(E_2)$ such that the integral on the right-hand side converges absolutely (this is the case, e.g., for the elements of the $\mathscr{F}^{\rho}$-spaces defined below).

Analogously one shows that the following reproducing property holds[20]:

$$\phi(c,d) = \iint da db\, \phi(a,b)\, \overline{\omega^{(c,d)}(a,b)} = \iint da db\, \phi(a,b) \omega^{(a,b)}(c,d) \qquad (12)$$

for all ϕ in $Z(E_2)$ for which the integrals converge absolutely. The $u_{[m,n]}$ and $\omega^{(c,d)}$ are related to the coherent states Ω^a in the following way:

$$(\Omega^a, \Omega^c)(\Omega^d, \Omega^b) = \omega^{(c,d)}(a,b),$$
$$\langle \Omega^a | [m] \rangle \langle [n], \Omega^b \rangle = u_{[m,n]}(a,b). \qquad (13)$$

In Ref. 20 we defined, $\forall\ \rho \in \mathbb{R}$, the spaces $\mathscr{F}^{\rho}$ and W^{ρ} as

$$\mathscr{F}^{\rho} = \left\{ \phi \in Z(E_2);\ \|\phi\|_{\rho}^2 = \iint da db\, (1 + |a|^2 + |b|^2)^{\rho} |\phi(a,b)|^2 < \infty \right\}$$

[where $|a|^2 = \frac{1}{2}(x_a^2 + p_a^2)$] (see also Ref. 14 for the definition of $\mathscr{F}^{\rho}$),

$W^{\rho} =$ closure under $\|\ \|_{\rho}^{s}$ of

$$\{ f \in \mathscr{S}(E);\ (\|f\|_{\rho}^{s})^2 = (f, (q^2 - \tfrac{1}{4}\Delta_q + p^2 - \tfrac{1}{4}\Delta_p + \nu)^{\rho} f) < \infty \}$$

(the W^{ρ} are Sobolev-type space with respect to the operator $x^2 + p^2 - \frac{1}{4}\Delta_x - \frac{1}{4}\Delta_p$; they are the same spaces as used in the N representation of $\mathscr{S}$ and $\mathscr{S}'$[21]). One can then show that, $\forall\ \rho \in \mathbb{R}$, I defines an isomorphism from W^{ρ} to $\mathscr{F}^{\rho}$.

We shall now proceed to derive some properties of the Weyl correspondence from these properties of the integral transform I.

B. The Weyl transform as a map from the (tempered) distributions to quadratic forms on the span of the coherent states

For notational convenience, we define

$D_{\rm coh}$ = linear span of the coherent states Ω^a.

$D_{\rm coh}$ is obviously dense in $\mathscr{H}$. For any quadratic form ϕ on $D_{\rm coh}$, we shall use the notation:

$$K_{\phi}(a,b) = \phi(\Omega^a, \Omega^b).$$

In what follows we shall only consider quadratic forms ϕ for which the associated function K_{ϕ} is an element of $Z(E_2)$. In doing so, we do not put too severe a restriction on ϕ: For instance, all the quadratic forms associated with Schrödinger operators $p^2 + V(x)$, with V a tempered distribution, fall into this class (this includes, e.g., the Coulomb potential as soon as $\nu \geqslant 2$). The $\mathscr{F}^{\rho}$-topologies can then be used to define topologies on the quadratic forms on $D_{\rm coh}$:

$$G^{\rho} = \{\phi \text{ quadratic form on } D_{\rm coh};\ K_{\phi} \in \mathscr{F}^{\rho}\}$$

Equipped with the corresponding norm, $\|\phi\|_{\rho} = \|K_{\phi}\|_{\rho}$, the G^{ρ} constitute then a nested Hilbert space[22] of quadratic forms. One can now define the Weyl transform for all of $\mathscr{S}'$ by means of the integral transform I:

$$\forall\ T \in \mathscr{S}'(E),\quad QT \text{ is a quadratic form on } D_{\rm coh} \text{ defined by } K_{QT} = IT. \qquad (14)$$

It is easy to check that this definition of QT coincides with the direct definition by Eqs. (1) or (2) for $f \in L^1$ or $f \in L^2$, i.e., that

$$\int dv\, f(v)\{a,b\,|v\} = (\Omega^a, Qf\Omega^b)$$

for f in these classes, which justifies our definition (14) as an extension. This can be verified also for the pseudodifferential operators. The fact that I is an isomorphism from W_{ρ} to $\mathscr{F}_{\rho}$ now easily translates to Q:

$$\forall\ \rho \in \mathbb{R},\quad Q:\ W^{\rho} \rightarrow G^{\rho}$$

is an isomorphism; one has[20]

$$\|QT\|_{\rho} \leqslant K_{\rho} \|T\|_{\rho}^{s}$$

and (15)

$$\|Q^{-1}\phi\|_{\rho}^{s} \leqslant K'_{\rho} \|\phi\|_{\rho}$$

with

$$K_{\rho} = e^{-\rho/2} \cdot \begin{cases} 1, & \rho \leqslant 0, \\ \sqrt{e}(1 + \rho/2\nu)^{\nu + \rho/2}, & \rho \geqslant 0, \end{cases}$$

$$K'_{\rho} = \begin{cases} (1 + \rho/2\nu)^{1/2}, & \rho \geqslant 0, \\ \sqrt{e}e^{-|\rho|/2}(1 + |\rho|/2\nu)^{\nu + \rho/2}, & \rho \leqslant 0. \end{cases}$$

Remarks

1. The quadratic forms thus obtained need not be closable! A striking example of a nonclosable form is given by $T(q,p) = \delta(q)$. While both $T_1(q,p) = 1$ and $T_2(q,p) = \delta(p)\delta(q)$ lead to nice, in this case even bounded operators, the quadratic form QT [as defined by Eq. (14)] is not closable, and hence not associated with an operator.

