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Wiener measures for path integrals with affine kinematic variables

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The results obtained earlier have been generalized to show that the path integral for the affine coherent state matrix element of a unitary evolution operator $\exp(-iTH)$ can be written as a well-defined Wiener integral, involving Wiener measure on the Lobachevsky half-plane, in the limit that the diffusion constant diverges. This approach works for a wide class of Hamiltonians, including, e.g., $-d^2/dx^2 + V(x)$ on $L^2(\mathbb{R}_+)$, with V sufficiently singular at $x = 0$.

I. INTRODUCTION

The observables that are the quantum kinematical operators are usually defined to have commutation relations analogous to the Poisson bracket structure of the associated classical kinematical variables. Examples are a single canonical pair and the Heisenberg commutation relation, or angular momentum variables and the Lie algebra of angular momentum operators. We shall say that p, q are classical affine variables if $q > 0$ (or $p > 0$), for example, with the other variable p (or q) being unrestricted. Since one variable is the generator of translations of the other, it follows that some conflict with the range restriction is possible, a situation that reflects itself in the quantum theory by the fact that the operators Q and P cannot both be observables (self-adjoint operators) satisfying the Heisenberg commutation relation if $Q > 0$ (or $P > 0$). An acceptable substitute for the nonobservable operator is the dilation operator $D = \frac{1}{2}(QP + PQ)$, which can always be chosen self-adjoint along with the positive operator. The Lie-algebra relation $[Q,D] = iQ$ with $Q > 0$ is just the quantum image of the Poisson-bracket relation $\{q,d\} = q$, $q > 0$, where $d = qp$. The generator D preserves the positivity of Q just as the classical counterpart d preserves the positivity of q. The indicated Lie algebra relation is that of the affine group, sometimes called the $(ax + b)$-group, which is the group of translations (b) and scale changes without reflection ($a > 0$) of the real line into itself, $x \to x' = ax + b$. Thus we refer to Q (or P) and D as quantum affine kinematical variables, and in view of the simple relation between d, p, and q, we loosely refer to p, q with $q > 0$ (or $p > 0$) as classical affine kinematic variables as noted earlier.

Focusing on the $p > 0$ case for the moment, we may imagine a formal phase-space path integral quantization of such a system given by

$$\mathcal{N}^{-1} \int \exp\left\{ i \int [p\dot{q} - H(p,q)]\, dt \right\} \prod_t [dp_t dq_t], \qquad (1.1)$$

where all paths satisfy the condition $p(t) > 0$. This expression is plagued by two problems. The first problem relates to what (1.1) could possibly represent since it *cannot* be the propagator expressed in the Q-representation for the simple reason that if $[Q,P] = i$ and $P > 0$ then no Q-representation is possible. A satisfactory answer to the first problem was given earlier[1] in which (1.1) was formally interpreted as the propagator expressed in the affine coherent-state representation (which makes fundamental use of the operators P and D rather than P and Q; see Refs. 2, 3). The second problem with (1.1) pertains to the formal nature of the path integral. In Ref. 1 meaning was given to (1.1) as the limit of a fairly standard lattice-space regularization. This approach made little direct contact with paths defined for continuous time as in the classical theory, and besides, it was relatively heuristic. On the other hand, in recent work[4] pertaining to the usual canonical case (and also for spin kinematical variables), it was shown how the appropriate coherent-state representation of the propagator can be defined as the limit of well-defined path integrals over pinned Brownian-motion measures as the diffusion constant diverges. The purpose of the present paper is to extend this alternative form of regularization and its associated rigorous definition of a path-integral representation to systems involving affine variables. To begin with, however, it is useful to give a brief description of the construction in Ref. 4 for the canonical case.

For a given Hamiltonian H, we defined[4] the path integrals

$$2\pi e^{\nu(t''-t')/2} \int \exp\left[\frac{i}{2} \int (p\,dq - q\,dp) - i \int h(p,q)dt \right] d\mu_W^\nu(p) d\mu_W^\nu(q), \qquad (1.2)$$

where $d\mu_W^\nu(p)$ and $d\mu_W^\nu(q)$ are Wiener measures associated to two independent Brownian processes (one in p, one in q) with diffusion constant ν, and pinned at p',q' for $t = t'$, at p'',q'' for $t = t''$. The function h in (1.2) is the antinormal ordered symbol[2] of H. For finite ν, (1.2) is a perfectly well-defined path integral on phase space. It has been proved[4] that for a wide class of Hamiltonians, the limit for $\nu \to \infty$ of (1.2) gives the coherent state matrix element

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$\langle p'',q''|\exp[-i(t''-t')H]|p',q'\rangle$.

This procedure is not restricted to only the canonical kinematical variables. In Ref. 5 an outline is given of how the above construction can be extended to general semisimple Lie groups. One has then to use the corresponding generalized coherent states.[6] One can define a metric on the group manifold associated to these coherent states,[5] and use the corresponding Laplace–Beltrami operator to define a generalized Wiener measure. Examples of interest outlined in Ref. 3 are (i) the Weyl–Heisenberg group, (ii) the group SU(2), and (iii) the affine $(ax+b)$-group, corresponding to, respectively, canonical, spin, and affine kinematic variables. The first two were extensively discussed in Ref. 4. Here we present a more detailed study of the affine variable case. In particular, we derive explicit conditions characterizing the class of Hamiltonians that can be treated by our methods, and we give several examples as well.

This paper is organized as follows. In Sec. II we review the definition and some properties of the coherent states associated with the $(ax+b)$-group.[2,3] We shall adopt notation related to that in Ref. 3, which is different from the notation in Refs. 1 and 2. We shall also indicate how to pass from one notation to the other. It is convenient to break the construction into two parts. In Sec. III we study the path integral for zero Hamiltonian. We introduce the Brownian process on the half plane, use it to construct the path integral, and show that in the limit of diverging diffusion constant the path integral converges to the coherent state overlap function [as it should, since $\exp(-itH)=\mathbb{1}$ if $H=0$]. In Sec. IV we discuss the path integral with a nonzero Hamiltonian, and we derive sufficient conditions on the Hamiltonian so that the limit for diverging diffusion constant leads to the appropriate coherent-state matrix element of the evolution operator.

II. THE $(ax+b)$-GROUP AND THE AFFINE COHERENT STATES

Let us review the definition of the $(ax+b)$-group and the associated coherent states, and give some of their properties. Most of this discussion is analogous to what happens for the Weyl–Heisenberg group and its associated coherent states, the more familiar canonical coherent states. Both the affine and the canonical coherent states are examples of the construction of coherent states associated with general Lie-groups.[6]

A. The $(ax+b)$-group

The "$(ax+b)$-group" is the set $M_+ := \mathbb{R}_+^* \times \mathbb{R}$, where $\mathbb{R}_+^* = (0,\infty)$, with the group law

$$(a'',b'')(a',b') = (a''a', b''+a''b') .$$

This group has two (faithful) inequivalent irreducible unitary representations U_+ and U_-. We shall consider their following realizations on $L^2(\mathbb{R}_+)$. For $\psi\in L^2(\mathbb{R}_+)$, one defines

$$[U_\pm(a,b)\psi](x) = a^{1/2}e^{\pm ibx}\psi(ax) . \tag{2.1}$$

We shall mainly use U_+, except when specified otherwise. The subscript $+$ will often be dropped.

