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An integral transform related to quantization. II. Some mathematical properties

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We study in more detail the mathematical properties of the integral transform relating the matrix elements between coherent states of a quantum operator to the corresponding classical function. Explicit families of Hilbert spaces are constructed between which the integral transform is an isomorphism.

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1. INTRODUCTION

In a preceding paper,[1] two of us studied an integral transform giving a direct correspondence between a classical function on the one hand and matrix elements of the corresponding quantum operator between coherent states on the other hand:

$$(\Omega^a, Qf\,\Omega^b) = \int_E dv\, f(v)\{a,b\,|v\}. \tag{1.1}$$

Here E is the phase space (i.e., a $2n$-dimensional real vector space, where n is the number of degrees of freedom), and the Ω^a are the usual coherent states, labeled by phase space points (they can be considered as states centered round the phase space point a labelling them, and they minimize the uncertainty inequalities[2]).

Formula (1.1) was obtained in Ref. 1 from the correspondence formula

$$Qf = 2^n \int_E dv\, f(v) W(2v)\Pi \tag{1.2}$$

where the $W(v)$ are the Weyl operators (see Ref. 1) and Π is the parity operator. [This formula is not the original Weyl formula[3]; it gives a more direct correspondence $f \leftrightarrow Qf$ than the usual expression, since no Fourier analysis step is needed. It was shown in Ref. 4 that (1.2) is equivalent to the Weyl quantization formula.]

The integral kernel $\{a,b\,|v\}$ in (1.1) is then defined as

$$\{a,b\,|v\} = 2^n(\Omega^a, W(2v)\Pi\,\Omega^b). \tag{1.3}$$

This function was computed explicitly in Ref. 1, where we also gave some properties of both the function and the integral transform defined by it, together with some examples. A deeper mathematical study of the integral transform was, however, not intended in Ref. 1; we propose to fill this gap at least partially with the present article.

Ultimately our aim is to use the results of the mathematical study of the integral transform (1.1) to derive properties of the Weyl quantization procedure. One can indeed use the well-known "resolution of the identity" property of the coherent states,[2]

$$\int da\,|\Omega^a)\,(\Omega^a| = \mathbf{1} \tag{1.4}$$

to see that, at least formally, any operator A is characterized by its coherent state matrix elements

$$A = \int_E da \int_E db\, |\Omega^a)(\Omega^b|\,(\Omega^a, A\Omega^b). \tag{1.5}$$

A detailed knowledge of the properties of the integral transform with kernel $\{a,b\,|v\}$ might therefore be useful for

(1) giving a sense to the Weyl quantization formula for rather large classes of functions (essentially, once a precise sense is given to the integral transform on a certain class of functions, one can try to define the corresponding operators from their matrix elements between coherent states),

(2) deriving properties of the quantum operator Qf directly from properties of the corresponding function f (and vice versa).

As we shall show, the inverse of the integral transform (1.1) is given again by using the same kernel

$$f(v) = \int_E \int_E da\, db\, Qf(a,b)\{b,a|v\}. \tag{1.6}$$

(Actually, this integral does not converge absolutely in most cases, and some limiting procedure has to be introduced.) Therefore we shall also be able to use the results of our study of the integral transforms associated with the kernel function $\{a,b\,|v\}$ to obtain information on the "dequantization procedure" [i.e. the inverse map of the "quantization procedure" as defined by (1.2)]. Note that this dequantization procedure is actually the same map as the one associating to each density matrix the corresponding Wigner function[5] extended, however, to a much larger class of operators. These applications shall be further developed in a following paper (a first application was given in Ref. 6); in the present article we restrict ourselves to a study of the integral transforms

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$$(If)(a,b) = \int_E dv\, f(v)\{a,b\,|v\}, \tag{1.7}$$

$$(\tilde{I}\varphi)(v) = \int_E \int_E da\, db\, \varphi(a,b)\{b,a|v\}. \tag{1.8}$$

We shall see that though f may be locally quite singular (one can even consider some classes of nontempered distributions), its image If will always be very gentle locally, with analyticity properties in both its arguments. It therefore makes sense to study not only the function $If(a,b)$, but also the coefficients $If_{m,n}$ of the Taylor series for $If(a,b)$. It turns out that one can construct a family of functions giving directly the link between $f(v)$ and $If_{m,n}$,

$$If_{m,n} = \int_E dv\, f(v) h_{m,n}(v). \tag{1.9}$$

Actually these $h_{m,n}$ are just the functions occurring in the bilinear expansion of $\{a,b\,|v\}$ in Ref. 1: formally (1.9) can be seen as the result of commuting in (1.7) the integral and the series expansion for $\{a,b\,|v\}$. However, (1.9) holds true for many functions for which such an interchanging of summation and integral would be *a priori* pure heresy. The functions $h_{m,n}$ have lots of beautiful properties, most of which are a consequence of the fact that they form a complete orthonormal set of eigenfunctions for the "harmonic oscillator" $x^2 + p^2 - \frac{1}{4}\Delta_x - \frac{1}{4}\Delta_p$ on $L^2(E)$, i.e., on phase space, where we consider an explicit decomposition of the phase space into x space p space: $E \cong \mathbb{R}^{2n} \cong \mathbb{R}^n + \mathbb{R}^n = x$ space $+\, p$ space (see also Sec. 2); in the context of Weyl quantization the $h_{m,n}$ can be seen as the classical functions corresponding to the dyadics $|n\rangle\langle m|$, where $|n\rangle$ are the harmonic oscillator eigenstates (see Refs. 1, 7, and 8). Note that the $h_{m,n}$ are not the usual set of Hermite functions (though they can of course be written as linear combinations of Hermite functions); they are related to the Laguerre polynomials.[7,8]

One can then derive all kinds of results relating the growth of $If(a,b)$ or $If_{m,n}$ to the behavior of f, and analogously for $\tilde{I}\phi$ and ϕ. The derivation of such results amounts to the construction of suitable Banach or Hilbert spaces between which the integral transforms $I,\tilde{I}$ become continuous linear maps or even isomorphisms.

Our main tool for the study of $I,\tilde{I}$ will be the link between the integral transform I and the Bargmann integral transform as defined in Ref. 9 (see Sec. 6 in Ref. 1). Using this link we shall be able to translate bounds obtained in Ref. 9 to our present context, and to obtain other bounds (for other families of spaces) by similar techniques. As in Ref. 9, we can give a complete characterization of the images of $\mathscr{S}(E)$, $\mathscr{S}'(E)$ under I; by a suitable generalization we shall even go beyond the tempered distributions. (Related results, but in a completely different context, and concerning quantization restricted to functions with certain holomorphicity properties, can be found in Ref. 10.)

The paper is organized as follows. In Sec. 2 we give a survey of our notations and some properties of the kernel $\{a,b\,|v\}$, in Sec. 3 we reintroduce the h_{mn} and state some related results, in Sec. 4 we show how bounds on If can be obtained starting from bounds on f, and vice versa: in Sec. 4A we review the Banach space approach found in Bargmann[9b]; in Sec. 4B we go over to Hilbert spaces, which are better suited to our purpose, and in Sec. 4C we generalize the construction of Sec. 4B, which enables us to treat certain Hilbert spaces of distributions "of type S" which are larger than $\mathscr{S}'(E)$. In Sec. 5 we shortly discuss the integral transform I when restricted to functions on phase space which can be split up into a product of a function depending only on x with a function depending only on p. Essentially the same types of statements can be formulated, and a short survey of results is given. In Sec. 6 we give some concluding remarks.

2. NOTATIONS AND BASIC PROPERTIES OF $\{a,b/v\}$

In Ref. 1 we worked with an intrinsic coordinate-free notation system using a symplectic structure on the phase space (basically this is the bilinear structure underlying the Poisson brackets), and a complex structure yielding a Euclidean form on the phase space, compatible with the symplectic structure. By choosing a suitable basis, this could be seen to lead to a decomposition of the phase space into a direct sum of two canonically conjugate subspaces. This decomposition is not unique: for a given symplectic structure, several compatible complex structures can be constructed; different complex structures correspond then to different decompositions of phase space. This freedom in the choice of the splitting up of the phase space is particularly useful whenever (linear) canonical transformations are discussed[11] or used (as, e.g., in the presence of a constant magnetic field). Here we shall not need to use simultaneously different decomposition possibilities for the phase space, and we shall therefore fix the decomposition once and for all. We shall use this decomposition from the very start to introduce our notations in a way that is less intrinsic but probably more familiar to most readers. It goes without saying that the results we shall obtain are independent of this approach, and that they could as well be obtained in the more intrinsic setting of Ref. 1 (see Ref. 8).

The phase space E is a $2n$-dimensional real vector space, which we shall consider as a direct sum of two n-dimensional subspaces

$$E = x \text{ space} \oplus p \text{ space}, \qquad E \ni v = (x,p). \tag{2.1}$$

The x and p need not be the conventional position and momentum variables: any set of canonically conjugate coordinates which are linear combinations of position and momentum are equally good candidates for these x and p. On E we have a symplectic structure

$$\sigma(v,v') = \sigma((x,p),(x',p')) = \tfrac{1}{2}(p\cdot x' - x\cdot p') \tag{2.2}$$

and a Euclidean structure

$$s(v,v') = s((x,p),(x',p')) = \tfrac{1}{2}(x\cdot x' + p\cdot p') \tag{2.3}$$

[this is the Euclidean structure corresponding with the σ-compatible complex structure $J((x,p)) = (p, -x)$—see Ref. 1]. For further convenience we introduce a Gaussian in the phase space variables,

$$\omega(v) = \exp[-\tfrac{1}{2}s(v,v)] = \exp[-\tfrac{1}{4}(x^2 + p^2)], \tag{2.4}$$

and a family of analytic functions

$$h^{[m]}(v) = \prod_{j=1}^{n} \left(\frac{p_j + ix_j}{\sqrt{2}} \right)^{m_j}, \tag{2.5}$$

where we have used the multi-index notation $[m] = (m_1, \ldots, m_n)$.

Note: Whenever we use the term "analyticity" when speaking of a function for phase space, this means that $f(v) = f(x,p)$ is analytic in the variable $p + ix$; if f is analytic in the variable $p - ix$, we say that f is antianalytic on phase space. For a definition of these concepts without using an *a priori* decomposition of the phase space, see Ref. 1.

We shall often need the set of functions which can be written as a product of the Gaussian ω, (2.4), with an analytic function on E. We call these functions "modified holomorphic," and denote their set by $Z(E)$ or Z:

$$Z(E) = \{\phi: E \to \mathbb{C}; \phi = f \cdot \omega, \text{ with } f \text{ analytic on } E\}. \tag{2.6}$$

Note that the pointwise product of two modified holomorphic functions is not modified holomorphic, having a factor ω too many.

The square integrable modified holomorphic functions form a closed subspace of $L^2(E)$ (see Refs. 1 and 9); we shall denote this Hilbert space by $\mathscr{L}_0$:

$$\mathscr{L}_0(E) = \{\phi \in Z(E); \int dv |\phi(v)|^2 < \infty\}. \tag{2.7}$$

The measure on E used here is just the usual translationally invariant measure on E, with normalization fixed by the requirement

$$\int dv\, \omega^2(v) = \int dv \exp[-s(v,v)] = 1,$$

i.e.,

$$dv = \frac{1}{(2\pi)^n} d^n x\, d^n p. \tag{2.8}$$

For any function $\phi = f \cdot \omega$ in Z one can, of course, decompose the analytic function f into its Taylor series, which gives

$$\phi(v) = \sum_{[m]} a_{\phi,[m]} h^{[m]}(v) \omega(v), \tag{2.9}$$

where the convergence is uniform on compact sets. One can prove (see Ref. 9) that for $\phi \in Z$ one has

$$\phi \in \mathscr{L}_0 \Leftrightarrow \sum_{[m]} |a_{\phi,[m]}|^2 [m!] < \infty,$$

$$\phi, \psi \in \mathscr{L}_0 \Rightarrow (\phi, \psi) = \int dv\, \overline{\phi(v)}\, \psi(v) = \sum_{[m]} \overline{a_{\phi,[m]}}\, a_{\psi,[m]}\, [m!], \tag{2.10}$$

where

$$[m!] = \prod_{j=1}^{n} (m_j!).$$

Equation (2.10) implies that the set of functions $u_{[m]}$,

$$u_{[m]}(v) = \frac{1}{\sqrt{[m!]}} h^{[m]}(v) \omega(v), \tag{2.11}$$

constitutes an orthonormal base in $\mathscr{L}_0$, and that the series (2.9) converges not only uniformly on compact sets, but also in L^2 as long as $\phi \in \mathscr{L}_0$. We shall often rewrite (2.9) and (2.10) in the following way.