2. If however a quadratic form ϕ in G^{ρ} *is* closable, then all the eigenstates $|[m]\rangle$ of $\frac{1}{2}(P^2 + Q^2)$ are in the form domain of the closure $\bar{\phi}$ of ϕ, and

$$\bar{\phi}(|[m]\rangle, |[n]\rangle) = (K_{\phi})_{[n,m]}, \qquad (16)$$

where the $(K_{\phi})_{[m,n]}$ are the coefficients in a Taylor expansion for K_{ϕ} [see Eq. 10)]:

$$K_{\phi}(a,b) = \sum_{[m],[n]} (K_{\phi})_{[m,n]} u_{[m,n]}(a,b) \qquad (17)$$

[Eq. (16) can easily be proved from the fact that suitable differentiations of the Ω^a yield the eigenvectors $|[m]\rangle$, in the same way as the $(K_\phi)_{[m,n]}$ can be obtained from K_ϕ by differentiating].

3. Actually one can give a sense to $\phi(|[m]\rangle, |[n]\rangle)$ for all ϕ in G^ρ, even if ϕ is not closable, in the following way. For $\mu \geqslant 0$, we define

$$D_\mu = \left\{\psi \in \mathscr{H};\ |\psi|^2_\mu = \int da\ |(\psi, \Omega^a)|^2 (1 + |a|^2)^\mu < \infty\right\}$$
$$= \left\{\psi \in \mathscr{H}; |\psi|^2_\mu = \sum_{[m]} |\langle\psi|[m]\rangle|^2 (|m| + \nu)^\mu < \infty\right\}. \quad (18)$$

For $\mu < 0$ we define the norms $|\ |_\mu$, $|\ |_\mu$ in exactly the same way; in this case D_μ is defined to be the closure of $\mathscr{H}$ with respect to either $|\ |_\mu$ or $|\ |_\mu$ (these two norms are equivalent for all μ). From the definitions of the D_μ, one immediately sees that, for any μ, D_μ is the dual space of $D_{-\mu}$ with respect to a suitable extension of the inner product on $\mathscr{H}$:

$$(\psi_1, \psi_2) = \int da\ (\psi_1, \Omega^a)(\Omega^a, \psi_2).$$

On the other hand, one can show that for any $\psi \in G^\rho$

$$\forall\ \psi_1, \psi_2 \in D_{\rm coh},$$
$$\phi(\psi_1, \psi_2) = \iint dadb\ K_\phi(a,b)(\psi_1, \Omega^a)(\Omega^b, \psi_2) \quad (19)$$

[one has only to show this, for $\psi_1 = \Omega^c$, $\psi_2 = \Omega^d$; the statement then follows from Eq. (12)].

Since $(1 + |a|^2)^{1/2}(1 + |b|^2)^{1/2} \leqslant 1 + |a|^2 + |b|^2 \leqslant (1 + |a|^2)(1 + |b|^2)$, it is then obvious from our definitions and from the duality of D_μ and $D_{-\mu}$ that every ϕ in G^ρ can be extended, using Eq. (19), to a continuous map from $D_{-\rho}$ to D_ρ if $\rho \leqslant 0$, from $D_{-\rho/2}$ to $D_{\rho/2}$ if $\rho \geqslant 0$. Alternatively, one can also say that in this way we have extended $\phi \in G^\rho$ to a quadratic form on $D_{-\rho}$ if $\rho \leqslant 0$ and on $D_{-\rho/2}$ if $\rho \geqslant 0$. In particular, since the $|[m]\rangle$ are elements of all the D_μ, this means we have given a sense to $\phi(|[m]\rangle, |[n]\rangle)$, which is again given by $(K_\phi)_{[n,m]}$:

$$\phi(|[m]\rangle, |[n]\rangle) = \iint dadb\ K_\phi(a,b)\langle[m]|\Omega^a\rangle\langle\Omega^b|[n]\rangle$$
$$= \iint dadb\ K_\phi(a,b) u_{[m,n]}(a,b) \quad \text{[see Eq. 13)]}$$
$$= (K_\phi)_{[n,m]} \quad \text{[see Eq. (11)]}.$$

4. From Eq. (18) one easily sees that D_μ is exactly the domain of $(P^2 + Q^2)^{\mu/2}$, or equivalently, the form domain of $(P^2 + Q^2)^\mu$.

Take $\rho \geqslant 0$. Then, by the extension defined above, $\phi \in G^{-\rho}$ defines a continuous, more precisely a Hilbert–Schmidt map from D_ρ to $D_{-\rho}$. This means that ϕ is a quadratic form, relatively form-compact with respect to $(P^2 + Q^2)^\rho$. (See Ref. 23 for the definition of relatively form-compactness).

5. Translating all this to the Weyl transform, we see now that $\forall\ \rho \geqslant 0$, $\forall\ T \in W^{-\rho}$: QT is a quadratic form, relatively form-compact with respect to $(P^2 + Q^2)^\rho$.