Both representations U_+ and U_- are square integrable. This means[7] that there exists an (unbounded) positive self-adjoint operator C on $L^2(\mathbb{R}_+)$ such that

$\forall\psi_1,\psi_2\in D(C^{1/2}),\quad \forall\phi_1,\phi_2\in L^2(\mathbb{R}_+)$:

$$\int d\mu(a,b)\langle\phi_1,U_\pm(a,b)\psi_1\rangle\langle U_\pm(a,b)\psi_2,\phi_2\rangle = \langle C^{1/2}\psi_2,C^{1/2}\psi_1\rangle\langle\phi_1,\phi_2\rangle . \tag{2.2}$$

Here $d\mu(a,b) = (1/2\pi)a^{-2}\,da\,db$ is the left-invariant measure on the $(ax+b)$-group. The operator C is given by

$$(C\psi)(x) = x^{-1}\psi(x) . \tag{2.3}$$

In particular (2.2) implies that, for all $\psi\in D(C^{1/2})$, $\|\psi\|=1$,

$$\int d\mu(a,b)U_\pm(a,b)|\psi\rangle\langle\psi|U_\pm(a,b)^* = c(\psi)\mathbb{1} , \tag{2.4}$$

with

$$c(\psi) = |C^{-1/2}\psi|^2 = \int_0^\infty dx(1/x)|\psi(x)|^2 . \tag{2.5}$$

The closed spaces $\mathscr{H}_\pm$ spanned by the sets

$$\{\langle U_\pm(\cdot,\cdot)\psi,\phi\rangle;\quad \psi\in D(C^{1/2}),\quad \phi\in L^2(\mathbb{R}_+)\}$$

are mutually orthogonal subspaces of $L^2(M_+) := L^2(M_+;d\mu)$. Together, $\mathscr{H}_+$ and $\mathscr{H}_-$ span the whole space $L^2(M_+)$. This can easily be checked by explicit calculation.

All this enables us to build orthonormal bases of $L^2(M_+)$, starting from orthonormal bases in $L^2(\mathbb{R}_+)$. Let $\{\phi_j;\ j\in\mathbb{N}\}$, $\{\psi_j;\ j\in\mathbb{N}\}$ be two orthonormal bases in $L^2(\mathbb{R}_+)$ such that $\psi_j\in D(C^{-1/2})$ for all j. Define elements $f_{ij}^\pm$ of $L^2(M_+)$ by

$$f_{ij}^\pm(a,b) = \langle U_\pm(a,b)C^{-1/2}\psi_i,\phi_j\rangle . \tag{2.6}$$

It is clear that for all i, j, $f_{ij}^\epsilon\in\mathscr{H}_\epsilon$ ($\epsilon = +$ or $-$). On the other hand, both $\{f_{ij}^+;\ i,j\in\mathbb{N}\}$ and $\{f_{ij}^-;\ i,j\in\mathbb{N}\}$ are orthonormal sets, as a consequence of (2.2). One easily checks that, for $\epsilon = +$ or $-$, $\{f_{ij}^\epsilon;\ i,j\in\mathbb{N}\}$ constitutes a basis for $\mathscr{H}_\epsilon$. The set $\{f_{ij}^\epsilon;\ i,j\exists\mathbb{N},\ \epsilon = +$ or $-\}$ is therefore an orthonormal basis for $L^2(M_+)$.

Let now B be a Hilbert–Schmidt operator on $L^2(\mathbb{R}_+)$ such that $C^{-1/2}B$ is trace class. Then

$$B = \sum_j \lambda_j|\psi_j\rangle\langle\phi_j| ,$$

where $\{\phi_j;\ j\in\mathbb{N}\}$, $\{\psi_j;\ j\in\mathbb{N}\}$ are orthonormal bases in $L^2(\mathbb{R}_+)$, with $\psi_j\in D(C^{-1/2})$ for all j, $\Sigma_j|\lambda_j|^2<\infty$. Since $C^{-1/2}B$ is trace class we can define

$$[F(B)](a,b) = (1/\sqrt{2})\mathrm{Tr}[(U_+(a,b)+U_-(a,b))C^{-1/2}B] = (1/\sqrt{2})\sum_{j,\epsilon}\lambda_j\langle\phi_j,U_\epsilon(a,b)C^{-1/2}\psi_j\rangle . \tag{2.7}$$

From the preceding paragraph it is clear that (2.7) can be considered as an expansion of $F(B)$ with respect to an orthonormal base in $L^2(M_+)$. Since the sequence of coefficients is square summable, $\Sigma_{j,\epsilon}|\lambda_j|^2 = 2\,\mathrm{Tr}(B^*B)$, we immediately see that $F(B)\in L^2(M_+)$, with

$$\int d\mu(a,b)\,|[F(B)](a,b)|^2 = \frac{1}{2}\sum_{j,\epsilon}|\lambda_j|^2 = \mathrm{Tr}(B^*B)\,. \tag{2.8}$$

The set of Hilbert–Schmidt operators B for which $C^{-1/2}B$ is trace class is dense in the space τ_2 of Hilbert–Schmidt operators. One can use this to extend the map $B\to F(B)$ to all of τ_2. This extension is a unitary map from τ_2 to $L^2(M_+)$. This is the $(ax+b)$-group analog of a well-known result for the Weyl–Heisenberg group.[8]

B. The affine coherent states

A special role in our path integral results below will be played by extremal-weight vectors for the unitary representation under consideration (see Ref. 5). In our case these are the vectors[1] (normalized to 1)

$$\psi_\beta(x) = 2^\beta\Gamma(2\beta)^{-1/2}x^{\beta-1/2}e^{-x}\,. \tag{2.9a}$$

In order for $c_\beta \equiv c(\psi_\beta)$ to be finite, one has to impose $\beta > \frac{1}{2}$. One finds

$$c_\beta = (\beta - \tfrac{1}{2})^{-1}\,. \tag{2.9b}$$

We shall use these minimal weight vectors ψ_β as "fiducial vectors"[6] for the construction of the affine coherent states,

$$|a,b;\beta\rangle = U(a,b)\psi_\beta\,.$$

From (2.4) one now immediately has the affine coherent state resolution of the identity

$$c_\beta^{-1}\int d\mu(a,b)\,|a,b;\beta\rangle\langle a,b;\beta| = \mathbb{1}\,. \tag{2.10}$$

The "overlap function" of different coherent states (same value of β) is given by

$$\begin{aligned}\langle a'',b'';\beta|a',b';\beta\rangle &= \left[\frac{a''+a'+i(b''-b')}{2\sqrt{a''a'}}\right]^{-2\beta}\\ &= \left[\frac{2}{1+\cosh d(a'',b'';a',b')}\right]^{\beta}\\ &\quad\times\exp\left(-2\beta i\tan^{-1}\frac{b''-b'}{a''+a'}\right),\end{aligned} \tag{2.11}$$

where d denotes the metric distance[9] on the Lobachevsky half-plane M_+

$$d(a'',b'';a',b') = \cosh^{-1}\left[1+\frac{(a''-a')^2+(b''-b')^2}{2\,a''a'}\right]. \tag{2.12}$$

For every $\beta > \frac{1}{2}$ one can define the following map on $L^2(\mathbb{R}_+)$:

$$\begin{aligned}(U_\beta\phi)(a,b) &= c_\beta^{-1/2}\langle a,b;\beta|\phi\rangle\\ &= c_\beta^{-1/2}2^\beta[\Gamma(2\beta)]^{-1/2}a^\beta\\ &\quad\times\int_0^\infty dx\,x^{\beta-1/2}e^{-(a+ib)x}\phi(x)\,.\end{aligned} \tag{2.13}$$

It is clear from (2.10) that U_β is an isometry from $L^2(\mathbb{R}_+)$ to $L^2(M_+)$. These maps U_β are the analogs of the Bargmann transform for the Weyl–Heisenberg case.[10] The image $\mathscr{H}_\beta \equiv U_\beta L^2(\mathbb{R}_+)$ consists of exactly those elements f of $L^2(M_+)$ that can be written as

$$f(a,b) = a^\beta\phi(a+ib)\,,$$

where $\phi(z)$ is an entire analytic function on the half-plane $\mathrm{Re}\,z>0$.