$\forall \phi \in Z$: we write a "modified Taylor expansion"

$$\phi(v) = \sum_{[m]} \phi_{[m]} u_{[m]}(v), \tag{2.12}$$

with uniform convergence on compact sets,

$$\forall \phi, \psi \in \mathscr{L}_0 : (\phi, \psi) = \sum_{[m]} \overline{\phi_{[m]}}\, \psi_{[m]}, \tag{2.13}$$

in particular

$$\int dv\, \overline{u_{[m]}(v)} \phi(v) = (u_{[m]}, \phi) = \phi_{[m]}. \tag{2.14}$$

The same construction can be made in the space $E \times E$. Most functions on $E \times E$ in this study will have the property that they are *modified holomorphic in one variable and modified antiholomorphic in the other one. We denote the set of functions having this property by* $Z(E_2)$ (or shorter Z_2):

$$Z(E_2) = \{\phi: E \times E \to \mathbb{C}; \phi(v,v') = f(v,v') \omega(v) \omega(v'), \text{ with } f(v,v') \text{ analytic in } (p + ix, p' - ix')\}. \tag{2.15}$$

Again we can restrict ourselves to the square integrable functions in $Z(E_2)$:

$$\mathscr{L}_0(E_2) = Z(E_2) \cap L^2(E \times E); \tag{2.16}$$

again this is a closed subspace of $L^2(E \times E)$, with orthonormal basis

$$u_{[m_1, m_2]}(v_1, v_2) = u_{[m_1]}(v_1)\, \overline{u_{[m_2]}(v_2)}. \tag{2.17}$$

The analogs of (2.12) and (2.13) are now

$$\forall \phi \in Z(E_2) : \phi(\zeta) = \sum_{[m_1],[m_2]} \phi_{[m_1,m_2]} u_{[m_1,m_2]}(\zeta), \tag{2.18}$$

with uniform convergence on compact sets,

$$\forall \phi, \psi \in \mathscr{L}_0(E_2) : (\phi, \psi) = \sum_{[m],[m_2]} \overline{\phi_{[m_1,m_2]}}\, \psi_{[m_1,m_2]}, \tag{2.19}$$

in particular

$$\phi_{[m_1,m_2]} = \int d\zeta\, \overline{u_{[m_1,m_2]}(\zeta)}\, \phi(\zeta), \tag{2.20}$$

where we have used the notation $\zeta = (v_1, v_2)$ (in general, the Greek letters ζ, ξ will denote elements of $E \times E$).

Both the spaces $\mathscr{L}_0(E)$ and $\mathscr{L}_0(E_2)$ have "reproducing vectors" (this is a common feature for Hilbert spaces of analytic functions[13]):

$$\forall a \in E, \quad \forall \zeta = (a_1, a_2) \in E \times E,$$

$$\exists \omega^a \in \mathscr{L}_0(E), \quad \exists \omega^\zeta \in \mathscr{L}_0(E_2),$$

such that

$$\forall \phi \in \mathscr{L}_0(E) : (\omega^a, \phi) = \phi(a),$$

$$\forall \phi \in \mathscr{L}_0(E_2) : (\omega^\zeta, \phi) = \phi(\zeta). \tag{2.21}$$

These ω^a, ω^ζ are given explicitly by

$$\omega^a(v) = e^{i\sigma(a,v)} \omega(v - a) = \sum_{[m]} \overline{u_{[m]}(a)}\, u_{[m]}(v), \tag{2.22}$$

$$\omega^\zeta(\xi) = \omega^{a_1,a_2}(b_1,b_2) = \omega^{a_1}(b_1)\,\overline{\omega^{a_2}(b_2)} = \sum_{[k],[l]} \overline{u_{[k,l]}(\zeta)}\, u_{[k,l]}(\xi). \tag{2.23}$$

The series in (2.22) and (2.23) converge uniformly on compact sets, but also in $L^2(E)$ (separately in a and v), respectively, $L^2(E\times E)$ (separately in ζ and ξ). Note that the reproducing properties (2.21), (2.14), and (2.20) can be proved for much more general classes of ϕ than only $\mathscr{L}_0$ (see below).

The integral kernel {a,b|v}

For any three points $a,b,v,\in E$ we define the function $\{a,b\,|v\}$ as follows (see Ref. 1):

$$\begin{aligned}\{a,b\,|v\} &= 2^n e^{i[\sigma(a,b)+2\sigma(b,v)+2\sigma(v,a)]}\,\omega(2v-a-b)\\ &= 2^n \exp\Big[i(\tfrac12 p_a x_b - \tfrac12 p_b x_a + p_b x_v - p_v x_b + p_v x_a - p_a x_v)\\ &\quad -\Big(x_v - \frac{x_a+x_b}{2}\Big)^2 - \Big(p_v - \frac{p_a+p_b}{2}\Big)^2\Big]\\ &= 2^n \exp[(p_v - ix_v)(p_a+ix_a) - \tfrac12(p_b - ix_b)\\ &\quad \times(p_a + ix_a) + (p_b - ix_b)(p_v+ix_v)]\\ &\quad \times\omega(a)\omega(b)\omega(2v).\end{aligned} \tag{2.24}$$

From the last expression in (2.24) it is obvious that $\{a,b\,|v\}$ is modified holomorphic in a, modified antiholomorphic in b, i.e.,

$$\forall v\in E: \{\cdot,\cdot|v\}\in Z_2. \tag{2.25}$$

Moreover, one can easily check (see Refs. 1 and 8) that $\{a,b\,|v\}$ has the following properties:

$$|\{a,b\,|v\}|<2^n,$$

$$\int dv \{a,b\,|v\}\ \{c,d\,|v\} = \omega^a(c)\omega^d(b). \tag{2.26}$$

This function $\{a,b\,|v\}$ will be used to define two integral transforms

$$(If)(\zeta) = \int dv\, f(v)\{\zeta\,|v\}, \tag{2.27a}$$

$$(\tilde I\varphi)(v) = \int d\zeta\, \phi(\zeta)\ \overline{\{\zeta\,|v\}}. \tag{2.27b}$$

It is our purpose here to investigate some of the properties of these integral transforms and their extensions.

3. BILINEAR EXPANSION OF {a,b|v}—THE FUNCTIONS $h_{[k,l]}$

A. Bilinear expansion of {a,b|v}

As elements of $Z(E_2)$, the functions $\{\cdot|v\}$ can be developed in a series with respect to the $u_{[k,l]}$ [see (2.13)]:

$$\{a,b\,|v\} = \{\zeta\,|v\} = \sum_{[k],[l]} u_{[k,l]}(\zeta) h_{[k,l]}(v). \tag{3.1}$$

The $h_{[k,l]}$ are defined, up to a factor $\sqrt{[k!]\ [l!]}$, as derivatives of the function $\{\zeta\,|v\}\ (\omega_2(\zeta))^{-1}$ in $\zeta=0$:

$$h_{[k,l]}(v) + 2^n\sqrt{[k!][l!]}\ 2^{(|k|+|l|)/2} \times\left[\frac{d^{[k]}}{dp_a^{[k]}}\frac{d^{[l]}}{dp_a^{[l]}}\{a,b\,|v\}\omega(a)^{-1}\omega(b)^{-1}\right]_{a=b=0}. \tag{3.2}$$

Because of the explicit form (2.24) of $\{\zeta\,|v\}$ it is obvious that every $h_{[k,l]}$ is a polynomial in v multiplied by the Gaussian $\omega(2v) = \exp[-(x_v{}^2 + p_v{}^2)]$. This automatically implies that all the $h_{[k,l]}$ are elements of $\mathscr{S}(E)$, the Schwartz space of C^∞ function which decrease faster than any negative power of (x,p).

B. Orthonormality of the $h_{[k,l]}$

On the other hand, we have [see (2.26)]

$$\int dv\ \overline{\{a,b\,|v\}}\ \{c,d\,|v\} = \omega^a(c)\omega^d(b).$$

Multiplying both sides with $\omega(a)^{-1}\omega(b)^{-1}$, and computing derivatives with respect to p_a,p_b, we obtain (it is obvious from the explicit form of $\{a,b\,|v\}$ that these derivatives can be commuted with the integral in the left hand side)

$$\int dv\ \overline{h_{[k,l]}(v)}\ \{c,d\,|v\} = u_{[k]}(c)\ \overline{u_{[l]}(d)}. \tag{3.3}$$

Repeating the same operations in the variables c and d, we obtain

$$\int dv\ \overline{h_{[k,l]}(v)}\ h_{[k',l']}(v) = \delta_{[k],[k']}\,\delta_{[l][l']}, \tag{3.4}$$

implying that the $h_{[k,l]}$ form an orthonormal set in $L^2(E)$.

C. Completeness of the $h_{[k,l]}$

From (3.3) we see that the coefficients with respect to the orthonormal set $h_{[k,l]}$ of the orthogonal projection of any $\{\zeta\,|\}$ (ζ fixed) onto the closed linear span of the $h_{[k,l]}$ are exactly the $u_{[k,l]}(\zeta)$; comparing this with (3.1) we conclude that for any ζ the function $\{\zeta\,|\cdot\}$ is an element of the closed span of the $h_{[k,l]}$; (3.1) can now be considered to be the composition in L^2 of $\{\zeta\,|\cdot\}$ with respect to the orthonormal set $h_{[k,l]}$. From this it is now easy to see that the closed span of the $h_{[k,l]}$ is all of $L^2(E)$. Indeed, let ψ be orthogonal to all $h_{[k,l]}$:

$$\forall[k],[l]: (\psi,h_{[k,l]}) = 0.$$

Then

$$\forall a,b: (\psi,\{a,b\,|\cdot\}) = 0$$

$$\Rightarrow\forall c: \int dv\ \psi(v)\ e^{i\sigma(c,v)}\,\omega(2v) = 0$$

$$\Rightarrow\psi(v)\cdot\omega(2v) = 0\ \text{a.e.}\ \Rightarrow\psi = 0$$

[ω is bounded, and the Fourier transform is unitary on $L^2(E)$]. Hence the $h_{[k,l]}$ constitute an orthornormal base for $L^2(E)$.

Note: The properties in Secs. 3B and 3C were already stated in Ref. 1, in more generality (valid also for the coefficient functions of other bilinear expansions of $\{a,b\,|v\}$), without proof. It is possible to prove them (see Ref. 8) using Godement's theorem on irreducible square integrable representations of unimodular locally compact groups. In the special case of the $h_{[k,l]}$, however, one can also prove them with very simple arguments, as shown here.

D. Unitarity of the integral transform I

It is now easy to show that the integral transform I, as defined by (2.27), defines a unitary operator from $L^2(E)$ onto $\mathscr{L}_0(E_2)$.

Proposition 3.1: The integral transform I:

$$If(\zeta) = \int dv\{\zeta|v\} f(v) \tag{3.5}$$

with $\{\zeta|v\} = \{a,b|v\}$ as defined by (2.24), defines a unitary operator from $L^2(E)$ to $\mathscr{L}_0(E_2)$; in particular

$$Ih_{[k,l]} = u_{[l,k]}. \tag{3.6}$$

Proof: We start by defining a linear operator on the span of the $h_{[k,l]}$ by putting

$$Uh_{[k,l]} = u_{[l,k]}.$$

Since the $h_{[k,l]}$, $u_{[k,l]}$ constitute orthonormal bases in $L^2(E)$, $\mathscr{L}_0(E_2)$, respectively, this U can be extended to a unitary operator from $L^2(E)$ onto $\mathscr{L}_0(E_2)$. In particular,

$$U(\{b,a|\cdot\}) = U\Big(\sum_{[k][l]} u_{[k,l]}(b,a)h_{[k,l]}\Big)$$
$$= \sum_{[k][l]} \overline{u_{[l,k]}(a,b)}\, u_{[l,k]} = \omega^{(a,b)},$$

where we have used (2.23). Take now any ϕ in $L^2(E)$. Then $U\phi \in \mathscr{L}_0(E_2)$, and its value at any point is given by the reproducing property (2.21),

$$(U\varphi)(\zeta) = (\omega^\zeta, U\varphi) = (U^*\omega^\zeta, \varphi)$$
$$= (\{\bar\zeta|\cdot\}, \varphi)$$
$$= \int dv\; \overline{\{\bar\zeta|v\}}\varphi(v)$$

[for $\zeta = (a,b)$, we denote (b,a) by $\bar\zeta$].

Hence $(U\phi)(\zeta) = (I\phi)(\zeta)$, which proves the proposition.■

Remarks:

1. A different proof of the unitarity of I between $L^2(E)$ and $\mathscr{L}_0(E_2)$ was given in Sec. 4E in Ref. 1 (the argument given there is not completely rigorous, but it can easily be transformed into a rigorous one).

2. The integral in (3.5) converges absolutely for any $f \in L^2(E)$, since $\{\zeta|\cdot\}$ is in $L^2(E)$ for each fixed ζ. The situation is different if one tries to apply $\tilde I$ to $\mathscr{L}_0(E_2)$: since $\{\cdot|v\}$ is not square integrable on $E \times E$, the integral transform (2.27b) cannot be defined on all of $\mathscr{L}_0(E_2)$. One has, however,

$$\int d\zeta\; \overline{\{\zeta|v\}}\, u_{[k,l]}(\zeta) = h_{[l,k]}(\zeta), \tag{3.7}$$

where the integral converges absolutely because $u_{[l,k]}$ is absolutely integrable. So

$$\tilde I I h_{[k,l]} = h_{[k,l]}, \tag{3.8}$$

which leads one to believe that $\tilde I$ is the inverse of I. Indeed, if one tries to circumvent the problem of possible divergence of the integral by taking limiting procedures, one finds (as in Ref. 9a), e.g.,

$$\forall \varphi \in \mathscr{L}_0(E_2): I^{-1}\varphi = L^2 - \lim_{R\to\infty} \tilde I(\chi_R \varphi)$$
$$= L^2 - \lim_{\alpha\to 0} \tilde I(\omega(\alpha\cdot)\varphi), \tag{3.9}$$

where

$$\chi_R(\zeta) = \begin{cases} 1, & |\zeta| < R \\ 0, & |\zeta| > R \end{cases} \quad \text{[here } |\zeta|^2 = |a|^2 + |b|^2 \text{ and } |a|^2 = s(a,a)\text{]}.$$

The same is true for any other reasonable limiting procedure.