6. In Ref. 20 the action of I on some classes of distributions "of type S," which contain also nontempered distributions, was studied. One can perform the same constructions and extensions as above for these larger classes; as a result one gets that Q, when applied to a $H(\overline{\alpha}, \overline{A})^{20,24}$ yields quadratic forms relatively compact with respect to

$$\exp[\tau(P^2 + Q^2)^{1/2\alpha}], \quad \text{for } \alpha > \tfrac{1}{2},\ \forall\ \tau > 2A^{-1/\alpha}.$$

C. The distributions corresponding to bounded operators

Let A be a bounded operator. Equation (14) provides us with a simple rule to find the function or distribution $Q^{-1}A$, via the coherent states. We define $K_A(a,b) = (\Omega^a, A\Omega^b)$. One can easily check, from the properties of the coherent states, that K_A is an element of $Z(E_2)$. Moreover [use Eq. (4)],

$$\int_E da\ |K_A(a,b)|^2 = \|A\Omega^b\|^2 \leqslant \|A\|^2;$$

hence, $\forall\ \epsilon > 0$,

$$\int da \int db\ (1 + |a|^2 + |b|^2)^{-\nu-\epsilon}|K_A(a,b)|^2$$
$$\leqslant \int db\ (1 + |b|^2)^{-\nu-\epsilon}\|A\|^2$$
$$= \|A\|^2 \frac{2}{\Gamma(\nu)} \int_0^\infty dx\ \frac{x^{2\nu-1}}{(1 + x^2)^{\nu+\epsilon}}$$
$$= \|A\|^2 \Gamma(\epsilon)\Gamma(\nu + \epsilon)^{-1}.$$

This implies that A, considered as a quadratic form on $D_{\rm coh}$, is an element of $\cap_{\epsilon>0} G^{-(\nu+\epsilon)}$, with

$$\forall\ \epsilon > 0, \quad \|A\|_{-(\nu+\epsilon)} \leqslant [\Gamma(\epsilon)/\Gamma(\nu + \epsilon)]^{1/2}\|A\|.$$

As a consequence of this, we can formulate the following restriction on the distributions corresponding to bounded operators: The Weyl transform QT of a distribution $T \in \mathscr{S}'(E)$ can be a bounded operator only if

$$T \in \bigcap_{\epsilon>0} W^{-(\nu+\epsilon)}$$

and $\exists\ K \geqslant 0$ such that

$$\forall\ \epsilon > 0, \quad \|T\|^s_{-(\nu+\epsilon)}$$
$$\leqslant e^{-(\nu+\epsilon-1)/2}[(3 + \epsilon/\nu)/2]^{(3\nu+\epsilon)/2}|\Gamma(\epsilon)/\Gamma(\nu + \epsilon)|^{1/2}K. \quad (20)$$

The "only if" in this statement cannot be replaced by an "if and only if." This is again illustrated by the distribution $T(q,p) = \delta(q)$: it turns out that T is an element of $\cap_{\epsilon>0} W^{-(\nu+\epsilon)}$ and satisfies Eq. (20), though QT is not even closable, and certainly not bounded.

The topology induced on $B(\mathscr{H})$ by the $\|\ \|_{-(\nu+\epsilon)}$-norms is much weaker than the norm topology. Actually, it is even weaker than the strong topology: if $A_n \to_{n\to\infty} A$ strongly, then the $\|A_n\|$ are uniformly bounded (by the principle of uniform boundedness, e.g., Ref. 21).

Hence

$$\int da\ (1 + |a|^2 + |b|^2)^{-\nu-\epsilon}|K_A(a,b) - K_{A_n}(a,b)|^2$$
$$\leqslant (1 + |b|^2)^{-\nu-\epsilon}\|A\Omega^b - A_n\Omega^b\|^2$$
$$\leqslant (1 + |b|^2)^{-\nu-\epsilon}\sup(4\|A_n\|^2),$$

which is integrable in b. Together with

$\|A\Omega^b - A_n\Omega^b\| \to_{n\to\infty} 0$ this implies $\|A - A_n\|_{-(\nu+\epsilon)} \to_{n\to\infty} 0\ \forall\ \epsilon > 0$; hence $Q^{-1}A_n \to_{n\to\infty} Q^{-1}A$ in each $W^{-(\nu+\epsilon)}$, $\epsilon > 0$.

Remarks

1. Note that this argument can also be used for unbounded operators: Whenever A_n and A are unbounded operators such that

$$(\Omega^a, A_n\Omega^b) \underset{n\to\infty}{\to} (\Omega^a, A\Omega^b) \quad \forall\ a,b$$

and $\exists\ K, l$ for which $|(\Omega^a, A_n\Omega^b)|^2 \leqslant K(1 + |a|^2 + |b|^2)^l$ uniformly in n then $Q^{-1}A_n \to_{n\to\infty} Q^{-1}A$ in $\mathscr{S}'(E)$ (i.e., "in the sense of the distributions").

2. Actually, the bound Eq. (20) can still be sharpened a little bit. Defining, for $\rho \in \mathbb{R}$, the norm $|\ |_\rho^s$ (equivalent to $\|\ \|_\rho^s$) by

$$|T|_\rho^s = (T, (q^2 - \tfrac{1}{4}\Delta_q + p^2 - \tfrac{1}{4}\Delta_p)^\rho T)^{1/2} \tag{21}$$

(we omit the extra ν in the definition of $\|\ \|_\rho^s$, which makes this norm larger than $\|\ \|_\rho^s$, for $\rho \leqslant 0$), one can show[25] that (20) can be replaced by

QT bounded only if $\exists\ K \geqslant 0$ such that

$$|T|_{-(\nu+\epsilon)}^s \leqslant \Gamma(\nu)^{-1/2}\zeta(1+\epsilon)^{1/2}K,$$

where ζ is Riemann's zeta function: $\zeta(x) = \Sigma_{k=1}^\infty k^{-x}$.