The Hilbert space $\mathscr{H}_\beta$ is a reproducing kernel Hilbert space,[11] with reproducing kernel $c_\beta^{-1}\langle a'',b'';\beta|a',b';\beta\rangle$. In other words, for f in $\mathscr{H}_\beta$,

$$f(a,b) = c_\beta^{-1}\int d\mu(a',b')\langle a,b;\beta|a',b';\beta\rangle f(a',b')\,.$$

This means in particular that the orthogonal projection operator P_β mapping $L^2(M_+)$ onto $\mathscr{H}_\beta$ is an integral operator with integral kernel

$$P_\beta(a'',b'';a',b') = c_\beta^{-1}\langle a'',b'';\beta|a',b';\beta\rangle\,. \tag{2.14}$$

C. Correspondence with the *pq*-notation

We mentioned in the Introduction that our notation would not coincide with that in Ref. 1. To conclude this section we give the correspondence between our present notation and the pq-notation in Ref. 1.

For fixed β, define $p=\beta a^{-1}$, $q=-b$. We shall also rescale the measure; $d\tilde{\mu}(p,q)$ is the image of $c_\beta^{-1}d\mu(a,b)$, i.e.,

$$d\tilde{\mu}(p,q) = c_\beta^{-1}\beta^{-1}\frac{dp\,dq}{2\pi} = \frac{1-1/2\beta}{2\pi}\,dp\,dq\,.$$

With this change of notation, (2.13) becomes, for instance,

$$\begin{aligned}(\tilde{U}_\beta\psi)(p,q) &= (2\beta)^\beta[\Gamma(2\beta)]^{-1/2}p^{-\beta}\\ &\quad\times\int_0^\infty dk\,k^\beta e^{-k(\beta p^{-1}-iq)}\psi(k)\,.\end{aligned} \tag{2.15}$$

This corresponds exactly with Eq. (24) in Ref. 1.

Using this correspondence every result we shall obtain here can be translated into the pq-notation used in Ref. 1, and vice versa. At the end of Sec. IV D we shall state our main result in pq-notation as well as in the ab-notation which will be used throughout this paper.

III. THE PATH INTEGRAL FOR ZERO HAMILTONIAN

In the ab-notation, with the correspondence rules of Sec. II C, (1.1) becomes

$$\mathscr{N}^{-1}\int\exp\left[-i\beta\int a^{-1}db - \int h(a,b)\,dt\right]\prod_t\frac{da_t\,db_t}{a_t^2}, \tag{3.1}$$

where $A>0$ throughout the integration domain. We shall give a sense to this expression by a regularization that leads to a Wiener measure, on the Lobachevsky half-plane, for diffusion constant ν. In the end we take the limit $\nu\to\infty$. For related ideas (regularization by extra factors that formally disappear in the limit as a diffusion constant diverges), see Ref. 12.

In this section we restrict ourselves to the case $h=0$. The general case $h\neq 0$ will be handled in the next section.

Let us first define the Wiener measure on the Lobachevsky half-plane. The Laplace–Beltrami operator is given by

$$\Delta = a^2(\partial_a^2+\partial_b^2) \tag{3.2}$$

(in the pq-notation, $\Delta = \partial_p p^2 \partial_p + \beta^2 p^{-2} \partial_q^2$). This is a symmetric operator in $L^2(M_+)$, essentially self-adjoint on $C_0^\infty(M_+)$, the C^∞-functions on M_+ with compact support away from $a=0$ (this essential self-adjointness is most easily checked in the pq-notation).

The heat kernel for this Laplace–Beltrami operator is given by[9]

$$K_t(a'',b'';a',b') \equiv [\exp(t\Delta)](a'',b'';a',b')$$
$$= \frac{e^{-t/4}}{2\sqrt{2\pi} t^{3/2}} \int_\delta^\infty \frac{x e^{-x^2/4t}}{\sqrt{\cosh x - \cosh \delta}} dx, \quad (3.3)$$

where $\delta = d(a'',b'';a'b')$ is the metric distance (2.12). We define the affine (pinned) Wiener measure with diffusion constant ν, denoted $d\mu^{\nu,T}_{W;a'',b'';a',b'}$, as the measure on path space, pinned at a',b' for $t=0$, at a'',b'' for $t=T$, such that

$$\int d\mu^{\nu,T}_{W;a'',b'';a',b'} = K_{\nu T}(a'',b'';a',b'). \quad (3.4)$$

Requiring (3.4) for all (a'',b''), $(a',b')\in M_+$, and all $T>0$ defines $d\mu^\nu_W$ unambiguously. We shall drop the super- and subscripts T, a'', b'', a', and b' in the sequel.

We use this measure to regularize (3.1) in the following way. We define

$$\mathscr{P}^0_\nu(a'',b'';a',b';T)$$
$$= c_\beta e^{\nu T\beta} \int \exp\left(-i\beta \int a^{-1} db\right) d\mu^\nu_W(a,b). \quad (3.5)$$

The expression $\int a^{-1} db$ should be considered as a stochastic integral, to be calculated using the Stratonovich (midpoint rule) procedure. Formally (3.5) can be written as

$$\mathscr{P}^0_\nu(a'',b'';a',b';T)$$
$$= \mathscr{N} \int \exp\left[-i\beta\int a^{-1}\dot b\, dt - \frac{1}{\nu}\int a^{-2}(\dot a^2 + \dot b^2)\, dt\right] \times \prod_t \frac{da_t\, db_t}{a_t^2},$$

where the factors c_β and $e^{\nu TB}$ have been absorbed in the (infinite) normalization constant $\mathscr{N}$. This formal expression shows how (3.5) can indeed be viewed as a regularization of (3.1) (for the case $h=0$). In the final step of our regularization procedure we take the limit for $\nu\to\infty$; in this limit the regularizing factor in the above formal expression vanishes.

It is our aim in this section to prove that

$$\lim_{\nu\to\infty} \mathscr{P}^0_\nu(a'',b'';a',b';T) = \langle a'',b'';\beta | a',b';\beta\rangle. \quad (3.6)$$

This is exactly what the general expression (3.1) or (1.1) should lead to[1] in the case $h=0$.

We start by studying $\mathscr{P}^0_\nu$ for finite ν.

Lemma 3.1: $c_\beta^{-1}\mathscr{P}^0_\nu$ is the integral kernel of a semigroup on $L^2(M_+)$:

$$\mathscr{P}^0_\nu(a'',b'';a',b';T) = c_\beta[\exp(-\nu TA)](a'',b'';a',b'). \quad (3.7)$$

The operator A is given by

$$A = -\beta - a^2[\partial_a^2 + (\partial_b + i\beta/a)^2] \quad (3.8a)$$
$$= a^2(\partial_a + \partial_b + i\beta/a)(i\partial_a - \partial_b - i\beta/a). \quad (3.8b)$$

In particular, A is a positive self-adjoint operator, with domain $D(-\Delta)$.