E. Other properties of the $h_{[k,l]}$

One can show (see Refs. 1 and 8) that

$$|h_{[k,l]}(v)| \leqslant 1. \tag{3.10}$$

Explicit calculation of the $h_{[k,l]}$ yields (see Refs. 7 or 6)

$$h_{[k,l]}(x,p) = 2^n e^{-x^2-p^2} \sum_{[s]=0}^{\min([k],[l])} \Big[(-2)^{-|s|} 2^{(|k|+|l|)/2} \frac{\sqrt{[k!][l!]}}{[s!][(l-s)!][(k-s)!]}(p+ix)^{[l-s]}(p-ix)^{[k-s]}\Big]. \tag{3.11}$$

One can check (by direct calculation) that these $h_{[k,l]}$ are the eigenfunctions of a dilated harmonic-oscillator-type operator on phase space E:

$$(-\tfrac14(\Delta_x + \Delta_p) + x^2 + p^2)h_{[k,l]} = (|k| + |l| + m)h_{[k,l]}. \tag{3.12}$$

As a consequence of this, the $h_{[k,l]}$ are linear combinations of products of Hermite functions:

$$h_{[k,l]}(x,p) = \sum_{\substack{[r][s] \\ |r|+|s|=|k|+|l|}} \alpha_{[k,l],[r,s]} H_{[r,s]}(x,p), \tag{3.13}$$

with

$$H_{[r,s]}(x,p) = 2^n H_{[r]}(\sqrt2 x)H_{[s]}(\sqrt2 p), \tag{3.14}$$

$$\sum_{[r],[s]} |\alpha_{[k,l],[r,s]}|^2 = 1 \tag{3.15}$$

and where $H_{[r]}$ is the $[r]$th order Hermite function.

There exists also a relationship between the $h_{[k,l]}$ and the Laguerre polynomials (see Ref. 7). For $n = 1$, $k = l$ one has for instance

$$h_{kk}(x,p) = 2(-1)^k e^{-(x^2+p^2)} L_k(2x^2 + 2p^2), \tag{3.16}$$

where L_k is the Laguerre polynomial of order k.

One can also prove the following recurrence relation for the $h_{[k,l]}$:

$$(|k| + |l|)\, h_{[k,l]} = \sqrt{|k|} \sum_j k_j(p_j - ix_j)h_{[k-\delta_j,l]}$$
$$+ \sqrt{|l|}\sum_j l_j(p_j + ix_j)h_{[k,l-\delta_j]}$$
$$- \sqrt{|k|\,|l|}\sum_j h_{[k-\delta_j,l-\delta_j]} k_j l_j, \tag{3.17}$$

where $[\delta_j]$ is the multi-index $(\delta_j)_m = \delta_{jm}$.

4. THE INTEGRAL TRANSFORM I AS A CONTINUOUS MAP BETWEEN SUITABLE FAMILIES OF BANACH SPACES AND HILBERT SPACES

As was already mentioned in the Introduction, there exists a link between our integral transform I and the Bargmann integral transform as defined in Ref. 9. Since we shall use this link to derive some of our results, we shall first show here what exactly is the connection between the two integral transforms. For $z\in\mathbb{C}^n$, $q\in\mathbb{R}^n$, the Bargmann integral kernel $A(z,q)$ is given by[9]

$$A(z,q) = \pi^{-n/4}\exp[-\tfrac{1}{2}(z^2+q^2)+\sqrt{2}\,z\cdot q]. \tag{4.1}$$

Identifying z with $(1/\sqrt{2})(x-ip)$ (which makes of the multiplication by z—on a suitable Hilbert space of analytic functions—a representation of the harmonic oscillator creation operator: see Ref. 9a), we can rewrite (4.1) as

$$A(x,p;q)\cdot e^{-1/4(x^2+p^2)} = \pi^{-n/4}\,e^{(1/2)xp}\,e^{-ipq}\,e^{-(1/2)(x-q)^2}. \tag{4.2}$$

Comparing this with (2.24) we see that

$$\{a,b\,|v\} = 2^n\pi^{n/2}A\left(\frac{x_a+x_b}{\sqrt{2}},\frac{p_b-p_a}{\sqrt{2}};\sqrt{2}x_v\right) A\left(\frac{p_a+p_b}{\sqrt{2}},\frac{x_a-x_b}{\sqrt{2}};\sqrt{2}\,pv\right) e^{-(1/4)(x_a^2+x_b^2+p_a^2+p_b^2)} \tag{4.3}$$

or

$$\{a,b\,|v\} = 2^n\,r^{n/2}A(c_{ab};\sqrt{2}x_v)\,A(d_{ab};\sqrt{2}p_v)\omega(c_{ab})\cdot\omega(d_{ab}),$$

with

$$c_{ab} = \frac{1}{\sqrt{2}}(x_a+x_b,p_b-p_a), \tag{4.4}$$
$$d_{ab} = \frac{1}{\sqrt{2}}(p_a+p_b,x_a-x_b).$$

Actually, (4.3) implies that we can consider the integral transform I as a $2n$-dimensional Bargmann transform. The explicit Gaussian factors $\omega(c_{ab})$, $\omega(d_{ab})$ just compensate for the difference in definition between our Hilbert space $\mathscr{L}_0(E_2)$ and the Bargmann Hilbert space [we absorb the Gaussian in the functions in $\mathscr{L}_0(E_2)$, whereas in Ref. 9 it is always displayed as a weight function in the definition of the inner product]. The constant factors $2^n\pi^{n/2}$ account for the difference in normalization in the measure, and for the dilation in x_v,p_v. Moreover, one can easily check that analyticity in c_{ab},d_{ab} is equivalent to analyticity in a, antianalyticity in b. So, from a mathematical point of view, I can be assimilated with a $2n$-dimensional Bargmann transform. Physically however, the two integral transforms have a different meaning: I gives a correspondence between classical and quantum aspects, while the Bargmann transform gives the unitary transformation between two different but equivalent realizations (for a short discussion, see Sec. 6 in Ref. 1).

The remarks above will enable us to translate various results obtained by Bargmann in Ref. 9b to the present setting. An example of this is the following (we keep the same notations as in Ref. 9, even though the functions considered here are in fact modified holomorphic instead of holomorphic):

Define

$$\forall\varphi\in Z(E_2):|\varphi|_\rho = \sup_\zeta|(1+|\zeta|^2)^\rho\varphi(\zeta)|,$$
$$\mathfrak{E} = \{\varphi\in Z(E_2);\quad \forall\rho\in\mathbb{R};|\varphi|_\rho<\infty\},$$
$$\mathfrak{E}' = \{\varphi\in Z(E_2);\quad \exists\rho\in\mathbb{R}\ \text{ such that }\ |\varphi|_\rho<\infty\}.$$

The spaces $\mathfrak{E}$, $\mathfrak{E}'$ can be equipped with very natural locally convex topologies by means of the norms $||_\rho$, and one has then the following result.

I defines an isomorphism between $\mathscr{S}(E)$ and $\mathfrak{E}$, with

$$\forall f\in\mathscr{S}(E):(If)(\zeta) = \int dv\{\zeta\,|v\}\,f(v);$$

by duality, I defines also an isomorphism between $\mathscr{S}'(E)$ and $\mathfrak{E}'$, with

$$\forall T\in\mathscr{S}'(E):(IT)(\zeta) = \overline{T(\{\bar\zeta\,|\cdot\})}.$$

The results in Ref.9b also concern two families of Banach spaces interpolating between $\mathscr{S}$ and $\mathscr{S}'$, $\mathfrak{E}$ and $\mathfrak{E}'$, respectively, and between which the integral transform I or its inverse are continuous. We give a survey of these results, translated to our present setting, in Sec. 4A.

The chains of Banach spaces presented in Sec. 4A display, however, several inconveniences. As already mentioned in Ref. 9b the $\mathfrak{E}^\rho$ spaces are not separable, and the little space $\mathfrak{E}$ is not dense in any $\mathfrak{E}^\rho$. Moreover, in relation to the present setting, it turns out that though one can always choose suitably matched spaces in the two ladders to make either I or its inverse continuous, it is impossible to choose them in such a way that I is an isomorphism. None of these problems arises when one uses a suitable interpolating chain of Hilbert spaces instead of Banach spaces (see Sec. 4B). The resulting bounds on I are much more precise between these Hilbert spaces, and therefore more useful for applications to quantization than the results of Sec. 4A.

Generalizing the construction of the Hilbert spaces in Sec. 4B, one can obtain even larger families containing spaces smaller than $\mathscr{S}$ (or $\mathfrak{E}$) or larger than $\mathscr{S}'(\mathfrak{E}')$, on which the integral transform I can still be defined and has continuity properties. The results in Secs. 4B and 4C can be considered as extensions of the bounds in Ref. 9b (Sec. 4B uses some estimates made in Ref. 9b). Other results on the Bargmann transform can, of course, easily be translated to the present context and be useful in a Weyl quantization setting (see, e.g., Ref. 1, where a characterization of the images under the Bargmann transform of the Gel'fand–Shilov spaces S and S^* are given; in a sense this can be considered as complementary to our results in Sec. 4C).

A. The Banach spaces $\mathscr{S}^k$, $\mathfrak{E}^\rho$ and related results on the integral transform I

For any C^k function f on E, we define (this norm is the same as in Ref. 9b, up to a dilation:

$$|f|_k^S = ||f\left(\frac{\cdot}{\sqrt{2}}\right)||_{k,\text{Bargmann}}^S)$$

$$|f|_k^S = \max_{\substack{[m_1],[m_2] \\ |m_1|+|m_2|<k}} \sup_{x,p} |2^{-(|m_1|+|m_2|)/2} (1+2x^2+2p^2)^{(k-|m_1|-|m_2|)/2} (\nabla_x^{[m_1]} \nabla_p^{[m_2]} f)(x,p)|. \tag{4.5}$$

The Banach space $\mathscr{S}^k$ is then defined as

$$\mathscr{S}^k = \{ f: E \to \mathbb{C}; f \text{ is } C^k, |f|_k^S < \infty \}. \tag{4.6}$$

On the other hand, we define, $\forall \rho \in \mathbb{R}$, the following subspaces $\mathfrak{S}^\rho$ of $Z(E_2)$:

$$\mathfrak{S}^\rho = \{ \varphi \in Z(E_2); |\varphi|_\rho = \sup_\zeta |(1+|\zeta|^2)^\rho \varphi(\zeta)| < \infty \}. \tag{4.7}$$

The following theorem was proved in Ref. 9b.

Theorem 4.1:

1. $\forall f \in \mathscr{S}^k$, the function

$$If(\zeta) = \int dv \{\zeta|v\} f(v)$$

is well defined and an element of $Z(E_2)$. Moreover,

$$If \in \mathfrak{S}^k$$

and

$$|If|_k \leqslant b_k |f|_k^s, \tag{4.8}$$

with

$$b_k = \frac{3e}{2} 2^{n/2} (16n)^{k/2} \begin{cases} 1, & k \leqslant 2 \\ e^{-k} k^k, & k \geqslant 3 \end{cases}. \tag{4.9}$$

2. $\forall \varphi \in \mathfrak{S}^\mu$, with $\mu > 2n$, the function

$$\tilde{I}\varphi(v) = \int dv \{\xi|v\} \varphi(\zeta)$$

is well defined on E. Moreover,

$$\forall k \in \mathbf{N},$$

with

$$k < \mu - 2n : \tilde{I}\varphi \in \mathscr{S}^k$$

and

$$|\tilde{I}\varphi|_k^s < b'_{k,\mu} |\varphi|_k, \tag{4.10}$$

with

$$b'_{k,\mu} = 2^{n+k} (\tfrac{3}{2})^{k/2} \Gamma\left(\frac{k}{2}+m+1\right) \Gamma\left(\frac{\mu-k}{2}-m\right) \times \Gamma\left(\frac{\mu-k}{2}\right)^{-1} \int_E dv\, e^{-2|v|^2} (1+|v|^2)^k. \tag{4.11}$$

3. $f \in \bigcup_{k > 2n+1} \mathscr{S}^k \Rightarrow \tilde{I}If = f,$

$$\varphi \in \bigcup_{\mu > 2n} \mathfrak{S}^\mu \Rightarrow I\tilde{I}\varphi = \varphi. \tag{4.12}$$

Note: This theorem was used in Ref. 6 to derive some restrictions in the class of distributions corresponding to bounded operators.

It is obvious from the definition (4.6) of the $\mathscr{S}^k$ spaces that $\mathscr{S}(E) = \bigcap_{k \in \mathbf{N}} \mathscr{S}^k$, and that the locally convex topology on $\mathscr{S}$ defined by the $\|\|\|_k^S$-norms coincides with the usual Schwartz topology. Defining, on the other hand,

$$\mathfrak{S} = \{ \varphi \in Z(E_2); \forall \rho: |\varphi|_\rho < \infty \}, \tag{4.13}$$

and equipping this space with the locally convex topology induced by the norms $\|\|_\rho$, we have immediately the following corollary to Theorem 4.1.

Corollary 4.2: The integral transform I, restricted to $\mathscr{S}$, defines an isomorphism from $\mathscr{S}$ onto $\mathfrak{S}$, with inverse $\tilde{I}$ (restricted to $\mathfrak{S}$).

In Ref. 9b it was shown that $\mathfrak{S}'$, the dual of $\mathfrak{S}$, can be identified with $\cup_{\rho \in \mathbf{R}} \mathfrak{S}^\rho$ in the following way:

$$\forall L \in \mathfrak{S}': g_L(\zeta) = \overline{L(\omega^\zeta)} \Rightarrow g_L \in \bigcup_{\rho \in \mathbf{R}} \mathfrak{S}^\rho,$$

$$\forall \varphi \in \bigcup_{\rho \in \mathbf{R}} \mathfrak{S}^\rho : L_\varphi(\psi) = \int d\zeta\, \overline{\varphi(\zeta)} \psi(\zeta) \Rightarrow L_\varphi \in \mathfrak{S}', \tag{4.14}$$

with

$$g_{L_\varphi} = \varphi, \quad L_{g_L} = L.$$

The topology on $\mathfrak{S}'$ corresponds with the natural topology on $\cup_{\rho \in \mathbf{R}} \mathfrak{S}^\rho$ induced by the norms $||_\rho$. In what follows, we shall always identify $\mathfrak{S}'$ with $\cup_{\rho \in \mathbf{R}} \mathfrak{S}^\rho$ and implicitly use (4.14).