D. A class of functions yielding trace-class operators

The restriction (20) on the class of distributions corresponding to bounded operators is sharper than the one derived in Ref. 19. We can consequently also find a larger class of functions leading to trace-class operators.

To do this, we shall follow essentially the same method as in Ref. 19: we need as a preliminary lemma that

$$\forall\ B \text{ trace-class,} \quad \operatorname{Tr} B = \int da\, (\Omega^a, B\Omega^a). \tag{22}$$

(This is fairly easy to prove; see, e.g., Ref. 19. Note that the absolute convergence of the integral on the right-hand side for a given operator B does not imply that B is trace-class).

Take now $f \in \mathscr{S}(E)$. We know[10] that Qf is trace-class. Moreover, the trace-norm of Qf is given by

$$\|Qf\|_{\text{tr}} = \operatorname{Tr}|Qf| = \sup_{A\in B(\mathscr{H}),\, A\neq 0} \|A\|^{-1}\cdot|\operatorname{Tr}(A\cdot Qf)|.$$

But, by Eqs. (22) and (4),

$$\begin{aligned}
|\operatorname{Tr}(A\cdot Qf)| &= \left|\int da\, (\Omega^a, AQf\Omega^a)\right| \\
&= \left|\int da \int db\, (\Omega^a, A\Omega^b)(\Omega^b, Qf\Omega^a)\right| \\
&\leqslant \left(\iint dadb\, (1 + |a|^2 + |b|^2)^{-\nu-\epsilon}|(\Omega^a, A\Omega^b)|^2\right)^{1/2} \\
&\quad\times\left(\iint dadb\, (1 + |a|^2 + |b|^2)^{\nu+\epsilon}|(\Omega^b, Qf\Omega^a)|^2\right)^{1/2} \\
&\leqslant |\Gamma(\epsilon)/\Gamma(\nu+\epsilon)|^{1/2}\|A\|\ \|If\|_{\nu+\epsilon} \\
&\leqslant |\Gamma(\epsilon)/\Gamma(\nu+\epsilon)|^{1/2}e^{-(\nu+\epsilon-1)/2}[(3+\epsilon/\nu)/2]^{(3\nu+\epsilon)/2}\|A\|\ \|f\|_{\nu+\epsilon}^s,
\end{aligned}$$

which implies

$$\begin{aligned}
\|Qf\|_{\text{tr}} &\leqslant e^{-(\nu+\epsilon-1)/2}[(3+\epsilon/\nu)/2]^{(3\nu+\epsilon)/2} \\
&\quad\times[\Gamma(\epsilon)/\Gamma(\nu+\epsilon)]^{1/2}\|f\|_{\nu+\epsilon}^s.
\end{aligned} \tag{23}$$

Since $\mathscr{S}(E)$ is dense in $W^{\nu+\epsilon}$, this implies then that all functions in $\cup_{\rho>\nu} W^\rho$ yield trace-class operators, with a bound on their trace-norm given by Eq. (23).

Remarks

1. Again, the bound Eq. (23) can be sharpened to

$$\|Qf\|_{\text{tr}} \leqslant \Gamma(\nu)^{-1/2}\zeta(1+\epsilon)^{1/2}|f|_{\nu+\epsilon}^s \tag{24}$$

with $|f|_{\nu+\epsilon}^s$ defined as in Eq. (21).

2. Note that all the elements of W^ρ $(\rho > \nu)$ are L^1-functions (they are, of course, also square-integrable, which was to be expected since Q maps L^2 unitarily onto the Hilbert–Schmidt operators,[9,19] and since every trace-class operator is also a Hilbert–Schmidt operator).

3. In the same way as Eq. (23) one can also derive the following inequality[25]:

$$f_1(q,p) = f(q) \quad \text{(independent of } p\text{)},$$
$$f_2(q,p) = g(p) \quad \text{(independent of } q\text{)}.$$

Define

$$|||f_1|||_\rho^s = [(f, (q^2 - \tfrac{1}{4}\Delta_q + \nu/2)^\rho f)]^{1/2},$$

with an analogous definition for $|||f_2|||_\rho^s$. Then

$$\begin{aligned}
&\|Qf_1Qf_2\|_{\text{tr}} \\
&\quad\leqslant 2^\nu[\Gamma(\epsilon)/\Gamma(\nu+\epsilon)]e^{-(\nu+\epsilon-1)}[(3+\epsilon/\nu)/2]^{2\nu+\epsilon} \\
&\quad\times|||f_1|||_{\nu+\epsilon}^s|||f_2|||_{\nu+\epsilon}^s.
\end{aligned}$$

This condition on f_1 and f_2 is reminiscent of, but stronger (and our result therefore weaker) than the condition in Theorem XI.21 in Ref. 26. The reason why our condition is stronger is that our treatment, and hence this condition, are invariant under Fourier transforms (see below), while the condition in Ref. 26 is not.