Proof: It is clear that the $c_\beta^{-1}\mathscr{P}^0_\nu$ satisfy a semigroup property, i.e.,

$$\int d\mu(a,b)\mathscr{P}^0_\nu(a'',b'';a,b;t_2)\mathscr{P}^0_\nu(a,b;a',b';t_1)$$
$$= c_\beta \mathscr{P}^0_\nu(a'',b'';a',b';t_1+t_2).$$

On the other hand, we have

$$|c_\beta^{-1}\mathscr{P}^0_\nu(a,b;a',b';t)| \leqslant e^{\nu t\beta}K_{\nu t}(a,b,a',b').$$

This already implies that $c_\beta^{-1}\mathscr{P}^0_\nu$ is the integral kernel of a semigroup of operators, i.e., Eq. (3.7), with

$$A \geqslant -\beta + \tfrac{1}{4}. \quad (3.9)$$

Here we have used that $-\Delta \geqslant \frac{1}{4}$ on the Lobachevsky half-plane. Following the standard procedure, and using the midpoint rule for the stochastic integral $\int a^{-1} db$, one obtains the following differential equation for $\mathscr{P}^0_\nu$:

$$\partial_t \mathscr{P}^0_\nu(a,b;a',b';t)$$
$$= \{-\beta - a^2[\partial_a^2 + (\partial_b + i\beta/a)^2]\}\mathscr{P}^0_\nu(a,b;a',b';t).$$

This implies that the infinitesimal generator A is given by (3.8). We have

$$A = -\Delta - \beta + \beta^2 - 2i\beta a\partial_b.$$

Since, for all $\psi\in D(-\Delta)$, and for all $\epsilon>0$,

$$\|a\partial_b\psi\|^2 = -\langle\psi, a^2\partial_b^2\psi\rangle \leqslant \langle\psi,(-\Delta)\psi\rangle$$
$$\leqslant \epsilon\|-\Delta\psi\|^2 + (1/4\epsilon)\|\psi\|^2,$$

we see that $A-(-\Delta)$ is $(-\Delta)$-bounded with infinitesimally small bound. Hence A is self-adjoint, with domain $D(-\Delta)$. Finally it follows from (3.8b) that A is positive.

Note: It follows from the proof that every core for $-\Delta$ is a core for A. In particular, A is essentially self-adjoint on $C_0^\infty(M_+)$, the set of C^∞-functions on M_+ with compact support away from $a=0$.

We shall see below that we can do much better than Lemma 3.1. We shall see that A has an isolated eigenvalue at 0. If we denote by P_0 the projection onto the eigenspace of A for the eigenvalue 0, we then see that

$$e^{-\nu TA} \underset{\nu\to\infty}{\to} P_0 \quad (T>0).$$

This will then lead to statement (3.6).

To carry out this program, we have to determine the spectrum of A and the corresponding eigenspaces. We shall reduce this to a spectral problem on $L^2(\mathbb{R}_+)$ rather than on $L^2(M_+)$.

We first introduce the infinitesimal generators of $U_\pm(a,b)$. Both $V_\pm(b) = U_\pm(1,b)$ and $W(\alpha) = U(e^\alpha,0)$ are strongly continuous unitary one-parameter groups. Their generators are, respectively, Q and D, i.e.,

$$V_\pm(b) = e^{\pm ibQ}, \quad W(\alpha) = e^{i\alpha D},$$

where Q and D are defined by

$$(Q\psi)(x) = x\psi(x),$$
$$(D\psi)(x) = -ix\psi'(x) - (i/2)\psi(x).$$

One easily checks that these are indeed self-adjoint operators on $L^2(R_+)$. The set $C_0^\infty(\mathbb{R}_+)$ of all C^∞-functions with compact support away from 0 is a core for both D and Q. Then $U_\pm(a,b)$ can be written in terms of Q, D as follows:

$$U_\pm(a,b) = e^{\pm ibQ}e^{i(\log a)D} = e^{i(\log a)D}e^{\pm(ib/a)Q}. \quad (3.10)$$

Note that $C = Q^{-1}$. With the help of all this we prove the following lemma.

Lemma 3.2: On $L^2(\mathbb{R}_+)$ we define the operators $D^2 + Q^2 \mp 2\beta Q + (\beta - \frac{1}{2})^2$, with domain $C_0^\infty(\mathbb{R}_+)$. These are symmetric operators; we denote their closures by $H_\pm$. Then

(1) $H_\pm$ are self-adjoint,

(2) $\forall \psi,\phi \in C_0^\infty(R_+)$, $\langle U_\pm(\cdot,\cdot)C^{-1/2}\psi,\phi\rangle \in D(A)$,

and

$$A\langle U_\pm(a,b)C^{-1/2}\psi,\phi\rangle = \langle U_\pm(a,b)C^{-1/2}H_\pm\psi,\phi\rangle. \quad (3.11)$$

Proof: To prove the first statement it is convenient to make a unitary transformation from $L^2(\mathbb{R}_+)$ to $L^2(\mathbb{R})$. We define, for $\psi \in L^2(\mathbb{R}_+)$,

$$(U\psi)(s) = e^{s/2}\psi(e^s). \quad (3.12)$$

Accordingly $UC_0^\infty(\mathbb{R}_+) = C_0^\infty(\mathbb{R})$, the set of C^∞-functions with compact support. On the other hand,

$$U[D^2 + Q^2 \mp 2\beta Q + (\beta - \tfrac{1}{2})^2]U^{-1} = -\frac{d^2}{ds^2} + e^{2s} \mp 2\beta e^s + \left(\beta - \frac{1}{2}\right)^2. \quad (3.13)$$

Since the potential $V_\pm(s) = 2e^s \mp 2\beta e^s + (\beta - \frac{1}{2})^2$ is the sum of a bounded potential and a positive smooth potential, the operators (3.13) are essentially self-adjoint on $C_0^\infty(\mathbb{R})$ by Theorem X.29 in Ref. 13. This proves the first statement.

It is easy to check that for $\psi,\phi \in C_0^\infty(\mathbb{R}_+)$ the functions $f_{\psi,\phi}^\pm(a,b) = \langle U_\pm(a,b)C^{-1/2}\psi,\phi\rangle$ are well-defined C^∞-functions in a,b. Their support is contained in a set of the form $[c_1,c_2]\times\mathbb{R}$, with $c_1 > 0$; they decrease more rapidly in b than any inverse polynomial, and this uniformly in a. This is sufficient to ensure that $f_{\psi,\phi}^\pm \in D(-\Delta) = D(A)$, and also to justify the calculations below.

We have

$$\begin{aligned}
(Af_{\psi,\phi}^\pm)(a,b) &= a^2\langle(-i\partial_a + \partial_b - i\beta/a)(-i\partial_a - \partial_b + i\beta/a)U_\pm(a,b)Q^{1/2}\psi,,\phi\rangle \\
&= a^2\langle(-i\partial_a + \partial_b - i\beta/a)U_\pm(a,b)(1/a)(D \mp iQ + i\beta)Q^{1/2}\psi,\phi\rangle \\
&= a^2\langle U_\pm(a,b)[a^{-2}(D \pm iQ - i\beta)(D \mp iQ + i\beta) + ia^{-2}(D \mp iQ + i\beta)]Q^{1/2}\psi,\phi\rangle \\
&= \langle U_\pm(a,b)(D \pm iQ - i\beta + i)(D \mp iQ + i\beta)Q^{1/2}\psi,\phi\rangle \\
&= \langle U_\pm(a,b)Q^{1/2}(D \pm iQ - i\beta + i/2)(D \mp iQ + i\beta - i/2)\psi,\phi\rangle \\
&= \langle U_\pm(a,b)Q^{1/2}[D^2 + Q^2 \mp 2\beta Q + (\beta - \tfrac{1}{2})^2]\psi,\phi\rangle,
\end{aligned}$$

where we have repeatedly used that $[D,Q^\alpha] = -i\alpha Q^\alpha$. Hence (3.11) follows. As a consequence of (3.11) the subspaces $\mathscr{H}_\pm$ are invariant subspaces for A. Moreover the spectrum of $A|_{\mathscr{H}_\pm}$ is exactly the spectrum of $H_\pm$. □

Lemma 3.3: Let $A_\pm$ be the restrictions of A to $\mathscr{H}_\pm$, with domains $D(-\Delta)\cap\mathscr{H}_\pm$. Then $\sigma(A_\pm) = \sigma(H_\pm)$.

Proof: Let $P_\Omega^\pm$ be the family of spectral projection operators associated with $H_\pm$.