Since I is an isomorphism between $\mathscr{S}$ and $\mathfrak{S}$, it is obvious that by duality I also defines an isomorphism between $\mathscr{S}'$ and $\mathfrak{S}'$:

$$\forall T \in \mathscr{S}': \text{we define } (IT)(\varphi) = T(\tilde{I}\varphi), \quad \forall \varphi \in \mathfrak{S}. \tag{4.15}$$

By means of the identification $\mathfrak{S}' = \cup_{\rho \in \mathbf{R}} \mathfrak{S}^\rho$, we define the function $IT(\zeta)$ as

$$IT(\zeta) = \overline{IT(\omega^\zeta)} = \overline{T(\tilde{I}\omega^\zeta)} = \overline{T(\{\bar{\zeta}|\cdot\})}.$$

One can easily check that for $f \in \mathscr{S}$, this new definition of If coincides with the old If defined as an integral transform. We have now immediately

Theorem 4.3: $\forall T \in \mathscr{S}'$, the function $IT(\zeta) = T(\{\bar{\zeta}|\cdot\})$ is a well-defined function on $E \times E$, with $IT \in \mathfrak{S}'$. This map $I: \mathscr{S}' \to \mathfrak{S}'$ is an isomorphism extending the isomorphism in Corollary 4.2.

Remarks: 1. The inverse map of $I: \mathscr{S}' \to \mathfrak{S}'$, with I defined as in (4.15), can again be constructed by combining $\tilde{I}$ and a limiting procedure. For instance,

$$\forall \varphi \in \mathfrak{S}': I^{-1}\varphi = \text{s}' - \lim_{k \to \infty} \tilde{I}(\varphi \cdot \chi_R). \tag{4.16}$$

2. One can enter in some detail into a discussion of I as an isomorphism between $\mathscr{S}'$ and $\mathfrak{S}'$, and compute explicit bounds on $|IT|_\rho$ for T in $(\mathscr{S}^k)'$, using the bounds in Theorem 4.1 (see Refs. 9b or 8).

So finally I defines an isomorphism between $\mathscr{S}$ and $\mathfrak{S}$ and between $\mathscr{S}'$ and $\mathfrak{S}'$. Moreover, we have two sets of interpolating spaces: the $\mathscr{S}^k (\mathscr{S}^k)'$ between $\mathscr{S}$ and $\mathscr{S}'$ and the $\mathfrak{S}^\rho$ between $\mathfrak{S}$ and $\mathfrak{S}'$, and we have at hand continuity statements and bounds for I between elements of these two interpolating chains, giving more detailed information on the action of I. Except for the two ends of the chain we have, however, no bicontinuity of I, considered as a map from $\mathscr{S}^k$ [or $(\mathscr{S}^k)'$] to a suitably chosen $\mathfrak{S}^\rho$. This problem will not occur with the chains of Hilbert spaces in the next subsection.

B. The Hilbert spaces $\mathscr{F}^\rho, W^\rho$, and related results concerning I

The Hilbert spaces $\mathscr{F}^\rho, W^\rho$ we define below constitute again two chains interpolating $\mathfrak{E}$ with $\mathfrak{E}'$, $\mathscr{S}$ with $\mathscr{S}'$ respectively. Actually the $\mathscr{F}^\rho$ spaces were already introduced in Ref. 9b as a tool for studying $\mathfrak{E}'$; they are weighted L^2 spaces of modified holomorphic functions. Their inverse images under the Bargmann integral transform were not displayed in Ref. 9b; we call these spaces W^ρ spaces; essentially they are the Hilbert spaces associated to the N-representation of $\mathscr{S}(E)$, $\mathscr{S}'(E)$ with respect to the harmonic oscillator-type operator $x^2 + p^2 - \frac{1}{4}\Delta_x - \frac{1}{4}\Delta_p$ (see, e.g., Ref. 14).

The $\mathscr{F}^\rho$ spaces

The $\mathscr{F}^\rho$ spaces are defined as ($\rho \in \mathbb{R}$)

$$\mathscr{F}^\rho = \left\{ \varphi \in Z(E_2); \|\varphi\|_\rho^2 = \int d\zeta\, (1 + |\zeta|^2)^\rho |\varphi(\zeta)|^2 < \infty \right\}, \tag{4.17}$$

with associated inner product:

$$(\varphi,\psi)_\rho = \int d\zeta\, (1 + |\zeta|^2)^\rho\, \overline{\varphi(\zeta)}\, \psi(\zeta). \tag{4.18}$$

The $\mathscr{F}^\rho$ spaces are Hilbert spaces $[\mathscr{F}^0 = \mathscr{L}_0(E_2)]$; one can check (see Appendix A or Ref. 9b) that the $u_{[k,l]}$ are orthogonal elements of the $\mathscr{F}^\rho$:

$$(u_{[k,l]}, u_{[k',l']})_\rho = \delta_{[k][k']}\, \delta_{[l][l']}\, \tau(\rho; |k| + |l|) \tag{4.19}$$

with

$$\tau(\rho;m) = \Gamma(m + 2n)^{-1} \int_0^\infty dx\, x^{m+2n-1} e^{-x} (1 + x)^\rho.$$

Moreover, for any $\phi \in \mathscr{F}^\rho$ with series expansion (2.18) one has

$$\|\phi\|_\rho^2 = \sum_{[k][l]} |\phi_{[k,l]}|^2 \tau(\rho; |k| + |l|), \tag{4.20}$$

and

$$\phi_{[k,l]} = \int d\zeta\, \overline{u_{[k,l]}(\zeta)}\, \phi(\zeta). \tag{4.21}$$

Equations (4.19) and (4.20) imply that the $\tau(\rho;|k| + |l|)^{-1/2} u_{[k,l]}$ constitute an orthonormal base of $\mathscr{F}^\rho$.

The following estimates for $\tau(\rho;m)$ were computed in Ref. 9b:

$$c'_\rho \leqslant \tau(\rho;m)(m + 2n)^{-\rho} \leqslant c''_\rho, \tag{4.22}$$

with

$$\left.\begin{aligned} c'_\rho &= \left(1 + \frac{\rho}{2n}\right)^{-1} \\ c''_\rho &= \left(1 + \frac{\rho}{2n}\right)^{\rho + 2n} e^{1-\rho} \end{aligned}\right\} \rho \geqslant 0,$$

$$\left.\begin{aligned} c'_\rho &= \left(1 - \frac{\rho}{2n}\right)^{-2n+\rho} e^{-1-\rho} \\ c''_\rho &= e^{-\rho} \end{aligned}\right\} \rho \leqslant 0. \tag{4.23}$$

The W^ρ-spaces

We put $\forall f \in \mathscr{S}(E), \forall \rho \in \mathbb{R}$:

$$(\|f\|_\rho^s)^2 = (f, (x^2 + p^2 - \tfrac{1}{4}\Delta_x - \tfrac{1}{4}\Delta_p + n)^\rho f). \tag{4.24}$$

Note: Actually, the operator $x^2 + p^2 - \frac{1}{4}\Delta_x - \frac{1}{4}\Delta_p$ has spectrum $\mathbb{N} \cap [n, \infty]$, which implies we could drop the extra term n in (4.24): the resulting topology on W would be exactly the same. We nevertheless introduce the extra term n in order to obtain the sharpest possible estimates on the integral transform I: to obtain these estimates, we shall use (4.22), where this extra n is already present.

We define then W^ρ as the closure of $\mathscr{S}(E)$ with respect to the norm $\|\,\|_\rho^s$; equipped with this norm, W^ρ is a Hilbert space.

The renormalized Hermite functions $(|r| + |s| + 2n)^{-\rho/2} H_{[r,s]}$ [see (3.14)] constitute an orthonormal base in W^ρ; one has

$$\forall T \in \mathscr{S}'(E): T \in W^\rho \Leftrightarrow \sum_{[r][s]} |T(H_{[r,s]})|^2 (|r| + |s| + 2n)^\rho < \infty \tag{4.25}$$

and

$$T \in W^\rho \Rightarrow (\|T\|_\rho^s)^2 = \sum_{[r][s]} |T(H_{[r,s]})|^2 (|r| + |s| + 2n)^\rho$$

(see, e.g., Sec. V.5 in Ref. 13).

Because of (3.13) and (3.15), we can rewrite (4.25) in terms of the $h_{[k,l]}$:

$$\forall T \in \mathscr{S}'(E): \text{define } T_{[k,l]} = \overline{T(\overline{h_{[k,l]}})} = \overline{T(h_{[l,k]})}. \tag{4.26}$$

Then

$$\forall T \in W^\rho: (\|T\|_\rho^s)^2 = \sum_{[k][l]} |T_{[k,l]}|^2 (|k| + |l| + 2n)^\rho. \tag{4.27}$$

The integral transform I as a map from W^ρ onto $\mathscr{F}^\rho$

From the definitions of $W^\rho, \mathscr{F}^\rho$ one can check that

$$\bigcap_{\rho \in \mathbb{R}} W^\rho = \mathscr{S}(E), \quad \bigcup_{\rho \in \mathbb{R}} W^\rho = \mathscr{S}'(E),$$

$$\bigcap_{\rho \in \mathbb{R}} \mathscr{F}^\rho = \mathfrak{E}, \quad \bigcup_{\rho \in \mathbb{R}} \mathscr{F}^\rho = \mathfrak{E}'.$$

The extended definition (4.15) of the integral transform I can therefore be applied to all W^ρ; for any $T \in W^\rho$, the resulting IT will be in $\mathfrak{E}'$ and have series expansion

$$IT(\zeta) = \sum_{[m][n]} (IT)_{[m,n]} u_{[m,n]}(\zeta),$$

with

$$\begin{aligned} (IT)_{[m,n]} &= \int d\zeta\, \overline{u_{[m,n]}(\zeta)}\, IT(\zeta) \quad [\text{use } (4.21)] \\ &= \overline{IT(u_{[m,n]})} \quad [\text{use } (4.14)] \\ &= \overline{T(h_{[n,m]})} \quad [\text{use } (4.15) \text{ and Proposition 3.1}] \\ &= T_{[m,n]}. \end{aligned} \tag{4.28}$$

Using the definitions of the norms $\|\,\|_\rho$ and $\|\,\|_\rho^s$, and the estimates (4.22) we see now that

$$T \in W^\rho \Leftrightarrow IT \in \mathscr{F}^\rho \tag{4.29}$$

and

$$c_\rho'(\|T\|_\rho^s)^2 < \|IT\|_\rho^2 < c_\rho''(\|T\|_\rho^s)^2. \tag{4.30}$$

Hence the following theorem.

Theorem 4.4: The map I: $IT(\zeta) = \overline{T(\{\zeta|\})}$ defines an isomorphism from W^ρ onto $\mathscr{F}^\rho$, and this $\rho\in\mathbf{R}$. Estimates on the norms of this isomorphism and its inverse are given by

$$\|I\|_{W^\rho\to\mathscr{F}^\rho} < c_\rho''^{1/2}, \|I^{-1}\|_{\mathscr{F}^\rho\to W^\rho} < c_\rho'^{-1/2}, \tag{4.31}$$

where C_ρ' and C_ρ'' are defined by (4.23).

Remarks: As we announced before, the restriction of I to a W^ρ is a bijection onto $\mathscr{F}^\rho$, which means we have no qualitative loss of information when mapping to and fro (this was not the case for the $\mathscr{S}^k, \mathfrak{S}^\rho$). Due to the fact that the product of the estimates on the norms in (4.31) is larger than 1, we have, however, still a "quantitative" loss of information, which gets worse for large $|\rho|$.

Up to now, we have considered the spaces $\mathscr{S}^k\mathfrak{S}^\mu$ and later $\mathscr{F}^\rho, W^\rho$, in order to obtain some fine structure in the study of I as an isomorphism from $\mathscr{S}'$ to $\mathfrak{S}'$; it turns out that the Hilbert spaces $\mathscr{F}^\rho, W^\rho$ are better suited to this end than the Banach spaces $\mathscr{S}^k, \mathfrak{S}^\rho$. Our ultimate aim is to use these results to derive properties of the Weyl quantization procedure, using the fact, mentioned in the Introduction, that the integral transform I constitutes the link between a classical function and the coherent state matrix elements of its quantal counterpart. Theorems 4.1 and 4.4 can then be used to translate restrictions on a tempered distribution to growth restrictions on the coherent state matrix elements of the corresponding operator. A first application of Theorem 4.1 was given in Ref. 6, where it was also noted that stronger results could be obtained by means of Theorem 4.4. Other applications shall be given in Ref. 15.

C. The Hilbert spaces $\mathscr{F}^G, W^\phi$

We shall here generalize the structures of both $\mathscr{F}^\rho, W^\rho$ to obtain Hilbert spaces larger than $\mathscr{S}'$, and which can still be handled by I.

The Hilbert spaces $\mathscr{F}^G$

$\mathscr{F}^\rho$ was constructed as a weighted L^2 space of $Z(E_2)$ functions, with the special weight $(1+|\zeta|^2)^\rho$. To generalize this construction, we consider now more general weights.