E. The functions corresponding to the dyadics $|[m]\rangle\langle[n]|$

Again we use Eq. (14) to find the functions corresponding to these dyadics (remember that the $|[n]\rangle$ are the eigenstates of the harmonic oscillator $P^2+Q^2{:}\frac{1}{2}(P_j^2+Q_j^2)|[n]\rangle = (n_j+\frac{1}{2})|[n]\rangle$):

$$K_{[m],[n]}(a,b) = \langle \Omega^a|[m]\rangle\langle[n]|\Omega^b\rangle = u_{[m,n]}(a,b) \quad \text{[see Eq. (13)]}.$$

Hence

$$Q^{-1}(|[m]\rangle\langle[n]|) = I^{-1}u_{[m,n]} \underset{\text{def}}{=} h_{[n,m]} \quad \text{(see Refs. 13, 15)}.$$

These functions $h_{[n,m]}$ were studied in Refs. 15, 20, and 25 (among other things). They can be calculated explicitly:

$$h_{[k,l]}(q,p) = 2^\nu e^{-(q^2+p^2)} \prod_{j=1}^{\nu}\left[\sum_{s_j=0}^{\min(k_j,l_j)} (-2)^{-s_j}2^{(k_j+l_j)/2}\frac{\sqrt{k_j!l_j!}}{s_j!(k_j-s_j)!(l_j-s_j)!}(p_j+iq_j)^{l_j-s_j}(p_j-iq_j)^{k_j-s_j}\right],$$

and they have many interesting properties, as, e.g.,[25]

$$(-\tfrac{1}{4}\Delta_q - \tfrac{1}{4}\Delta_p + q^2+p^2)h_{[k,l]} = (|k|+|l|+\nu)h_{[k,l]}$$

which means they are related to the Hermite functions. Like the Hermite functions, they form an orthonormal base in L^2.[20] There exists also a connection with the Laguerre polynomials. One has, for instance, in the case $\nu=1$,

$$h_{kk}(q,p) = 2(-1)^k e^{-(x^2+p^2)}L_k(2x^2+2p^2), \tag{25}$$

while the nondiagonal h_{kl} can be related to generalized Laguerre polynomials.[3,27]

It is amusing to note that the functions we denote by $h_{[k,l]}$ have been discovered and rediscovered several times in the literature. The diagonal h_{kk} can already be found in Ref. 28 (25 years ago!); in Ref. 29 they are rediscovered, and used in a very elegant way to derive properties of the Laguerre polynomials from the properties of the Weyl transform. They were again found in Ref. 8. It is quite likely that these are not the only places in the literature where they were discovered... . The nondiagonal h_{kl} seem to be less popular; they can, nevertheless, also be found in Refs. 27 and 3. One can probably extend the methods used in Ref. 29 and use the h_{kl} and their relation with the Weyl transform to derive properties of the generalized Laguerre polynomials.

Their main interest to us, here, is that they can be used to compute matrix elements between harmonic oscillator eigenstates (such matrix elements are often used, e.g., in nuclear physics). Indeed, for any $T\in\mathscr{S}'(E)$ one has (see Sec. IIB)

$$\begin{aligned}\langle[m]|QT[n]\rangle &= (K_{QT})_{[n,m]} \\ &= (IT)_{[n,m]} \quad \text{[use Eq. (14)]} \\ &= \iint da\,db\; IT(a,b)u_{[m,n]}(a,b) \quad \text{[see Eq. (11)]} \\ &= \int dv\; T(v)h_{[n,m]}(v) \quad \text{(see Ref. 15)},\end{aligned} \tag{26}$$

where the last integral has to be understood as $T(h_{[n,m]})$, if the distribution T is not given by a genuine function. This means one can compute the matrix elements of QT between harmonic oscillator eigenstates by a direct integration on the classical phase space.

F. The relation between Q and the Fourier transform

As was pointed out in Refs. 15 and 20, the integral kernel $\{a,b|\cdot\}$ satisfies the following invariance property under Fourier transforms:

$$F_4(\{a,b|\cdot\})(v) = \{a,-b|v\},$$

where

$$(F_\alpha f)(v) = 2^{-\nu}|\alpha|^\nu \int dv'\, e^{i\alpha\sigma(v,v')}f(v')$$

(F_1 is exactly the symplectic Fourier transform ˜ already defined in the introduction). This then implies, $\forall\; T\in\mathscr{S}'(E)$ [also for all the nontempered distributions in the $H(\overline{\alpha},\overline{A})$ spaces; see Ref. 20]

$$\forall\; a,b,\quad I(F_{-4}T)(a,b) = IT(a,-b).$$

Translated to the Weyl transform language, this means that

$$Q(F_{-4}T) = QT\cdot\Pi \tag{27}$$

($QT\cdot\Pi$ can be defined as a quadratic form on D_{coh}, and, if $T\in W^\rho$, on $D_{-\rho}$, without any problems, since these domains are invariant under Π). This also means that one can enlarge every class of distributions (providing it is not already invariant under F_{-4}), yielding operators with a specific property (provided this property is invariant under multiplication by the parity operator), just by applying F_{-4}, and so produce a new set of distributions with the desired property.

The classes of distributions we introduced in the preceding sections are invariant under F_{-4}, but this is not the case with other characterizations found in the literature. For instance, we know[10] that

$$\forall\; f\in L^1(E),\; Qf \text{ compact and } \|Qf\|\leqslant 2^\nu\|f\|_1 \tag{28}$$

[this can also be derived from Eq. (23): the bound Eq. (28) obviously holds for all f in L^1—see Eq. (2)—; since $W^{\nu+\epsilon}$ is dense in L^1, Eq. (23) implies then that, $\forall\; f\in L^1$, Qf is the norm-limit of trace-class operators, and hence compact). Using Eq. (27), we see then that

$$\forall\ f \in L^1,\ Q(F_{-4}f) \text{ compact and } \|Q(F_{-4}f)\| \leqslant 2^\nu \|f\|_1. \tag{29}$$

Remark

Equation (29) can already be inferred from the formal expression (1); the invariance property of the kernel $\{a,b\,|\cdot\}$ under F_{-4} is another way of saying that Eqs. (1) and (2) are equivalent: formally $[F_4 W(2\cdot)](v) = W(2v)\Pi$.