Let ψ_j be an orthonormal base in $L^2(\mathbb{R}_+)$, with $\psi_j \in D(H_\pm^2)$. This ensures that $\psi_j \in D(C^{-1/2})$ and $H_\pm\psi_j \in D(C^{-1/2})$. Define now $\mathbf{P}_\Omega^\pm$ on $\mathscr{H}_\pm$ by

$$\mathbf{P}_\Omega^\pm\left(\sum_{j,k} c_{jk}\langle U_\pm(\cdot,\cdot)C^{-1/2}\psi_j,\psi_k\rangle\right) = \sum_{j,k} c_{jk}\langle U_\pm(\cdot,\cdot)C^{-1/2}P_\Omega^\pm\psi_j,\psi_k\rangle. \quad (3.14)$$

Using (2.2) one finds $|\mathbf{P}_\Omega^\pm| \leqslant 1$ and $(\mathbf{P}_\Omega^\pm)^* = \mathbf{P}_\Omega^\pm$. On the other hand clearly $(\mathbf{P}_\Omega^\pm)^2 = \mathbf{P}_\Omega^\pm$, $\mathbf{P}_\mathbb{R}^\pm = \mathbf{1}_{\mathscr{H}_\pm}$, and $\mathbf{P}_{\Omega_1}^\pm\mathbf{P}_{\Omega_2}^\pm = \mathbf{P}_{\Omega_1\cap\Omega_2}^\pm$. This implies that the family $\{\mathbf{P}_\Omega^\pm; \Omega$ Borel set in $\mathbb{R}\}$ is the set of spectral projection operators for some self-adjoint operator on $\mathscr{H}_\pm$. It follows from (3.11) that this self-adjoint operator is exactly $A_\pm$. Since it is clear from (3.14) that the two projection-valued measures $P^\pm$ and $\mathbf{P}^\pm$ have the same support, $\sigma(A_\pm) = \sigma(H_\pm)$ follows immediately. □

Remark: Suppose that λ is an isolated eigenvalue of H_+ (we shall see below that H_- has only continuous spectrum) with eigenvector ϕ_λ (we assume the multiplicity of λ to be 1). Then λ is an isolated eigenvalue of A_+. It follows from the proof of Lemma 3.3 that the associated eigenspace E_λ of A_+ is given by

$$E_\lambda = \{\langle U(\cdot,\cdot)C^{-1/2}\phi_\lambda,\phi\rangle;\ \phi \in L^2(\mathbb{R}_+)\}, \quad (3.15)$$

E_λ is an infinite-dimensional closed subspace of $\mathscr{H}_+$, and every eigenvalue of A_+ is infinitely degenerate. This is completely analogous to what happens in the Weyl–Heisenberg case.[4] In order to find the spectrum of A and the associated eigenspaces we have thus only to determine the spectrum and eigenspaces for $H_\pm$. This turns out to be very easy, because $H_\pm$ are related to the exactly solvable Morse Schrödinger operator.[14]

Lemma 3.4: (1) H_- has only the continuous spectrum

$$\sigma(H_-) = [(\beta - \tfrac{1}{2})^2, \infty), \quad (3.16)$$

and (2) H_+ has the same continuous spectrum, and $\lfloor\beta + \frac{1}{2}\rfloor$ eigenvalues lying below it:

$$\sigma(H_+) = \{(\beta - \tfrac{1}{2})^2 - (\beta - n - \tfrac{1}{2})^2;\ n = 0,1,\ldots,\lfloor\beta - \tfrac{1}{2}\rfloor\} \cup [(\beta - \tfrac{1}{2})^2, \infty). \quad (3.17)$$

Note: Here we have used the notation $\lfloor x\rfloor$ for the largest integer strictly smaller than x:

$$|x| = \max\{n\in\mathbb{N}\,;\quad n<x\}\,.$$

Proof: Again it will be convenient to consider $UH_\pm U^{-1}$ rather than $H_\pm$ itself, with U as defined by (3.12). We have [see (3.13)]

$$UH_\pm U^{-1} = -\frac{d^2}{ds^2} + V_\pm(s),$$

$$\text{with } V_\pm(s) = e^{2s} \mp 2\beta e^s + (\beta-\tfrac{1}{2})^2\,.$$

Since $\beta>0$, $V_-(s)$ is a continuous, monotonously increasing function of s, tending to $(\beta-\frac{1}{2})^2$ for $s\to-\infty$ and to ∞ for $s\to\infty$. It is clear therefore that $\sigma(H_-)\subset[(\beta-\frac{1}{2})^2,\infty)$. On the other hand wave functions with support in $[-2L,-L]$, with L very large, will "see" only the constant part $(\beta-\frac{1}{2})^2$ of the potential V_-. This means that the spectrum of H_- will at least contain

$$\sigma\Big(-\frac{d^2}{ds^2}\Big) + \Big(\beta-\frac{1}{2}\Big)^2 = \Big[\Big(\beta-\frac{1}{2}\Big)^2,\infty\Big)\,.$$

Hence (3.16).

The operator $-d^2/ds^2 + v_+(s)$ is really the Morse operator.[14] Putting several constants equal to 1, one finds in Ref. 14 that the operator

$$-\frac{d^2}{dy^2} + D(e^{-2y}-e^{-y}) \quad \text{on } L^2(\mathbb{R}) \tag{3.18}$$

has discrete spectrum

$$\{-[\sqrt{D}-(n+\tfrac{1}{2})]^2;\ n\in\mathbb{N}\,,\ n<\sqrt{D}-\tfrac{1}{2}\}\,.$$

Its continuous spectrum is $[0,\infty)$. Putting $s=-y+\log\beta$, $D=\beta^2$, one finds that $-d^2/ds^2+V_+(s)-(\beta-\frac{1}{2})^2$ reduces to (3.18). Hence

$$\sigma(H_+) = \sigma\Big(-\frac{d^2}{ds^2}+V_+(s)\Big)$$
$$= \{(\beta-\tfrac{1}{2})^2-(\beta-\tfrac{1}{2}-n)^2;\ n\in\mathbb{N},\ n<\beta-\tfrac{1}{2}\}$$
$$\cup[(\beta-\tfrac{1}{2})^2,\infty)\,. \qquad \square$$

Remark: Reference 14 also gives explicit formulas for the eigenvectors of $-d^2/dy^2 + D(e^{-2y}-e^{-y})$. We shall only need the ground state. This is given by

$$\phi_0(y) = [\Gamma(2\sqrt{D}-1)]^{-1/2}(2\sqrt{D}e^{-y})^{\sqrt{D}-1/2}e^{-\sqrt{D}e^{-y}}\,.$$

Substituting $y=-s-\log\beta$, and making the inverse transformation U^{-1}, we find the ground state ϕ_0 of H_+:

$$\phi_0(x) = [\Gamma(2\beta-1)]^{-1/2}2^{\beta-1/2}x^{\beta-1}e^{x}\,. \tag{3.19}$$

If we bring together the results of Lemmas 3.2, 3.3, and 3.4 we see indeed that $A\geqslant 0$ and that 0 is an isolated eigenvalue of A. The associated eigenspace E_0 is given by [see (3.15)]

$$E_0 = \{\langle U(\cdot,\cdot)C^{-1/2}\phi_0,\ \phi\rangle;\ \phi\in L^2(\mathbb{R}_+)\}\,.$$

Here ϕ_0 is the ground state of H_+, as defined by (3.19). Note that

$$(C^{-1/2}\phi_0)(x) = [\Gamma(2\beta-1)]^{-1/2}2^{\beta-1/2}x^{\beta-1/2}e^{-x}$$
$$= \sqrt{\beta-\tfrac{1}{2}}[\Gamma(2\beta)]^{-1/2}2^{\beta}x^{\beta-1/2}e^{-x}$$
$$= c_\beta^{-1/2}\psi_\beta(x)\,, \tag{3.20}$$

with c_β, ψ_β as defined by (2.9).