Let G be a function from $\mathbb{R}^+$ to $\mathbb{R}^+$. We define

$$\mathscr{F}^G = \{\phi\in Z(E_2); \|\phi\|_G^2 = \int d\zeta\, |\phi(\zeta)|^2 G(|\zeta|^2) < \infty\} \tag{4.32}$$

and we equip this space with the norm $\|\ \|_G$. Since one has to be careful with Hilbert spaces of analytic functions, we shall first investigate the conditions to impose on G to ensure that $\mathscr{F}^G$ is an infinitely dimensional Hilbert space (see also Ref. 16).

Proposition 4.5:

1. If $\forall r\in\mathbb{R}^+$: $\operatorname{ess\,inf}_{x<r} G(x) > 0, \operatorname{ess\,sup}_{x<r} G(x) < \infty$, $\qquad$ (4.33)

then $\mathscr{F}^G$ is complete.

2. Define

$$\lambda_m^G = \frac{1}{\Gamma(m+2n)}\int_0^\infty dx\, x^{m+2n-1}e^{-x}G(x). \tag{4.34}$$

A necessary and sufficient condition for $\mathscr{F}^G$ to be infinitely dimensional is

$$\forall m: \lambda_m^G < \infty. \tag{4.35}$$

If this condition is satisfied, then

$$\forall [k],[l]: u_{[k,l]}\in\mathscr{F}^G$$

and

$$\forall\phi = \sum_{[k][l]}\phi_{[k,l]}u_{[k,l]}\in\mathscr{F}^G: \|\phi\|_G^2 = \sum_{[k][l]}|\phi_{[k,l]}|^2\lambda^G_{|k|+|l|}. \tag{4.36}$$

The following three conditions involve not only G, but also $1/G$:

3. If

$$\forall m: \lambda_m^{1/G} < \infty, \text{ then } \forall\phi\in\mathscr{F}^G: \phi_{[k,l]} = \int d\zeta\, \overline{u_{[k,l]}(\zeta)}\phi(\zeta) \tag{4.37}$$

[i.e., (2.20) holds for all ϕ in $\mathscr{F}^G$].

4. If

$$\lim_{m\to\infty}\lambda_m^{1/G}(m\lambda_{m-1}^{1/G})^{-1} = 0, \tag{4.38}$$

then

$$\forall\phi\in\mathscr{F}^G: \phi(\zeta) = \int d\xi\, \overline{\omega^\zeta(\xi)}\phi(\xi) \tag{4.39}$$

[i.e., the reconstruction property (2.21) still holds for $\mathscr{F}^G$].

5. If

$$\exists K_G', K_G'' > 0 \text{ such that } \forall m: K_G' < \lambda_m^G\lambda_m^{1/G} < K_G'', \tag{4.40}$$

then $\mathscr{F}^{1/G}$ can be identified with the dual, $(\mathscr{F}^G)'$, of $\mathscr{F}^G$, by means of the map

$$\mathscr{F}^{1/G}\to(\mathscr{F}^G)',$$

$$\psi\mapsto L_\psi \text{ with } L_\psi(\phi) = \int d\zeta\, \overline{\psi(\zeta)}\phi(\zeta) \tag{4.41}$$

$$= \sum_{[k][l]}\overline{\psi_{[k,l]}}\phi_{[k,l]}. \tag{4.42}$$

Proof:

1. Using

$$\phi(\zeta) = (\pi r^2)^{-2n}\omega_2(\zeta)\int_{|\zeta'-\zeta|<r} d\zeta'\phi(\zeta')\omega_2(\zeta')^{-1},$$

and (4.33) one can check that

$$\forall R: \exists K_{G,R} \text{ such that } \forall\zeta, |\zeta|<R: |\phi(\zeta)| < K_{G,R}\|\phi\|_G. \tag{4.43}$$

Hence, convergence with respect to $\|\ \|_G$ automatically entails uniform convergence on compact sets. Therefore, any Cauchy sequence in $\mathscr{F}^G$ has a limit in $\mathscr{F}^G$, and $\mathscr{F}^G$ is complete.

2. Proposition A1 in Appendix A proves that $\forall\phi\in Z(E_2)$:

$$\int d\zeta\, |\phi(\zeta)|^2 G(|\zeta|^2) = \sum_{[k][l]}|\phi_{[k,l]}|^2\lambda^G_{|k|+|l|}, \tag{4.44}$$

where these expressions can be finite or infinite. If $\lambda_m^G < \infty$ for all m, then (4.44) shows that all the $u_{[k,l]} \in \mathscr{F}^G$; (4.36) is proved in Appendix A. If $\lambda_m^G = \infty$, then $\forall m' \geqslant m: \lambda_{m'}^G = \infty$ [use (4.33)]. Hence, $\forall \phi \in \mathscr{F}^G: \phi_{[k,l]} = 0$ if $|k| + |l| \geqslant m$ and $\mathscr{F}^G$ is finite-dimensional.
($\mathscr{F}^G \subset \operatorname{span}\{u_{[k,l]}; |k| + |l| < m\}$).

3. If $\forall m: \lambda_m^{1/G} < \infty$, then

$$\forall [k],[l]: u_{[k,l]} \in \mathscr{F}^{1/G} \subset L^2\left(E \times E; \frac{d\zeta}{G(|\zeta|^2)}\right),$$

(4.37) is then a consequence of Proposition A2 in Appendix A.

4. Equation (4.38) implies
$\forall \zeta: \omega^\zeta \in \mathscr{F}^{1/G} \subset (\mathscr{F}^G)'$

$$\left(\text{use} \sum_{\substack{[k][l] \\ |k| + |l| = m}} |u_{[k,l]}(\zeta)|^2 = e^{-|\zeta|^2} \frac{|\zeta|^{2m}}{m!}\right),$$

apply Proposition A2.

5. See Proposition 4.3 in Ref. 16.

Note that once (4.40) is satisfied, (4.39) holds automatically;

$$(4.40) \Rightarrow (\mathscr{F}^G)' \cong \mathscr{F}^{1/G}.$$

Because of (4.43), $\exists \tilde{\omega}^\xi \in \mathscr{F}^{1/G}$ such that

$$\forall \phi \in \mathscr{F}^G: \int \overline{\tilde{\omega}^\xi(\zeta)} \phi(\zeta) = \phi(\xi).$$

In particular

$$\int d\zeta\ \overline{\tilde{\omega}^\xi(\zeta)} u_{[k,l]}(\zeta) = u_{[k,l]}(\xi) \Rightarrow \tilde{\omega}^\xi = \omega^\xi. \qquad \blacksquare$$

Examples:

1. Take $G(x) = (1 + x)^\rho$. This weight satisfies all the conditions in Proposition 4.5; the corresponding $\mathscr{F}^G$ spaces are of course exactly the $\mathscr{F}^\rho$ of Sec. 4B.

2. Another possibility is $G(x) = e^{\beta x}$, with $|\beta| < 1$. This choice for G satisfies (4.33), (4.35), and (4.38); one has $\lambda_m^G = (1 - \beta)^{-m-2n}$, from which one clearly sees that the duality condition (4.40) is not satisfied.

3. $G(x) = e^{\beta\sqrt{x}}$. This corresponds to a simple exponential weight for $\mathscr{F}^G$:

$$\|\phi\|_G^2 = \int d\zeta\, |\phi(\zeta)|^2 e^{\beta|\zeta|}.$$

This choice also satisfies all the conditions in Proposition 4.5 (see below).

4. A rather general class of interesting weight functions is given by taking $G = F_\rho^{q,\tau}$,
with

$$F_\rho^{q,\tau}(x =)(1 + x)^\rho e^{\tau x^q} \rho \in \mathbb{R}, \tau \in \mathbb{R} \setminus \{0\}, q \in (0,1). \tag{4.45}$$

For all the values of the parameters indicated above, $F_\rho^{q,\tau}$ satisfies (4.33), (4.35), and (4.38).

A detailed analysis of the asymptotic behavior of $\lambda_m^{F_\rho^{q,\tau}}$ yields (see Appendix B)

$$\forall q \in \left(1 - \frac{1}{n}, 1 - \frac{1}{n+1}\right), \quad n = 1,2,\dots,$$

$$\lambda_m^{F_\rho^{q,\tau}} \underset{m \to \infty}{\sim} \text{const} \times m^\rho \exp[\tau m^q + A_1 m^{2q-1} + \cdots + A_{n-2} m^{nq-(n-1)}](1 + O(m^\alpha)), \tag{4.46}$$

with

$$\alpha = \max(q - 1; q + n(q - 1)).$$

For $q \leqslant \frac{1}{2}$ this specializes to

$$\lambda_m^{F_\rho^{q,\tau}} \sim \text{const} \times m^\rho e^{\tau m^q}[1 + O(m^\alpha)], \tag{4.47}$$

which implies the duality condition (4.40) is satisfied for $q \leqslant \frac{1}{2}$; for $q > \frac{1}{2}$ it is not.

The Hilbert spaces W^ϕ.

The $\mathscr{F}^G$ spaces were constructed on the same principle as the $\mathscr{F}^\rho$ spaces, with more general weight functions. We shall likewise generalize the construction of the W^ρ spaces. Let $\phi(m)$ be any sequence of strictly positive real numbers. We define

$$(\|f\|_\phi^s)^2 = \sum_{[r][s]} |(f, H_{[r,s]})|^2 \phi(|r| + |s| + n)$$
$$= \sum_{[k][l]} |(f, h_{[k,l]})|^2 \phi(|k| + |l| + n). \tag{4.48}$$

The set of all functions f in $\mathscr{S}$ for which $\|f\|_\phi$ is finite we call $\mathscr{S}_\phi$; W^ϕ is then the completion of $\mathscr{S}_\phi$ with respect to $\|\ \|_\phi$.

We can, of course, as in (4.24), consider W^ϕ as the natural domain of $\phi(x^2 + p^2 - \frac{1}{4}\Delta_x - \frac{1}{4}\Delta_p)^{1/2}$, and put $\|f\|_\phi^s = \|\phi(x^2 + p^2 - \frac{1}{4}\Delta_x - \frac{1}{4}\Delta_p)^{1/2} f\|$.

Examples:

1. Taking $\phi_\rho(l) = (l + n)^\rho$, one has $\mathscr{S}_{\phi_\rho} = \mathscr{S}$ (it is only for ϕ increasing faster than polynomials that $\mathscr{S}_\phi$ may become a proper subset of $\mathscr{S}$), and $W^{\phi_\rho} = W^\rho$.

2. The $H(\alpha, A)$, $H(\overline{\alpha}, \overline{A})$ spaces, introduced in Ref. 17, are a special case of a W^ϕ-structure. For the detailed definition we refer to Ref. 17; a survey is given in Appendix C (our definitions are slightly adjusted to deal with the dilation in $x^2 + p^2 - \frac{1}{4}\Delta_x - \frac{1}{4}\Delta_p$ with respect to a normal harmonic oscillator). Essentially the $H(\alpha, A)$ form a scale of spaces of test functions "of type S" and their duals $H(\overline{\alpha}, \overline{A})$ a scale of Hilbert spaces of distributions or generalized functions of type S.[18] They are defined (see Appendix C) as W^ϕ spaces with:

for $H(\alpha, A)$:

$$\phi(k + n) = \gamma_k^{-2}(\alpha, A) = \sum_{m=0}^{\infty} \frac{a(m;k)}{A^{2m}\Gamma^2(\alpha m)}, \tag{4.49}$$

for $H(\overline{\alpha}, \overline{A})$:

$$\phi(k + n) = \gamma_k^2(\alpha, A),$$

where the $a(m;k)$ are numbers satisfying

$$2^{-m} \frac{\Gamma(k + 2n + m)}{\Gamma(k + 2n)} \leqslant a(m;k) \leqslant \frac{\Gamma(k + n + m)}{\Gamma(k + n)} \tag{4.50}$$

[for the exact definition of $a(m;k)$, see Appendix C]. For all (α, A) with $\alpha < \frac{1}{2}$, A arbitrary, or $\alpha = \frac{1}{2}$, $A > \sqrt{2}$, $H(\alpha, A)$ is an infinitely dimensional Hilbert space, with orthonormal basis $\gamma_{|k|+|l|}(\alpha, A) h_{[k,l]}$; $H(\overline{\alpha}, \overline{A})$ is its dual: for any $f \in H(\alpha, A)$, the action of $T \in H(\overline{\alpha}, \overline{A})$ on f is simply the natural extension of the action of elements of $\mathscr{S}'$ on $H(\alpha, A)$:

$$T(f) = \sum_{[k][l]} T(h_{[k,l]})(h_{[k,l]}, f). \tag{4.51}$$

In Ref. 17 it was shown that $\forall(\alpha, A)$ satisfying the restriction above, $\exists C(\alpha, A)$ such that $\forall f \in H(\alpha, A)$:

$$|f(x,p)| < C(\alpha, A) \|f\|_{\alpha,A} \prod_{j=1}^{n} (x_j p_j)^{n+2} \times \exp\left[-\frac{1}{2n(2A)^{1/\alpha}} \sum_{j=1}^{n} (x_j^{1/\alpha} + p_j^{1/\alpha}) \right]. \qquad (4.52)$$

In the extreme case $\alpha = \frac{1}{2}$, $A > \sqrt{2}$ this becomes

$$|f(x,p)| < C(\tfrac{1}{2}, A) \|f\|_{1/2,A} \prod_{j=1}^{n} (x_j p_j)^{n+2} \exp\left[-\frac{x^2 + p^2}{8nA^2} \right], \qquad (4.53)$$

i.e., f has a Gaussian-like behavior at infinity.

Another property proved in Ref. 17 is the following.