III. NEGATIVE RESULTS: PATHOLOGIC BEHAVIOR

In this second half of this paper we want to present some results showing one should be careful about Wigner functions and the Weyl transform, and not always trust one's first intuition. These results complement the continuity statements in part A in showing which continuity properties can definitely *not* be expected.

A. A positive function leading to a nonpositive operator

This first section is really only a remark.

It is well known that the Wigner distribution corresponding to a density matrix (i.e., a positive, trace-class operator with trace 1) need not be positive everywhere.[4] {Actually, one can prove[30] that the *only* pure states for which the Wigner function is positive everywhere are the Gaussian states, i.e., the states of form $\psi(x) = N\exp[i\alpha x + \beta(x-x_0)^2]$ in the Schrödinger representation.} Note that the same phenomenon occurs for the so-called diagonal representation with respect to the coherent states, where one represents operators by an integral over dyadics $|\Omega^a)(\Omega^a|$: $A = \int da\, \phi_A(a)|\Omega^a)(\Omega^a|$. Here too it may happen that A is positive, even though ϕ_A is not positive everywhere. However, one always has $\phi_A \geqslant 0 \Rightarrow A \geqslant 0$. This is not true for the Weyl transform, where one *can* have a nonpositive operator stemming from a positive function.

Using the parity operator Π, we build here an example of a positive function f for which the Weyl transform Qf is not positive. Take any ψ such that $\Pi\psi = -\psi$, $\psi \neq 0$, (e.g., $\psi = |[n]\rangle$ with $|n|$ odd). Since the Wigner operators $\Pi(v)$ are strongly continuous, there exists an $r \geqslant 0$ such that

$$|v| \leqslant r \Rightarrow (\psi, \Pi(v)\psi) \leqslant -\tfrac{1}{2}\|\psi\|^2.$$

Define

$$f(v) = \begin{cases} 1, & |v| \leqslant r, \\ 0, & |v| > r. \end{cases}$$

Then

$$\begin{aligned}(\psi, Qf\psi) &= 2^\nu \int dv\ f(v)(\psi, \Pi(v)\psi) \\ &= 2^\nu \int_{|v|\leqslant r} dv\,(\psi, \Pi(v)\psi) \\ &\leqslant -2^{\nu-1}\|\psi\|^2 \int_{|v|\leqslant r} dv \\ &= -2^\nu r^{2\nu}[1/\Gamma(\nu+1)]\|\psi\|^2,\end{aligned}$$

which shows that Qf is not positive.

Remarks

1. The function f we have constructed here is clearly in L^2, which means Qf is a Hilbert–Schmidt operator, and therefore bounded. We shall show later on that there exist positive, bounded functions f for which $\sigma(Qf)$ is not even bounded below.

2. Since the function f as we constructed it is discontinuous, it cannot be the Wigner distribution of a trace-class operator. It is obvious, however, that we could also have chosen f positive, C^∞, with support in $\{v, |v| \leqslant r\}$, without invalidating the conclusion that $\langle\psi, Qf\psi\rangle < 0$. This shows that there exist positive functions in $\mathscr{S}(E)$ with nonpositive, trace-class Weyl transforms.

B. Bounded functions leading to unbounded operators

As long as we concern ourselves with functions depending only on q or on p, we know that the operator corresponding to such a bounded function will always be bounded, with

$$\|Qf\| \leqslant \|f\|. \tag{30}$$

We shall show here that this is no longer the case once one considers functions depending on both q and p: Not only does the bound Eq. (29) not hold any longer, even with an extra constant K on the right-hand side, but there actually exist bounded functions whose Weyl transform is an unbounded operator.

We shall prove this in two steps. In a first step we construct a sequence of functions f_n such that $\|f_n\|_\infty = 1$, $\|Qf_n\| \to_{n\to\infty} \infty$, showing thereby that no K exists for which $\|Qf\| \leqslant K\|f\|_\infty$. We then prove (ad absurdum) that this implies $Q(L^\infty) \not\subset B(\mathscr{H})$.

Take, for simplicity reasons, $\nu = 1$. Define

$$f_n(v) = \begin{cases} 1 & \text{if } h_{nn}(v) \geqslant 0, \\ -1 & \text{if } h_{nn}(v) < 0. \end{cases}$$

Then

$$\begin{aligned}\langle n|Qf_n|n\rangle &= \int dv\, f_n(v) h_{nn}(v) \quad [\text{see Eq. (26)}] \\ &= \int dv\, |h_{nn}(v)| \\ &= (1/2\pi)\int dq \int dp\ 2e^{-(q^2+p^2)}|L_n(2q^2+2p^2)| \\ &\qquad [\text{see Eq. (25)}] \\ &= \tfrac{1}{2}\int du\ e^{-u/2}|L_n(u)|.\end{aligned}$$

To put a lower bound on this integral, we shall use asymptotic formulas for the Laguerre polynomials derived by Tricomi.[31] Let η be a small positive number (more specifically $\eta \in]0, \frac{4}{5}[$). Then, for $x \in [\eta(n+\frac{1}{2}), (1-\eta)(4n+2)]$,

$$L_n(x) = e^{x/2}(1/\sqrt{\pi k}\,\sin 2\theta)[\sin(\Theta + n\pi) + O(k^{-1})]$$

where

$$k = n + \tfrac{1}{2}, \quad \theta = \cos^{-1}\sqrt{x/4k},$$
$$\Theta = k(2\theta - \sin 2\theta) + \pi/4$$

and where the term $O(k^{-1})$ can be bounded by Mk^{-1}, *uniformly* in x for $x \in [\eta k, 4(1-\eta)k]$. (This last fact is not stated explicitly in Ref. 31, but it can be derived from the details of Tricomi's proof). Consequently,