Hence, with the notations of Sec. II B,

$$E_0 = \{\langle a,b;\beta\,|\phi\rangle;\ \phi\in L^2(\mathbb{R}_+)\}$$
$$= U_\beta L^2(\mathbb{R}_+) = \mathscr{H}_\beta\,.$$

This implies that the spectral projection operator $\mathbf{P}_0^+$ of A associated with the eigenvalue 0 is exactly P_β. Since $A\geqslant 0$, and since the eigenvalue 0 of A is isolated, we have therefore

$$\operatorname*{s-lim}_{\nu\to\infty}\exp(-\nu TA) = P_\beta \quad (T>0)\,.$$

This implies at least in a distributional sense, convergence of the corresponding integral kernels. In other words, and taking into account (2.14) and (3.7),

$$\mathscr{P}_\nu^0(a'',b'';a',b';T) \underset{\nu\to\infty}{\to} \langle a'',b'';\beta\,|a',b';\beta\rangle \quad (T>0)\,.$$

This is exactly what we set out to prove [see (3.6)].

We can do better, however, than only distributional convergence. In order to prove pointwise convergence of the $\mathscr{P}_\nu^0$, we first derive a formula relating the integral kernel of $\exp(-\nu AT)$ with $H_\pm$. This is done in the following two lemmas.

Lemma 3.5: For $t>0$, the operators $C^{-1/2}\times\exp[-tH_\pm]C^{-1/2}$ are trace class.

Lemma 3.6:

$$[\exp(-At)](a'',b'';a',b')$$
$$= \sum_{\epsilon=+,-}\operatorname{Tr}[U_\epsilon(a'',b'')^{-1}U_\epsilon(a',b')C^{-1/2}$$
$$\times\exp(-tH_\epsilon)C^{-1/2}]\,. \tag{3.21}$$

Proof of Lemma 3.6: We shall first derive (3.21), already assuming that $C^{-1/2}\exp[-tH_\pm]C^{-1/2}$ are trace class.

Let $\{\psi_j;\ j\in\mathbb{N}\}$ be an orthonormal base of $L^2(\mathbb{R}_+)$ such that $\psi_j\in D(C^{+1/2})\cap D(C^{-1/2})$ for all j. Define, as in (2.6),

$$f_{ij}^\pm(a,b) = \langle U_\pm(a,b)C^{-1/2}\psi_i,\psi_j\rangle\,.$$

The $f_{ij}^\pm$ constitute an orthonormal base of $L^2(M_+)$. Hence, at least in a distributional sense,

$$[\exp(-At)](a'',b'';a',b')$$
$$= \sum_{i,j,\epsilon}\sum_{k,l,\epsilon'} f_{ij}^\epsilon(a'',b'')\langle f_{ij}^\epsilon,\ e^{-At}f_{kl}^{\epsilon'}\rangle\ \overline{f_{kl}^{\epsilon'}(a',b')}\,. \tag{3.22}$$

It is clear from the proof of Lemma 3.3 that

$$(e^{-At}f_{kl}^{\epsilon'})(a,b) = \langle U_{\epsilon'}(a,b)C^{-1/2}e^{-H_{\epsilon'}t}\psi_k,\psi_l\rangle\,. \tag{3.23}$$

Note that $C^{-1/2}e^{-H_{\epsilon'}t}\psi_k$ is well defined, since $\psi_k\in D(C^{1/2})$, hence $\psi_k = C^{-1/2}\phi_k$ for some ϕ_k, and since $C^{-1/2}\exp(-H_{\epsilon'}t)C^{-1/2}$ is a bounded (even trace-class) operator. From (3.23), (2.2), and the orthogonality of $\mathscr{H}_+$ and $\mathscr{H}_-$ we obtain

$$\langle f_{ij}^\epsilon, e^{-At}f_{kl}^{\epsilon'}\rangle = \delta_{\epsilon,\epsilon'}\delta_{jl}\langle e^{-H_\epsilon t}\psi_k,\psi_i\rangle\,.$$

Substituting this into (3.22) leads to

$$[\exp(-At)](a'',b'';a',b')$$
$$= \sum_{\epsilon,i,j,k}\langle U_\epsilon(a'',b'')C^{-1/2}\psi_i,\psi_j\rangle$$
$$\times\langle\psi_j,U_\epsilon(a',b')C^{-1/2}\psi_k\rangle\langle e^{-H_\epsilon t}\psi_k,\psi_i\rangle$$
$$= \sum_\epsilon\operatorname{Tr}[U_\epsilon(a'',b'')^{-1}U_\epsilon(a',b')C^{-1/2}e^{-H_\epsilon t}C^{-1/2}]\,.$$

Since the final result of this calculation is clearly a continuous function in (a'',b''), (a',b'), we may conclude (3.21) pointwise, even though *a priori* (3.22) was true only in a distributional sense. □

We now turn to the proof of Lemma 3.5. In the course of the proof we shall not only prove that $C^{-1/2}\exp(-tH_\pm)C^{-1/2}$ are trace class, but also calculate an estimate of the trace. The method used in this estimation will be useful again in the next section, as well as the estimate itself.

Proof of Lemma 3.5: Again it is convenient to use the unitary transform (3.12). We have

$$UC^{-1/2}U^{-1}=(2\pi)^{-1/2}e^{s/2},$$

$$UH_\pm U^{-1}=-\frac{d^2}{ds^2}+V_\pm(s),$$

with $V_\pm(s)=e^{2s}\mp 2\beta e^s+(\beta-\frac{1}{2})^2$.

We thus have to study

$$e^{s/2}\exp[-T(-\Delta+V_\pm)]e^{s/2}$$

on $L^2(\mathbb{R})$. By the Feynman–Kac formula $\exp[-t(-\Delta+V_\pm)]$ and therefore also $e^{s/2}[-T(-\Delta+V_\pm)]e^{s/2}$ has a positive integral kernel. It is therefore trace class if and only if this integrable kernel is integrable, i.e., if

$$\int_{-\infty}^{\infty}ds\, e^{s/2}\{\exp[-T(-\Delta+V_\pm)]\}(s,s)e^{s/2}<\infty\,. \tag{3.24}$$

By the Feynman–Kac formula we have (see, e.g., Refs. 13 and 15)

$$\{\exp[-T(-\Delta+V_\pm)]\}(s,s)=\int d\rho_{W,T;s,s}\exp\left\{-\int_0^T dt\, V_\pm[\omega(t)]\right\}. \tag{3.25}$$

Here $d\rho_{W,T;s_1,s_2}$ is the familiar pinned Wiener measure. We have denoted it by ρ in order to distinguish it from our Wiener measure $d\mu_W^\nu$ on the Lobachevsky half-plane. The measure $d\rho_{W,T;s_1,s_2}$ is pinned at s_1 for $t=0$, at s_2 for $t=T$. It is a Gaussian measure with normalized connected convariance $(t_1\leqslant t_2)$

$$\langle\omega(t_1)\omega(t_2)\rangle^c=\langle\omega(t_1)\omega(t_2)\rangle-\langle\omega(t_1)\rangle\langle\omega(t_2)\rangle=2t_1(1-t_2/T)\,.$$

Substituting (3.25) into (3.24) gives

$$\int_{-\infty}^{\infty}ds\, e^s\int d\rho_{W,T;s,s}\exp\left\{-\int_0^T dt\, V_\pm[\omega(t)]\right\}$$

$$\leqslant\int_{-\infty}^{\infty}ds\, e^s\int d\rho_{W,T;s,s}\,T^{-1}\int_0^T dt\times\exp\{-TV_\pm[\omega(t)]\}\quad\text{(by Jensen's inequality)}$$

$$=\int_{-\infty}^{\infty}ds\, e^s\int d\rho_{W,T;0,0}\,T^{-1}\int_0^T dt\times\exp\{-TV_\pm[\omega(t)+s]\}$$

$$=\int d\rho_{W,T;0,0}\,T^{-1}\int_0^T dt\int_{-\infty}^{\infty}ds\, e^s\times\exp\{-TV_\pm[\omega(t)+s]\}$$

$$=\int d\rho_{W,T;0,0}\,T^{-1}\int_0^T dt\int_{-\infty}^{\infty}ds\, e^{s-\omega(t)}\times\exp[-TV_\pm(s)]$$

$$[\text{translates } s\to s+\omega(t) \text{ for every } t]. \tag{3.26}$$

(This technique, using first Jensen's inequality and then, after permuting the integrals over t and s, shifting s by $\omega(t)$, was used by Lieb[16] to derive bounds on the number of bound states for $-\Delta+V$; see also the discussion of the Lieb inequality in Ref. 15.)