$\forall \alpha \in (\frac{1}{2}, 1), \forall f \in H(\alpha, A)$:

f is the restriction to the reals of an entire analytic function of order $\rho < (1 - \alpha)^{-1}$.

The integral transform I as a map from W^ϕ to $\mathscr{F}^G$ and vice versa

Looking back at the arguments leading to the formulation of Theorem 4.4, we see that the estimates (4.22) played a crucial role in the proof of the bijectivity of I between W^ρ and $\mathscr{F}^\rho$. In the case of a general W^ϕ–$\mathscr{F}^G$ pair, we shall use again such estimates.

Theorem 4.5: Let $\mathscr{F}^G$, W^ϕ be two Hilbert spaces as defined above [with G satisfying (4.33), (4.35)]:

(1) If $\exists\, K_1 > 0$ such that $\forall m \in \mathbb{N}: K_1 \phi(m+n) > \lambda_m^G$, $\qquad$ (4.54)

then I can be considered as a bounded linear map from W^ϕ to $\mathscr{F}^G$, with

$$\forall T \in W^\phi: IT(\zeta) = \sum_{[k][l]} \overline{T(h_{[l,k]})}\, u_{[k,l]}(\zeta), \qquad (4.55)$$

where the series converges uniformly on compact sets. Moreover,

$$\forall T \in W^\phi: \|IT\|_G < K_1^{1/2} \|T\|_\phi. \qquad (4.56)$$

(2) *If* $\exists\, K_2 > 0$ such that $\forall m \in \mathbb{N}: K_2 \phi(m+n) < \lambda_m^G$, (4.57)

then $\tilde{I}$ can be extended to a bounded linear map from $\mathscr{F}^G$ to W^ϕ with

$$\forall \Phi = \sum_{[k][l]} \Phi_{[k,l]} u_{[k,l]} \in \mathscr{F}^G: \tilde{I}\Phi = W^\phi - \lim_{m\to\infty} \sum_{\substack{[k][l] \\ |k| + |l| < m}} \Phi_{[k,l]} h_{[l,k]}. \qquad (4.58)$$

One has

$$\|\tilde{I}\Phi\|_\phi < K_2^{-1/2} \|\Phi\|_G.$$

(3) If $\exists\, K_1, K_2 > 0$ such that

$$\forall m: K_1 \phi(m+n) < \lambda_m^G < K_2 \phi(m+n), \qquad (4.59)$$

then I as defined by 4.55) is an isomorphism from W^ϕ onto $\mathscr{F}^G$, with inverse $\tilde{I}$ [as defined by (4.58)]

Proof: $I, \tilde{I}$ are already defined on the finite linear combinations of the $h_{[k,l]}, u_{[k,l]}$, respectively. The bounds (4.54), (4.57) ensure that $I, \tilde{I}$ can be extended as indicated. Formula (4.55) is a consequence of the fact that $\|\ \|_G$ convergence implies uniform and absolute convergence on compact sets. ∎

Note: Of course, one can always define $\mathscr{F}^G$ first, and then take $\phi(m+n) = \lambda_m^G$; for this particular W^ϕ space, the theorem becomes trivial. A W^ϕ space defined in this way would, however, be rather useless because of its too intrinsic definition: we are more interested in the situation where $\mathscr{F}^G$ and W^ϕ are defined separately, but where nevertheless a link can be established via I or $\tilde{I}$. This was the case for the $\mathscr{F}^\rho$ and the W^ρ spaces: the W^ρ spaces made sense as ladder spaces in the N-representation of $\mathscr{S}'(E)$, and the $\mathscr{F}^\rho$ spaces were polynomially weighted L^2 spaces of modified holomorphic functions. Results such as Theorem 4.4 (or Theorem 4.5 with explicit $\mathscr{F}^G - W^\phi$ pairs) can then be used to characterize the behavior of the coherent state matrix elements of an operator by means of the properties of the corresponding classical function (or distribution) or vice versa.

An example of corresponding W^ϕ–$\mathscr{F}^G$ pairs different from the W^ρ–$\mathscr{F}^\rho$ pairs in Sec. 4B is given in the following subsection.

The action of the integral transform I on Hilbert spaces of distributions of type S

We shall study in this subsection the action of I on the Hilbert spaces $H(\alpha, A)$, $H(\bar{\alpha}, \bar{A})$ defined above. In order to be able to apply Theorem 4.5, we have to find suitable weight functions ${}_jG_{\alpha,A}$, ${}_j\bar{G}_{\alpha,A}$ such that

$$K_2' \lambda_m^{{}_2G_{\alpha,A}} < \gamma_m^{-2}(\alpha, A) < K_1' \lambda_m^{{}_1G_{\alpha,A}},$$
$$\bar{K}_2' \lambda_m^{{}_2\bar{G}_{\alpha,A}} < \gamma_m^2(\alpha, A) < \bar{K}_1' \lambda_m^{{}_1\bar{G}_{\alpha,A}} \qquad (4.60)$$

for some $K_1', K_2', \bar{K}_1', \bar{K}_2' > 0$.

Using the bounds (4.50) we can easily construct the ${}_jG_{\alpha,A}$ functions. Indeed we have

$$\sum_{l=0}^{\infty} \frac{2^{-l}}{A^{2l}\Gamma^2(\alpha l)} \frac{\Gamma(m+2n+l)}{\Gamma(m+2n)} < \gamma_m^{-2}(\alpha, A) < \sum_{l=0}^{\infty} \frac{1}{A^{2l}\Gamma^2(\alpha l)} \frac{\Gamma(m+n+l)}{\Gamma(m+n)}.$$

Since it is clear from (4.34) that for

$$\hat{G}_{\beta,B}(x) = \sum_{j=0}^{\infty} \frac{x^j}{\Gamma^2(\beta j) B^{2j}},$$

the corresponding λ_m^G are given by

$$\lambda_m^{\hat{G}_{\beta,B}} = \sum_{j=0}^{\infty} \frac{1}{B^{2j}\Gamma^2(2j)} \frac{\Gamma(m+2n+j)}{\Gamma(m+2n)},$$

we immediately have

$$\lambda_m^{\hat{G}_{\alpha,\sqrt{2}A}} \leqslant \gamma_m^{-2}(\alpha, A) < \lambda_m^{\hat{G}_{\alpha,A}}. \qquad (4.61)$$

To find candidates for the functions ${}_j\bar{G}_{\alpha,A}$, we have to do a little more work. We shall study the asymptotic behavior of the $\lambda_m^{G_{\beta,B}}$, then invert (4.61) and try to find suitable ${}_j\bar{G}_{\alpha,A}$.

$\hat{G}_{\beta,B}$ as defined above is typically an entire function of finite order. Computing its order and type we find[19]

$$\rho(\beta, B) = \overline{\lim_{n\to\infty}} \frac{m \ln n}{\ln(\Gamma^2(\beta n) B^{2n})} = \frac{1}{2\beta},$$

$$\tau(\beta, B) = \frac{1}{2\rho(\beta, B)} \overline{\lim_{n\to\infty}}\, n\, [\Gamma^2(\beta n) B^{2n}]^{-1/2\beta n} = 2B^{-1/\beta}.$$

So $\hat{G}_{\beta,B}$ is an entire function of growth $\leqslant (1/2\beta, 2B^{-1/\beta})$. Since the positive real axis is the direction of fastest growth for $\hat{G}_{\beta,B}$, this implies that

$$\forall \tau' < \tau(\beta,B),\ \forall \tau'' > \tau(\beta,B),\ \exists K',K'' > 0$$

such that

$$\forall x \in \mathbb{R}^+ : K' F^{(2\beta)^{-1},\tau'}(x) \leqslant \hat{G}_{\beta,B}(x) \leqslant K'' F^{(2\beta)^{-1},\tau''}(x)$$

[we define $F^{q,\tau}(x) = F_0^{q,\tau}(x) = \exp(\tau x^q)$]

and hence

$$K' \lambda_m^{F^{(2\beta)^{-1},\tau'}} \leqslant \lambda_m^{\hat{G}_{\beta,B}} \leqslant K'' \lambda_m^{F^{(2\beta)^{-1},\tau''}}.$$

Using now the estimates (4.46), inverting (4.61), these inequalities imply that

$$\forall \alpha > \tfrac{1}{2}; \forall \tau_2 > \tau(\alpha, A) = 2A^{-1/2}, \forall \tau_1 < \tau(\alpha, \sqrt{2}A) = 2(\sqrt{2}A)^{-1/2} : \exists K_1, K_2 > 0$$

such that

$$K_2 \lambda_m^{F^{(2\alpha)^{-1},-\tau_2}} \leqslant \gamma_m^2(\alpha, A) \leqslant K_1 \lambda_m^{F^{(2\alpha)^{-1},-\tau_1}}. \qquad (4.62)$$

Since this inequality has exactly the right form of (4.60), we are now in a position to apply Theorem 4.5; we get the following results.

Theorem 4.6: For any $q \in (0,1), \tau \in \mathbb{R}$, we define $F^{q,\tau}(x) = e^{\tau x^q} (x \in \mathbb{R}_+)$; for any (β,B) with $\beta > \frac{1}{2}$ or $\beta = \frac{1}{2}$, $B > \sqrt{2}$, we define

$$\hat{G}_{\beta,B}(x) = \sum_{n=0}^{\infty} \frac{x^n}{\Gamma^2(\beta n) B^{2n}}.$$

Take any $[\alpha, A)$ with $\alpha > \frac{1}{2}$ or $\alpha = \frac{1}{2}, A > \sqrt{2}$. Then

(1) The integral transform I defines a continuous linear map from the Hilbert space $H(\alpha, A)$ of functions of type S to the weighted L^2 space of holomorphic functions $\mathscr{F}^{\hat{G}_{\alpha,\sqrt{2}A}}$: we have

$$\forall f \in H(\alpha, A) : If(\zeta) = \int dv\, \{\zeta | v\} f(v)$$

and

$$\|If\|^2_{\hat{G}_{\alpha,\sqrt{2}A}} = \int d\zeta\, |If(\zeta)|^2 \hat{G}_{\alpha,\sqrt{2}A}(|\zeta|^2) \leqslant \|f\|^2_{\alpha,A}. \qquad (4.63)$$

(2) The integral transform $\tilde{I}$ defines a continuous linear map from $\mathscr{F}^{\hat{G}_{\alpha,A}}$ to

$$H(\alpha, A) : \forall \phi \in \mathscr{F}^{\hat{G}_{\alpha,A}} : (\tilde{I}\phi)(v) = \int d\zeta\, \{\bar{\zeta} | v\} \phi(\zeta)$$

and

$$\|\tilde{I}\phi\|^2_{\alpha,A} \leqslant \int d\zeta\, |\phi(\zeta)|^2 \hat{G}_{\alpha,A}(|\zeta|^2) = \|\phi\|^2_{\hat{G}_{\alpha,A}}. \qquad (4.64)$$

For the next two results, we restrict ourselves to the case $\alpha > \frac{1}{2}$.

(3) $\forall \tau_2 > \tau(\alpha, A) = 2A^{-1/2}$, the integral transform I extends to a continuous linear map from the Hilbert space $H(\bar{\alpha}, \bar{A})$ of distributions of type S to the weighted L^2 space of holomorphic functions $\mathscr{F}^{F^{(2\alpha)^{-1},-\tau_2}}$:

$$\forall T \in H(\bar{\alpha}, \bar{A}) : IT(\zeta) = \sum_{[k][l]} \overline{T(h_{[l,k]})} u_{[k,l]}(\zeta) = \overline{T(\{\bar{\xi} | \cdot\})} \qquad (4.65)$$

and

$$\exists K \text{ such that } \|IT\|^2_{F^{(2\alpha)^{-1},-\tau_2}} = \int d\zeta\, |IT(\zeta)|^2 e^{-\tau_2 |\zeta|^{1/\alpha}} \leqslant K \|T\|^2_{\bar{\alpha},\bar{A}}.$$

(4) $\forall \tau_1 < \tau(\alpha, \sqrt{2}A) = 2(\sqrt{2}A)^{-1/\alpha}$, $\tilde{I}$ extends to a continuous linear map from $\mathscr{F}^{F^{(2\alpha)^{-1},-\tau_1}}$ to $H(\bar{\alpha}, \bar{A})$:

$$\forall \phi \in \mathscr{F}^{F^{(2\alpha)^{-1},-\tau_1}} : \tilde{I}\phi = H(\bar{\alpha}, \bar{A}) - \lim_{R \to \infty} \int_{|\zeta| < R} d\zeta\, \{\bar{\xi} \cdot\} \phi(\zeta) \qquad (4.66)$$

and

$$\exists K' > 0 \text{ such that } K' \|\tilde{I}\phi\|^2_{\bar{\alpha},\bar{A}} \leqslant \int d\zeta\, |\phi(\zeta)|^2 e^{-\tau_1 |\zeta|^{1/\alpha}} = \|\phi\|^2_{F^{(2\alpha)^{-1},-\tau_1}}.$$

For $\alpha = \frac{1}{2}, A > \sqrt{2}$ we have

(5) $\forall y > (A^2/2 - 1)^{-1}$, the integral transform I extends to a continuous linear map from $H(\bar{\frac{1}{2}}, \bar{A})$ to the weighted L^2 space $\mathscr{F}^{G_{-y}}$, where

$$G_a(x) = e^{ax} \quad (a < 1).$$

We have $\forall T \in H(\bar{\frac{1}{2}}, \bar{A}) : IT(\zeta) = \overline{T(\{\bar{\zeta} | \cdot\})}$ and

$$\|IT\|^2_{G_{-y}} = \int d\zeta\, |IT(\zeta)|^2 e^{-y|\zeta|^2} \leqslant (1+y)^{-n+1} K_{\alpha,A}(y^{-1})^{-1} \|T\|^2_{\overline{1/2},\bar{A}},$$

with

$$K_{\alpha,A}(z) = \sum_{m=0}^{\infty} \frac{m!}{\Gamma^2(\alpha m) A^{2m}} (z+2)^m. \qquad (4.67)$$

Proof: (1)–(4) were essentially proven above. Since $\forall (\alpha, A) : \{\zeta | \cdot\} \in H(\alpha, A)$, we can always write $IT(\zeta)$ as $\overline{T(\{\bar{\zeta} | \cdot\})}$. For (5) we use the estimate $\gamma_m^{-2}(\frac{1}{2}, A) \leqslant (1+y)^{m+n-1} K_{\alpha,A}(y^{-1})$ (see Appendix C). Since $\lambda_m^{G_a} = (1-a)^{-(m+2n)}$, (4.67) follows. ■

Remarks: As we already mentioned previously, our motivation for this detailed study of the integral transforms $I, \tilde{I}$ is their relation with the Weyl quantization procedure [see (1.1) and (1.6)]. Possible applications of Theorem 4.6 in this quantization context are, e.g., the following.