$$\int_0^\infty du\, e^{-u/2}|L_n(u)|$$
$$\geqslant \int_{\eta(n+1/2)}^{(1-\eta)(4n+2)} dx\,(1/\sqrt{\pi k}\sin 2\theta)$$
$$\times|\sin(\Theta+n\pi)+O(k^{-1})|$$
$$\geqslant (8k/\sqrt{2\pi k})\int_{\cos^{-1}\sqrt{1-\eta}}^{\cos^{-1}\sqrt{\eta/4}} d\theta\,\sqrt{\sin 2\theta}$$
$$\times\{|\sin[k(2\theta-\sin 2\theta)+\pi/4]|-Mk^{-1}\}$$
$$\underset{n\to\infty}{\longrightarrow}(8\sqrt{k}/\sqrt{2\pi})\left[(2/\pi)\int_{\cos^{-1}\sqrt{1-\eta}}^{\cos^{-1}\sqrt{\eta/4}}\right.$$
$$\left.\times d\theta\,\sqrt{\sin 2\theta}\,(1-\pi M/2k)\right]$$

(by an extension of the Riemann–Lebesgue lemma). This implies there exists an $\alpha>0$ such that

$$\forall\ n,\quad \int_0^\infty du\, e^{-u/2}|L_n(u)|\geqslant\alpha\sqrt{n}\ ;$$

hence (31)

$$\|Qf_n\|\geqslant\langle n|Qf_n|n\rangle\geqslant\alpha\sqrt{n}\ .$$

Suppose now that $Q(L^\infty)\subset B(\mathscr{H})$. We show next that this implies that Q is closed from L^∞ to $B(\mathscr{H})$. Indeed, take $g_n\in L^\infty$ such that

$$\|g_n-g\|_\infty\underset{n\to\infty}{\longrightarrow}0,\quad \|Qg_n-B\|\underset{n\to\infty}{\longrightarrow}0\,.$$

Then,

$$\forall\ a,b,\quad (\Omega^a,B\Omega^b)$$
$$=\lim_{n\to\infty}(\Omega^a,Qg_n\Omega^b)$$
$$=\lim_{n\to\infty}\int dv\,g_n(v)\{a,b\,|v\}$$
$$=\int dv\,g(v)\{a,b\,|v\}\quad[\{a,b\,|\cdot\}\in L^1(E)]$$
$$=(\Omega^a,Qg\Omega^b)\,;$$

hence $B=Qg$. This implies Q is closed from L^∞ to $B(\mathscr{H})$; hence, by the closed graph theorem, $\exists\ K>0$ such that $\|Qf\|\leqslant K\|f\|_\infty$. This is clearly in contradiction with Eq. (31), which allows us to conclude that $Q(L^\infty)\not\subset B(\mathscr{H})$; i.e., there exist bounded functions with unbounded Weyl transforms.

Remark

If f is a bounded function with unbounded Weyl transform Qf, then $g=f+\|f\|_\infty$ is a positive bounded function with unbounded Weyl transform. From g one can then construct a positive bounded function h such that Qh is not even bounded below. Let us suppose that Qg is bounded below (otherwise, we simply take $h=g$). Then Qg is not bounded above, since Qg is unbounded. Take now $h=\|g\|_\infty-g$. This is clearly a positive, bounded function; moreover, $Qg+Qh=\|g\|_\infty\mathbb{1}$, which implies Qh is not bounded below.

C. Non-absolutely-integrable Wigner distributions corresponding to positive, trace-class operators

All the functions leading to trace-class operators we have encountered here till now were absolutely integrable (e.g., the $h_{\{m,n\}}$, the class $\cup_{\rho>\nu}W^\rho$ in Sec. IID). Of course, this does not mean that every Wigner distribution for a trace-class operator must necessarily be in L^1. On the other hand, if Qf is trace-class, then

$$\lim_{R\to\infty}\int_{|v|\leqslant R}dv\,f(v)=\operatorname{Tr}Qf\leqslant\|Qf\|_{\rm tr}\ .$$

One knows, of course, that f need not be positive everywhere, even if Qf is positive, and hence that the integral of $|f|$ will be larger, in most cases, than the integral of f itself. The connected components of the domain where f is negative (for Qf positive, trace-class) have to be rather "small," however (otherwise, negative expectation values of Qf would be possible), and are physically thought of as being caused by "quantum fluctuations."[7] It thus does not seem unreasonable, at first sight, to hope that, even though

$$\|f\|_1\leqslant\|Qf\|_{\rm tr}$$

cannot possibly hold for all trace-class operators Qf, one still would retain the property that for every trace-class operator A the associated Wigner distribution $Q^{-1}A$ would be in $L^1(E)$. It is a direct consequence of the result in the preceding section that this argument turns out to be deceiving: There do exist positive trace-class operators A for which $Q^{-1}A\notin L^1(E)$. Indeed, suppose there were none. Then

$$Q^{-1}(\tau_1(\mathscr{H}))\subset L^1(E)\,.$$

Again, this implies that $Q^{-1}:\tau_1(\mathscr{H})\to L^1(E)$ is closed: Take A_n such that

$$\|A_n-A\|_{\rm tr}\underset{n\to\infty}{\longrightarrow}0\,,\quad \|Q^{-1}A_n-f\|_1\underset{n\to\infty}{\longrightarrow}0\,;$$

then

$$(\Omega^a,A\Omega^b)=\lim_{n\to\infty}(\Omega^a,A_n\Omega^b)$$
$$=\lim_{n\to\infty}\int dv\,(Q^{-1}A_n)(v)\{a,b\,|v\}$$
$$=\int dv\,f(v)\{a,b\,|v\}\quad[\{a,b\,|\cdot\}\in L^\infty(E)]\ ;$$

hence $Q^{-1}A=f$ [$I:\mathscr{S}'\to Z(E_2)$ is injective]. By the closed graph theorem this then implies

$$\exists\ K>0\quad\text{such that}\quad \|Q^{-1}A\|_1\leqslant K\|A\|_{\rm tr}\ .$$