One easily calculates

$$\int d\rho_{W,T;0,0}\,e^{-\omega(t)}=\frac{1}{\sqrt{\pi T}}e^{t(T-t)/T}, \tag{3.27}$$

and

$$T^{-1}\int_0^T dt\, e^{t(T-t)/T}=\int_0^1 dr\, e^{Tr(1-r)}\leqslant e^{T/4}. \tag{3.28}$$

On the other hand,

$$\int_{-\infty}^{\infty}ds\, e^s\exp[-TV_\pm(s)]=\int_0^\infty dx\exp\left\{-T\left[x^2\mp 2\beta x+\left(\beta-\frac{1}{2}\right)^2\right]\right\}\leqslant\frac{\sqrt{\pi}}{\sqrt{T}}\exp\left[\left(\beta-\frac{1}{4}\right)T\right]. \tag{3.29}$$

Putting together (3.26), (3.27), (3.28), and (3.29) shows that condition (3.24) is fulfilled. This means that $C^{-1/2}\exp(-H_\pm T)C^{-1/2}$ is trace class, and

$$\mathrm{Tr}[C^{-1/2}\exp(-H_\pm T)C^{-1/2}]\leqslant(1/T)e^{\beta T}. \tag{3.30}$$

□

With the help of Lemmas 3.5 and 3.6, and of estimate (3.30), we can prove (3.6) pointwise.

Proposition 3.7: Let $\mathscr{P}_\nu^0$ be defined by (3.5). Then, for all $T>0$, and for all (a'',b''), $(a',b')\in M_+$,

$$\lim_{\varphi\to\infty}\mathscr{P}_\nu^0(a'',b'';a',b';T)=\langle a'',b'';\beta|a',b';\beta\rangle\,.$$

Proof: By the definition of the affine coherent states in Sec. II B, and by (3.20), we have

$$\begin{aligned}\langle a'',b'';\beta|a',b';\beta\rangle&=\langle\psi_\beta|U_+(a'',b'')^{-1}U_+(a',b')\psi_\beta\rangle\\&=c_\beta\langle C^{-1/2}\phi_0|U_+(a'',b'')^{-1}U_+(a',b')C^{-1/2}\phi_0\rangle\\&=c_\beta\,\mathrm{Tr}[U_+(a'',b'')^{-1}U_+(a',b')C^{-1/2}P_0C^{-1/2}]\,,\end{aligned} \tag{3.31}$$

where $P_0=|\phi_0\rangle\langle\phi_0|$ is the zero-eigenvalue spectral projection operator of H_+.

Comparing (3.31) with Lemma 3.6, we find

$$\begin{aligned}&|c_\beta^{-1}[\mathscr{P}_\nu^0(a'',b'';a',b';T)-\langle a'',b'';\beta|a',b';\beta\rangle]|\\&\leqslant|\mathrm{Tr}[U_-(a'',b'')^{-1}U_-(a',b')C^{-1/2}e^{-\nu H_-t}C^{-1/2}]|\\&\quad+|\mathrm{Tr}[U_+(a'',b'')^{-1}U_+(a',b')\times C^{-1/2}(e^{-\nu H_+T}-P_0)C^{-1/2}]|\\&\leqslant|\mathrm{Tr}[C^{-1/2}e^{-\nu H_-T}C^{-1/2}]|\\&\quad+|\mathrm{Tr}[C^{-1/2}(e^{-\nu H_+T}-P_0)C^{-1/2}]|\,.\end{aligned} \tag{3.32}$$

The estimate (3.30) is not sufficient to conclude that this converges to 0 for $\nu\to\infty$. We can improve this estimate in the following way. For all $\lambda\in[0,1]$,

$$e^{-H_-t}\leqslant e^{H_-\lambda t}\|e^{-H_-(1-\lambda)t}\|$$
$$\leqslant e^{-H_-\lambda t}e^{-(1-\lambda)(\beta-1/2)^2t}.$$

Hence, for all $\lambda\in[0,1]$,

$$\mathrm{Tr}[C^{-1/2}e^{-H_-t}C^{-1/2}]$$
$$\leqslant e^{-(1-\lambda)(\beta-1/2)^2t}\,\mathrm{Tr}[C^{-1/2}e^{-\lambda H_-t}C^{-1/2}]$$
$$\leqslant e^{-(\beta-1/2)^2t}\,\frac{\exp\{[(\beta-1/2)^2+\beta]\lambda t\}}{2\lambda t}.$$

If $t\geqslant[\beta^2+1/4]^{-1}$, we can choose $\lambda=[t(\beta^2+1/4)]^{-1}\leqslant 1$, and we find

$$\mathrm{Tr}[C^{-1/2}e^{-H_-t}C^{-1/2}]\leqslant(\beta^2+\tfrac{1}{4})e^{1-(\beta-1/2)^2t}. \tag{3.33}$$

If $t\leqslant[\beta^2+\frac{1}{4}]^{-1}$, we take $\lambda=1$, and we find

$$\mathrm{Tr}[C^{-1/2}e^{-H_-t}C^{-1/2}]\leqslant(e/t)\,e^{-(\beta-1/2)^2t}. \tag{3.34}$$

Combining (3.33) and (3.34) we find that there exists a constant $\not{c}$ such that, for all $t>0$,

$$\mathrm{Tr}[C^{-1/2}e^{-H_-t}C^{-1/2}]\leqslant\not{c}(1+t^{-1})e^{-(\beta-1/2)^2t}. \tag{3.35}$$

The same can be done for $e^{-H_+t}-P_0$. There, the basic inequality is

$$e^{-H_+t}-P_0\leqslant(e^{-H_+\lambda t}-P_0)\cdot\|e^{-H_+(1-\lambda)t}-P_0\|$$
$$\leqslant e^{-H_+\lambda t}\cdot\exp[-(1-\lambda)B(\beta)t],$$

with

$$B(\beta)=\begin{cases}(\beta-\frac{1}{2})^2, & \text{if } \beta<\frac{3}{2},\\ 2(\beta-1), & \text{if } \beta>\frac{3}{2}.\end{cases} \tag{3.36}$$

This distinction is due to the fact that H_+ has more than one bound state if $\beta>\frac{3}{2}$. In this case $2(\beta-1)$ is the energy difference between the ground state and the first excited state. The estimate for H_+, corresponding to the inequality (3.35) for H_-, is then

$$\mathrm{Tr}[C^{-1/2}(e^{-H_+t}-P_0)C^{-1/2}]\leqslant\not{c}(1+t^{-1})e^{-B(\beta)t}. \tag{3.37}$$

Substituting the estimates (3.37) and (3.35) into (3.32) leads to

$$|\mathscr{P}^0_\nu(a'',b'';a',b';T)-\langle a'',b'';\beta|a',b';\beta\rangle|$$
$$\leqslant\not{c}[1-(\nu T)^{-1}]\exp[-B(\beta)\nu T], \tag{3.38}$$

where $\not{c}$ denotes a constant [not the same as in (3.37) or (3.35)] which depends on β, but not on ν or T. It is clear that $(3.38)\to 0$ for $\nu\to\infty$. This concludes our proof.