1. In Ref. 17 it was shown that for $\alpha > 1$, the functions in $H(\alpha, A)$ are the restrictions to the real line of entire functions of order $(1-\alpha)^{-1}$. On these $H(\alpha, A)$ one can therefore define the complex δ functions $\delta_{v+iw} : \mapsto (v+iw)$ as continuous linear forms, i.e., as elements of $H(\bar{\alpha}, \bar{A})$ (see Ref. 17). By means of the integral transform I, and applying Theorem 4.6, one can therefore quantize these δ functions with complex argument. The same can be done for the real exponentials e^{ax+bp}; the quantal operators corresponding to both these functions are actually complex translation operators, and can therefore be useful in the study of certain resonance problems. Complex dilations also can be obtained as quantizations of $H(\bar{\alpha}_{\frac{1}{2}}, \bar{A})$-objects (at least for the dilation parameter θ in some strip of the complex plane).

2. Using the $\tilde{I}$ results, the statements in Theorem 4.6 enable one to dequantize certain families of operators with coherent state matrix elements with fast growth (up to Gaus-

sian-like growth) in the coherent state labels, and to derive properties of the corresponding classical functions.

5. THE INTEGRAL TRANSFORM I ACTING ON FUNCTIONS FACTORIZING INTO A PRODUCT OF A FUNCTION DEPENDING ON x WITH A FUNCTION DEPENDING ON p

Whereas for "dequantization procedures" it may be useful to know how to treat an operator in which the x and p parts cannot be disentangled, for quantization purposes one is mostly interested in functions depending only on x or on p or linear combinations of such functions. We shall therefore indicate here how the additional information that a given function is factorizable, $f(x,p) = f_1(x)\cdot f_2(p)$ or, depending only on x or on p, $f(x,p) = f_1(x)$, $f(x,p) = f_2(p)$, can be used to sharpen the results derived in the preceding section. To achieve this, we shall use the decomposition (4.3) of the integral kernel $\{a,b\,|\,v\}$:

$$\{a,b\,|\,v\} = K_B(c_{ab};x_v)K_B(d_{ab};p_v), \tag{5.1}$$

where

$$K_B(c;y) = K_B((c_1,c_2);y) = 2^{n/2}\pi^{n/4}A\,(c_1,c_2;\sqrt{2}y)e^{-(1/4)(c_1^2+c_2^2)} \tag{5.2}$$

($y\in\mathbb{R}^n; c\in\mathbb{R}^{2n} = \mathbb{R}^n\oplus\mathbb{R}^n$), with A given by (4.1), and where

$$c_{ab} = \frac{1}{\sqrt{2}}(x_a + x_b, p_b - p_a),$$

$$d_{ab} = \frac{1}{\sqrt{2}}(p_a + p_b, x_a - x_b). \tag{5.3}$$

One immediately sees from (5.1) that the integral transform I, when applied to a factorizable function $f(x,p) = f_1(x)f_2(p)$, splits into two pieces:

$$If(a,b) = I_B f_1(c_{ab}) I_B f_2(d_{ab}), \tag{5.4}$$

with $\forall g$ function on $\mathbb{R}^n$, $\forall d = (d_1, d_2)\in\mathbb{R}^{2n}$:

$$(I_B\, g)(d_1, d_2) = (2\pi)^{-n/2}\int_{\mathbb{R}^n} d^n y K_B(d;y)\,g(y). \tag{5.5}$$

It is not difficult to check that the integral transform I_B with kernel K_B has exactly the same properties as the integral transform I, except that all the dimensions have to be halved. Since the exact value of the dimension n plays no role whatever in the results derived up until now, we see that all the results for I hold also for I_B, provided we replace each n by $n/2$.

We give below a list of bounds on $I(f_1\cdot f_2)$ which can be obtained in this way. For all the cases where the images $I_B f_1$, $I_B f_2$ cannot be defined directly (i.e., $f_1, f_2 \notin L^\infty + L^2$), we define I_B as a continuous extension of the integral transform with kernel K_B (just as we did for I).

In the case where $f_2 = 1$, i.e., where the function f depends only on x (the case $f_1 = 1$ is completely similar), one of the factors in (5.4) can be calculated explicitly:

$$I(f_1\cdot 1)(a,b) = I_B f_1)(c_{ab}) \times\exp\left[-\frac{(x_a - x_b)^2}{4} + \frac{i(p_a + p_b)\cdot(x_a - x_b)}{4}\right]. \tag{5.6}$$

We also give some bounds for this special case.

Examples:

1. Define $\mathscr{S}^k = \{\, g: \mathbb{R}^n \to \mathbb{C}; g$ is C^k and

$$|\,g\,|_k^s = \max_{|m|<k} \sup |2^{-|m|/2}(1 + 2|\,y|^2)^{(k-|m|)/2} \nabla^{[m]} g(y)| < \infty \}.$$

Take $f_1\in\mathscr{S}^{k_1}$, $f_2\in\mathscr{S}^{k_2}$. Then

$$|\,I(f_1\cdot f_2)(a,b)| \leqslant \tilde b_{k_1}\tilde b_{k_2}|f_1|_{k_1}^s|f_2|_{k_2}^s \times\left(1 + \frac{|x_a + x_b|^2 + |p_a - p_b|^2}{2}\right)^{-k_1/2} \times\left(1 + \frac{|x_a - x_b|^2 + |p_a + p_b|^2}{2}\right)^{-k_1/2},$$

with

$$\tilde b_k = \frac{3e}{2}2^{n/4}(8n)^{k/2}\begin{cases}1 & \text{if} \quad k\leqslant 2,\\ e^{-k}k^k & \text{if} \quad k\geqslant 3.\end{cases}$$

2. Define $\widetilde W^\rho$ as the closure of $\mathscr{S}(\mathbb{R}^n)$ with respect to $\|\;\|_\rho^s$, with $(\|\,g\,\|_\rho^s)^2 = (g,(\tilde H + n/2)^\rho g)$ and $\tilde H = y^2 - \frac{1}{4}\Delta_y$. Take $f_1\in\widetilde W^{\rho_1}$, $f_2\in\widetilde W^{\rho_2}$. Then

$$\int\int da\,db\,|I(f_1\cdot f_2)(a,b)|^2 \times\left(1 + \frac{|x_a + x_b|^2 + |p_a - p_b|^2}{2}\right)^{\rho_1} \times\left(1 + \frac{|x_a - x_b|^2 + |p_a + p_b|^2}{2}\right)^{\rho_2} \leqslant \tilde c''_{\rho_1}\cdot\tilde c''_{\rho_2}(\|f_1\|_{\rho_1}^s\cdot\|f_2\|_{\rho_2}^s)^2,$$

with

$$c''_\rho = \begin{cases} e^{1-\rho}\left(1 + \dfrac{\rho}{n}\right)^{\rho+n}, & \rho > 0\\ e^{-\rho}, & \rho\leqslant 0.\end{cases}$$

3. Take $f_1\in\mathscr{S}^{-k}$. Then

$$|I(f_1\cdot 1)(a,b)| \leqslant \tilde b_k e^{-(1/4)|x_a - x_b|^2} \times|f_1|_k^s\left(1 + \frac{|x_a + x_b|^2 + |p_a - p_b|^2}{2}\right)^{-k/2}.$$

4. Take $T_1\in\mathscr{S}'(\mathbb{R}^n)$, with $\forall g\in\mathscr{S}(\mathbb{R})^n$: $|T_1(g)| < K_{T_1}|\,g\,|_k^s$. Then $\forall\mu > k + n$:

$$|I(T_1\cdot 1)(a,b)| < 2c_\mu\tilde b'_{k,\mu}K_{T_1}e^{-(1/4)|x_a - x_b|^2} \times\left(1 + \frac{|x_a + x_b|^2 + |p_a - p_b|^2}{2}\right)^{\mu/2},$$

with

$$c_\mu = \mu^{\mu/2}\exp[-\tfrac{1}{2}(\mu - 1)],$$

$$\tilde b'_{k,\mu} = \pi^{-n/2}(\tfrac{8}{3})^{k/2}\Gamma\left(\frac{k+n}{2} + 1\right)\Gamma\left(\frac{\mu - k - n}{2}\right)\Gamma\left(\frac{\mu - k}{2}\right)^{-1} \times\int_{\mathbb{R}^n} d^n y\, e^{-2y^2}(1 + y^2)^k.$$

5. Take $f_1\in\widetilde W^\rho$; let g be any function in $L^2(\mathbb{R}^n)$. Then

$$\int\int da\, db\, |I(f_1\cdot 1)(a,b)|^2 \left| g\left(\frac{p_a+p_b}{\sqrt{2}}\right)\right|^2 \times\left(1+\frac{|x_a+x_b|^2+|p_a-p_b|^2}{2}\right)^\rho < \tilde{c}''_\rho \,\| g\|_2^2 (\| f_1\|_\rho^s) .$$

By taking other combinations for f_1, f_2 one can easily derive other bounds of this kind.

6. CONCLUDING REMARKS

Using intensively the analyticity properties of the function $\{a, b\,|v\}$ we have derived several families of bounds on the action of I and $\tilde{I}$. These results will be used in Ref. 15 to derive properties of the Weyl quantization procedure. Examples of such results are

$$\left.\begin{array}{l}\forall f\in W^{\nu+\epsilon}: Qf \text{ is trace-class}\\ \forall A\in\mathscr{B}(\mathscr{H}): Q^{-1}A\in W^{-\nu-\epsilon}\end{array}\right\} \text{ see also Ref. 5,}$$

$\forall f_1, f_2\in \tilde{W}^{\nu+\epsilon}$: Qf_1Qf_2 trace-class,

$f_1, f_2\in L^2(\mathbf{R}^n)\Rightarrow Qf_1Qf_2$ Hilbert–Schmidt,

$\forall T\in W^\mu$: QT is a quadratic form, relatively form-bounded with respect to a power of the harmonic oscillator Hamiltonian QH.

$\forall f\in W^\rho, g\in W^{-\rho}$: the twisted product $f\circ g$ is defined, and $\in W^{-2\rho}$.

The bounds derived here can also be used to show that all the operations in, e.g., Ref. 11 were well defined.

Because of the link of our integral transform I with the Bargmann integral transform in Ref. 9 any result on the Bargmann integral transform (such as, e.g., in Ref. 1) can be translated to give properties of I, and hence of the Weyl quantization procedure. Note also that analogous bounds can be obtained if one starts not with the coherent state family $\{\Omega^a\}$, but with any other overcomplete family depending analytically on its label, and having the reproducing property (1.4); an example of such a family would be given by[20] $\{\Omega^a_{[m]}\}$, where $\Omega^a_{[m]} = W(a)u_{[m]}$ [$W(a)$ are the Weyl operators; to obtain the usual coherent states one takes $[m]=[0]$]. This would give rise to another integral kernel $\{a, b\,|v\}_{[m]} = 2^n(\Omega^a_{[m]}, \Pi(v)\Omega^b_{[m]})$ but essentially the same theorems could be derived (with some adjustments). Finally, it is important to note that the integral transform I has the following invariance property with respect to the symplectic Fourier transform (see also Ref. 1).

Define

$$(F_\alpha f)(v) = 2^{-n}|\alpha|^n \int dv'\, e^{i\alpha\sigma(v,v')} f(v'),$$

then

$$F_4(\{a, b\,|\cdot\})(v) = \{a, -b\,|v\}$$

and hence

$$\forall T: I(F_{-4}T)(a,b) = IT(a,-b)$$

and this for T in any of the classes considered above (all the spaces we have introduced are invariant under the Fourier transform). This leads to the property $Q(F_{-4}T) = QT\cdot\Pi$ (Π is the parity operator) for the Weyl quantization procedure, but it also implies that the same Fourier invariance will turn up in all results, thereby weakening some of them (e.g., the result of the trace-class properties of Qf_1Qf_2: see Ref. 15).

APPENDIX A

We prove some results on $\mathscr{F}^G$ spaces.

Proposition A.1: Let G be a function $\mathbf{R}^+\to\mathbf{R}^+$, such that

$$\forall m: \lambda^G_m = \frac{\int_0^\infty dx\, x^{m+2n-1}e^{-x}G(x)}{\Gamma(m+2n)}$$

is finite. Then

$$\forall\phi = \sum_{[k][l]} \phi_{[k,l]}u_{[k,l]}\in Z_2,$$

$$\int d\zeta\, |\phi(\zeta)|^2 G(|\zeta|^2) < \infty \Leftrightarrow \sum_{[k][l]} |\phi_{[k,l]}|^2 \lambda^G_{|k|+|l|} < \infty .$$

If one of these expressions is finite, they are equal.