But we already calculated that for $A_n=|n\rangle\langle n|$ (take $\nu=1$)

$$\|Q^{-1}A_n\|_1=\int dv\,|h_{nn}(v)|\geqslant\alpha\sqrt{n}\ ,$$

while $\|A_n\|_{\rm tr}=1$. This is clearly a contradiction, implying that $Q^{-1}(\tau_1(\mathscr{H}))\not\subset L^1(E)$.

Remark

The inclusion in the other direction does not hold either:

$$Q(L^1(E)) \not\subset \tau_1(\mathscr{H}).$$

This is an immediate consequence of the fact that all Wigner distributions of trace-class operators are continuous. Moreover, $Q^{-1}(\tau_1(\mathscr{H})) \subset L^2(E)$ (since every trace-class operator is Hilbert–Schmidt), which shows that not even all continuous L^1 functions lead to trace-class operators.

ACKNOWLEDGMENTS

It is a pleasure for me to thank A. Grossmann for his constant interest and his encouraging support. I would also like to thank O. Heilman for a helpful remark concerning Laguerre polynomials and J.-P. Antoine for indicating Ref. 26 to me and for his interest.

[1]H. Weyl, *The Theory of Groups and Quantum Mechanics* (Dover, New York, 1950).

[2]A Grossmann, Comm. Math. Phys. **48**, 191 (1976).

[3]J. P. Amiet and P. Huguenin, *Mécaniques classique et quantique dans l'espace de phase* (Université de Neuchâtel, Neuchâtel, Switzerland, 1981).

[4]E. Wigner, Phys. Rev. **40**, 749 (1932).

[5]A. Voros, "Développements semi-classiques," Thèse de Doctorat, Université de Paris-Sud, 1977.

[6]A. Grossmann, Comm. Math. Phys. **48**, 194 (1976).

[7]L. G. Suttorp and S. R. de Groot, Nuovo Cimento A **65**, 245 (1970); S. R. de Groot and L. G. Suttorp, *Foundations of Electrodynamics* (North-Holland, Amsterdam, 1972).

[8]F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowics, and D. Sternheimer, Ann. Phys. (N.Y.) **111**, 111 (1978).

[9]I. Segal, Math. Scand. **13**, 31 (1963); J. C. Pool, J. Math. Phys. **7**, 66 (1966).

[10]G. Loupias, "Sup la convolution gauche," Thèse de Doctorat, Université d'Aix—Marseille, 1966; G. Loupias and S. Miracle-Sole, Ann. Inst. H. Poincaré **6**, 39 (1967).

[11]A. Grossmann, G. Loupias, and E. M. Stein, Ann. Inst. Fourier **18**, 1 (1969).

[12]L. Hörmander, Comm. Pure Appl. Math. **32**, 359 (1979).

[13]J. R. Klauder, Ann. Phys. (N.Y.) **11**, 123 (1960); J. R. Klauder and E. C. G. Sudarshan, *Fundamentals of Quantum Optics* (Benjamin, New York, 1968).

[14]V. Bargmann, Comm. Pure Appl. Math. **14**, 187 (1961); **20**, 1 (1967).

[15]I. Daubechies and A. Grossmann, J. Math. Phys. **21**, 2080 (1980).

[16]N. Aronszajn, Trans. Am. Math. Soc. **68**, 337 (1950).

[17]J. McKenna and J. R. Klauder, J. Math. Phys. **5**, 878 (1964); J. R. Klauder, J. Math. Phys. **4**, 1055 (1965).

[18]C. L. Mehta and E. C. G. Sudarshan, Phys. Rev. B **138**, 274 (1965).

[19]I. Daubechies, Comm. Math. Phys. **75**, 229 (1980).

[20]I. Daubechies, A. Grossmann, and J. Reignier, J. Math. Phys., (to be published).

[21]M. Reed and B. Simon, *Modern Methods in Mathematical Physics, Vol. I: Functional Analysis* (Academic, New York, 1972).

[22]A. Grossmann, Comm. Math. Phys. **2**, 1 (1966).

[23]M. Reed and B. Simon, *Modern Methods in Mathematical Physics, Vol. IV: Analysis of Operators* (Academic, New York, 1978).

[24]A. Grossmann, J. Math. Phys. **6**, 54 (1965).

[25]I. Daubechies, "Representation of quantum mechanical operators by kernels on Hilbert spaces of analytic functions," doctoral thesis, Vrije Universiteit Brussel, 1980.

[26]M. Reed and B. Simon, *Modern Methods in Mathematical Physics, Vol. III: Scattering Theory* (Academic, New York, 1979).

[27]A. M. Nachin, Thèse de 3^e cycle, Université d'Aix—Marseille, 1972.

[28]U. Uhlhorn, Arkiv Fysik **11**, 87 (1957).

[29]J. Peetre, Le Matematiche (Catania) **27**, 301 (1972).

[30]R. L. Hudson, Rep. Math. Phys. **6**, 1249 (1974); C. Piquet, C. R. Acad. Sci. Paris A **279**, 107 (1974).

[31]F. G. Tricomi, Comm. Math. Helv. **22**, 150 (1949).