Proof: From

$$\int d\zeta\, \overline{u_{[k,l]}(\zeta)}u_{[k',l']}(\zeta) = \delta_{[k][k']}\delta_{[l][l']}$$

one sees that (see Ref. 9b)

$$\int_{|\zeta|=1} d\zeta\, \overline{u_{[k,l]}(\zeta)}u_{[k',l']}(\zeta) = \frac{2}{\Gamma(|k|+|l|+2n)}\delta_{[k][k']}\delta_{[l][l']} .$$

Hence

$$\int_{|\zeta|<R} d\zeta\, G(|\zeta|^2)\, \overline{u_{[k,l]}(\zeta)}u_{[k',l']}(\zeta) = \delta_{[k][k']}\delta_{[l][l']} A_G(|k|+|l|;R),$$

with

$$A_G(m;R) = \frac{1}{\Gamma(m+2n)}\int_0^{R^2} dy\, e^{-y}y^{m+2n-1}G(y) \underset{R\to\infty}{\nearrow} \lambda^G_m$$

Then

$$\begin{aligned}\int d\zeta\, |\phi(\zeta)|^2 G(|\zeta|^2) &= \lim_{R\to\infty}\int_{|\zeta|<R} d\zeta\, |\phi(\zeta)|^2 G(|\zeta|^2)\\ &= \lim_{R\to\infty}\sum_{\substack{[k][l]\\ [k'][l']}} \overline{\phi_{[k,l]}}\phi_{[k',l']}\delta_{[k][k']}\delta_{[l][l']} \times A_G(|k|+|l|;R)\\ &= \lim_{R\to\infty}\sum_{[k][l]} |\phi_{[k,l]}|^2 A_G(|k|+|l|;R)\\ &= \sum_{[k][l]} |\phi_{[k,l]}|^2\lambda^G_{|k|+|l|} .\end{aligned}$$

Proposition A.2: Let $\Sigma_{[k][l]}\,\phi_{[k,l]}u_{[k,l]}$ be an element of $Z(E_2)$.

1. If $\int d\zeta\, |u_{[k,l]}(\zeta)\phi(\zeta)| < \infty$, then

$$\int d\zeta\, \overline{u_{[k,l]}(\zeta)}\phi(\zeta) = \phi_{[k,l]} .$$

2. If $\int d\zeta\, |\omega^\xi(\zeta)\phi(\zeta)| < \infty$, then

$$\int d\zeta\, \overline{\omega^\xi(\zeta)}\delta(\zeta) = \phi(\xi) .$$

Proof:

1. We have

$$\int d\zeta\ \overline{u_{[k,l]}(\zeta)}\,\phi(\zeta)$$
$$= \lim_{R\to\infty}\int_{|\zeta|<R} d\zeta\ \overline{u_{[k,l]}(\zeta)}\,\phi(\zeta)$$
$$= \lim_{R\to\infty}\sum_{[k'][l']}\phi_{[k',l']}\delta_{[k][k']}\delta_{[l][l']}A_1(|k|+|l|;R)$$
$$= \lim_{R\to\infty}\phi_{[k,l]}A_1(|k|+|l|;R)=\phi_{[k,l]}\,.$$

2. Analogously,

$$\int d\zeta\ \overline{\omega^{\xi}(\zeta)}\phi(\zeta)=\lim_{R\to\infty}\int_{|\zeta|<R} d\zeta\ \overline{\omega^{\xi}(\zeta)}\phi(\zeta)$$
$$=\lim_{R\to\infty}\sum_{\substack{[k][l]\\ [k'][l']}}\phi_{[k,l]}u_{[k',l']}(\xi)\delta_{[k][k']}\delta_{[l][l']}\times A_1(|k|+|l|;R)$$
$$=\lim_{R\to\infty}\sum_{[k][l]}\phi_{[k,l]}u_{[k,l]}(\xi)A_1(|k|+|l|;R)$$
$$=\phi(\xi)\,.$$

APPENDIX B

We compute the asymptotic behavior of

$$\lambda_m^{F_\rho^{q,\tau}}=\frac{1}{\Gamma(m+2n)}\int_0^\infty dx\,x^{m+2n-1}e^{-x}(1+x)^\rho e^{\tau x^q}\qquad\text{(B1)}$$

for $m\to\infty$. To estimate the asymptotic behavior in m of the integral

$$I_{\tau,\rho,q,m}=\int_0^\infty dx\exp[-x+\tau x^q+(m+2n-1)\ln x+\rho\ln(x+1)]\,,\qquad\text{(B2)}$$

we shall use a stationary point method. The exponent

$$X(x)=-x+\tau x^q+(m+2n-1)\ln x+\rho\ln(x+1)$$

has a unique maximum in $x_0=m(1+\tau q m^{q-1}+O(m^{q-1}))$. One can use this to estimate that

$$I_{\tau,\rho,q,m}=\tilde I_{\tau,\rho,q,m}(1+O(m^{-1}))\,,\qquad\text{(B3)}$$

where

$$\tilde I_{\tau,\rho,q,m}=\int_0^\infty dx\exp[-x+\tau x^q+(m+2n+p-1)\ln x]\,.\qquad\text{(B4)}$$

We shall therefore restrict ourselves to this last integral. The exponent in (B4),

$$Y(x)=-x+\tau x^q+(m+2n+\rho-1)\ln x\,,\qquad\text{(B5)}$$

has a unique maximum defined by the equation

$$x=-(m+2n+\rho-1)=\tau q x^q\,.\qquad\text{(B6)}$$

The solution to this equation can be computed using perturbation techniques:

$$\tilde x=m(1+\tau q m^{q-1})(1+O(m^{\max(-1,2q-2)}))\,.\qquad\text{(B7)}$$

One has then

$$Y''(\tilde x)=\frac{1}{m}(1+O(m^{\max(-1,2q-2)}))\,,\qquad\text{(B8)}$$
$$\forall j>2:\ Y^{(j)}(\tilde x)=O(m^{1-j})\,,\qquad\text{(B9)}$$
$$Y(\tilde x)=(-m+(m+2n-\tfrac12)\ln m)+(\rho-\tfrac12)\ln m+m\tau m^{q-1}+O(m^{\max(0,2q-1)})\,,$$

hence

$$-\ln\Gamma(m+2n)+Y(\tilde x)=(\rho-\tfrac12)\ln m+\tau m^q+O(m^{\max(0,2q-1)})\,.\qquad\text{(B10)}$$

Collecting all these results, we see now that

$$\frac{1}{\Gamma(m+2n)}I_{\tau,\rho,q,m}\underset{m\to\infty}{\sim}\frac{1}{\sqrt{m^{-1}}}\times\exp[(\rho-\tfrac12)\ln m+\tau m^q+O(m^{\max(0,2q-1)})]\times(1+O(m^{\max(-1/2,2q-2)}))\qquad\text{(B11)}$$

[the higher derivatives contribute only a factor $(1+O(m^{-q/2}))$ because of the estimate (B9)—see Ref. 21].

If $q<\frac12$, we can rewrite (B11) as [being a little more careful in estimating $\tilde x$ in (B7)]

$$\lambda_m^{F_\rho^{q,\tau}}\underset{m\to\infty}{\sim}\text{const}\,x m^\rho e^{\tau m^q}(1+O(m^{\max(-1/2,2q-1)}))\,.\qquad\text{(B12)}$$

For $q>\frac12$, the estimate (B7) is too coarse. The next term in the perturbation gives

$$\tilde x=m(1+\tau q m^{q-1}+\tau^2q^3m^{2q-2})(1+O(m^{\max(-1,3q-3)}))$$

yielding, for $\frac12<q<\frac23$,

$$\lambda_m^{F_\rho^{q,\tau}}\underset{m\to\infty}{\sim}\text{const}\,x m^\rho e^{\tau m^q+(1/2)\tau^2q^2m^{2q-1}}(1+O(m^{\max(q-1,3q-2)}))\,.\qquad\text{(B13)}$$

It is easy to see that for $q\in[1+1/n,1-1/(n+1)]$, n extra terms have to be introduced in the perturbation series for $\tilde x$, and that finally

$$1-\frac{1}{n}<q<1-\frac{1}{n+1}\quad(n\geqslant2)$$
$$\Rightarrow\lambda_m^{F_\rho^{q,\tau}}\underset{m\to\infty}{\sim}\text{const}\,x m^\rho\times\exp[\tau m^q+A_1m^{2q-1}+A_2m^{3q-2}+\cdots+A_{n-1}m^{nq-(n-1)}]\times[1+O(m^{\max[q-1,q+n(q-1)]})]\,,\qquad\text{(B14)}$$

where $A_1=\frac12\tau^2q^2$ [as in (B13)].

APPENDIX C

We indicate here how the definitions of Ref. 17 have been adjusted to fit the $h_{[k,l]}$.

Define on $\mathscr{S}(E)$ two sequences of operators by the following recursion:

$$M_m=\sum_\delta\Big(x_j\,M_{m-1}\,x_j+p_j\,M_{m-1}\,p_j-\frac14\frac{\partial}{\partial x_j}M_{m-1}\frac{\partial}{\partial x_j}-\frac14\frac{\partial}{\partial p_j}M_{m-1}\frac{\partial}{\partial p_j}\Big),$$

$$\tilde{M}_m = \sum_j \Big(\tfrac{1}{4} x_j \tilde{M}_{m-1} x_j + \tfrac{1}{4} p_j \tilde{M}_{m-1} p_j - \frac{\partial}{\partial x_j} \tilde{M}_{m-1} \frac{\partial}{\partial x_j} - \frac{\partial}{\partial p_j} \tilde{M}_{m-1} \frac{\partial}{\partial p_j} \Big),$$

$$M_0 = \tilde{M}_0 = \mathbb{1}.$$

Both M_m and $\tilde{M}_m$ are positive; one can easily check that $\forall f, g \in \mathscr{S}(E)$: $(\tilde{f}, \tilde{M}_m \tilde{g}) = (f, M_m g)$, where $\sim$ denotes the Fourier transform F_1 (see Sec. 6). The M_m conserve the orthogonality of the $h_{[k,l]}$ (see Ref. 17),

$$(h_{[k',l']}, M_m h_{[k,l]}) = \delta_{[k][k']} \delta_{[l][l']} a(m; |k| + |l|);$$

the $a(m;k)$ satisfy the following relations:

$$\forall k:\ a(0;k) = 1,$$

$$m \geqslant 1 \Rightarrow a(m;k) = \frac{k+2n}{2} a(m-1;k+1) + \frac{k}{2} a(m-1;k-1).$$

This last recursion relation implies

$$2^{-m} \frac{\Gamma(k+2n+m)}{\Gamma(k+2n)} \leqslant a(m;k) \leqslant \frac{\Gamma(k+n+m)}{\Gamma(k+n)}.$$

The Hilbert space $H(\alpha, A)$ is then defined as

$$H(\alpha, A) = \Big\{ f \in \mathscr{S}(E);\ \|f\|^2_{\alpha,A} = \sum_m \frac{1}{A^{2m}\Gamma^2(\alpha m)} (f, M_m f) < \infty \Big\}.$$

Since

$$\|h_{[k,l]}\|^2_{\alpha,A} = \sum_m \frac{1}{A^{2m}\Gamma^2(\alpha m)} a(m;|k|+|l|) \leqslant (1+y)^{|k|+|l|+n-1} \sum_m \frac{(1+y^{-1})^m}{A^{2m}\Gamma^2(\alpha m)} \Gamma(m+1)$$

and this $\forall\, y > 0$, we see that if $\exists z > 0$ such that

$$k_{\alpha,A}(z) = \sum_m \frac{(1+z)^m m!}{A^{2m}\Gamma^2(\alpha m)}$$

converges, then $h_{[k,l]} \in H(\alpha, A)\ \forall [k], [l]$. This convergence is guaranteed for any A if $\alpha > \frac{1}{2}$, for $A > (2(1+z))^{1/2}$ if $\alpha = \frac{1}{2}$. Hence $H(\alpha, A)$ is an infinitely dimensional Hilbert space with orthonormal basis $\|h_{[k,l]}\|^{-1}_{\alpha,A} h_{[k,l]}$ if $\alpha > \frac{1}{2}$, or $\alpha = \frac{1}{2}, A > \sqrt{2}$. One can check that the topology on $H(\alpha, A)$ defined by the norm $\|\ \|_{\alpha,A}$ is really stronger than the topology on $\mathscr{S}(E)$, and that $H(\alpha, A)$ is a proper subset of $\mathscr{S}(E)$ (see Ref. 17).

The norm $\|\ \|_{\alpha,A}$ on $H(\alpha, A)$ can also be written as

$$\|f\|^2_{\alpha,A} = \sum_{[k][l]} |(f, h_{[k,l]})|^2 \gamma^{-2}_{|k|+|l|}(\alpha, A),$$

with

$$\gamma_k^{-2}(\alpha, A) = \sum_{m=0}^{\infty} \frac{a(m;k)}{\Gamma^2(\alpha m) A^{2m}}.$$

The Hilbert space $H(\bar{\alpha}, \bar{A})$ is then defined as the dual of $H(\alpha, A)$ with respect to the normal action of $\mathscr{S}'$ on $\mathscr{S}$. It can be constructed as the closure of $\mathscr{S}'(E)$ for the norm $\|\ \|_{\bar{\alpha},\bar{A}}$:

$$\|T\|^2_{\bar{\alpha},\bar{A}} = \sum_{[k][l]} |T(h_{[k,l]})|^2 \gamma^2_{|k|+|l|}(\alpha, A).$$

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