For zero Hamiltonian, we have thus achieved our aim. We have given a sense to the formal expression (3.1) by regularizing it by means of a Wiener measure with diffusion constant ν, and we have proved that we obtain the expected result for $\nu\to\infty$.

IV. THE PATH INTEGRAL FOR NONZERO HAMILTONIAN

For nonzero Hamiltonians our strategy will essentially be the same as for the zero-Hamiltonian case. We regularize (3.1) by means of a Wiener measure with diffusion constant ν, i.e., we define

$$\mathscr{P}^h_\nu(a'',b'';a',b';T)=c_\beta e^{\nu T\beta}\int\exp\Big[-i\beta\int a^{-1}\,db$$
$$-i\int h(a,b)dt\Big]\,d\mu^\nu_W(a,b). \tag{4.1}$$

Again the stochastic integral $\int a^{-1}\,db$ should be understood in the Stratonovich sense. We shall show that in the limit for ν tending to ∞, $\mathscr{P}^h_\nu$ tends to the affine coherent state matrix element $\langle a'',b'';\beta|\exp(-iTH)|a',b';\beta\rangle$, where

$$H=c_\beta^{-1}\int d\mu(a,b)|a,b;\beta\rangle h(a,b)\langle a,b;\beta|. \tag{4.2}$$

Our proof of this statement will run along the same lines as for the Weyl–Heisenberg case, in Ref. 2. We shall therefore not repeat the whole argument. We shall prove some basic estimates and show how, given these estimates, the proofs in Ref. 4 carry over to the affine path integrals studied here.

The proof, in Ref. 4 of the convergence, for $\nu\to\infty$, of the ν-dependent path integral $\mathscr{P}^h_\nu$ proceeded in essentially three steps. First it was shown that $\mathscr{P}^h_\nu$ was the integral kernel of a contraction semigroup. Then strong convergence, as $\nu\to\infty$, of these contraction operators was proved; this led to convergence of the $\mathscr{P}^h_\nu$ in a distributional sense. Finally, pointwise convergence of the $\mathscr{P}^h_\nu$ was proved. For these three steps, different conditions of a technical nature were imposed on the function h.

We shall distinguish these same three steps here. We start however with a subsection listing different conditions on h and estimates following from these conditions. These estimates will be needed in the following three subsections, outlining the proof of our main result.

A. Conditions on the function *h* and various estimates

The first estimate will ensure that $\mathscr{P}^h_\nu$ is a well-defined expression, i.e., that

$$\exp\Big\{-i\int_0^T dt\;h\,[a(t),b(t)]\Big\}$$

is integrable with respect to $d\mu^\nu_W$. For this it is sufficient that

$$\int d\mu^\nu_W\int_0^T dt\,|h\,[a(t),b(t)]|<\infty. \tag{4.3a}$$

This can be rewritten as

$$\int_0^T dt\int d\mu(a,b)K_{T-t}(a'',b'';a,b)$$
$$\times|h(a,b)|K_t(a,b;a',b')<\infty, \tag{4.3b}$$

with K_t as defined by (3.3).

The following lemma gives a sufficient condition on h for (4.3) to hold.

Lemma 4.1: Define

$$D(a,b):=d(a,b;1,0)=\cosh^{-1}\left[\frac{1+a^2+b^2}{2a}\right] \tag{4.4}$$

[see (2.12)]. If, for all $\alpha>0$,

$$k_\alpha(h)=\int d\mu(a,b)|h(a,b)|^2\exp[-\alpha D(a,b)^2]<\infty , \tag{4.5}$$

then, for all (a',b'), $(a'',b'')\in M_+$, and all $T>0$,

$$\int_0^T dt\int d\mu(a,b)K_{T-t}(a'',b'';a,b)|h(a,b)|K_t(a,b;a'b') \leqslant ¢\,[k_{(16T)^{-1}}(h)]^{1/2} \times \exp\{1/16T\,[D(a',b')^2+D(a'',b'')^2]\}. \tag{4.6}$$

Note: We shall, throughout this section, denote all constants by ¢ without further identification. A constant ¢ may depend on β. Occasionally as in (4.6) the constant ¢ may also depend on T. In all the cases where the T dependence is important, however, we shall explicitly keep track of it.

Proof: By (3.3) we have

$$\int_0^T dt\,K_{T-t}(a'',b'';a,b)K_t(a,b;a',b') \leqslant ¢\int_{\delta'}^\infty dx\frac{xe^{-x^2/8T}}{\sqrt{\cosh x-\cosh\delta'}} \times\int_{\delta''}^\infty dy\frac{ye^{-y^2/8T}}{\sqrt{\cosh y-\cosh\delta''}}I(x,y),$$

where $\delta'=d(a,b;a',b')$, $\delta''=d(a,b;a'',b'')$, and with $I(x,y)$ given by

$$\begin{aligned}I(x,y)&=\int_0^T dt\,[t(T-t)]^{-3/2}e^{-x^2/8t}e^{-y^2/8(T-t)}\\&\leqslant\left[\frac{T}{2}\right]^{-3/2}e^{-y^2/8T}\int_0^{T/2}dt\,t^{-3/2}e^{-x^2/8t}\\&\quad+\left[\frac{T}{2}\right]^{-3/2}e^{-x^2/8T}\int_0^{T/2}dt\,t^{-3/2}e^{-y^2/8t}\\&\leqslant ¢T^{-3/2}(x^{-1}+y^{-1})\int_0^\infty ds\,s^{-3/2}e^{-1/8s}\\&\leqslant ¢T^{-3/2}(x^{-1}+y^{-1}).\end{aligned}$$

On the other hand

$$\begin{aligned}\int_\delta^\infty dx\frac{e^{-\alpha x^2}}{\sqrt{\cosh x-\cosh\delta}}&\leqslant\frac{1}{\sqrt{\sinh\delta}}\int_0^\infty du\frac{e^{-\alpha(u+\delta)^2}}{\sqrt{u}}\\&\leqslant ¢\alpha^{-1/4}\frac{e^{-\alpha\delta^2}}{\sqrt{\sinh\delta}}\end{aligned}$$

$$\begin{aligned}\int_\delta^\infty dx\frac{xe^{-\alpha x^2}}{\sqrt{\cosh x-\cosh\delta}}&=\int_0^\infty du\frac{(u+\delta)e^{-\alpha(u+\delta)^2}}{\sqrt{\cosh\delta(\cosh u-1)+\sinh\delta\sinh u}}\\&\leqslant\frac{e^{-\alpha\delta^2}}{\sqrt{\cosh\delta}}\int_0^\infty du\frac{u}{\sqrt{\cosh u-1}}+\frac{e^{-\alpha\delta^2}\delta}{\sqrt{\sinh\delta}}¢\alpha^{-1/4}\\&\leqslant(1+\alpha^{-1/4}\delta^{1/2})e^{-\alpha\delta^2}\leqslant ¢(1+\alpha^{-1/2})e^{-\alpha\delta^2/2}.\end{aligned}$$

Hence

$$\int_0^T dt\,K_{T-t}(a'',b'';a,b)K_t(a,b;a',b') \leqslant ¢T^{-5/4}(1+T^{-1/2})\left[\frac{1}{\sqrt{\sinh\delta'}}+\frac{1}{\sqrt{\sinh\delta''}}\right] \times e^{-(\delta'^2+\delta''^2)/16T}. \tag{4.7}$$

This implies

$$\begin{aligned}&\int_0^T dt\int d\mu(a,b)K_{T-t}(a'',b'';a,b)|h(a,b)|K_t(a,b;a',b')\\&\leqslant ¢T^{-5/4}(1+T^{-1/2})\\&\quad\times\left\{\left[\int d\mu(a,b)|h(a,b)|^2e^{-d(a,b;a',b')^2/8T}\right]^{1/2}\right.\\&\quad\times\left.\left[\int d\mu(a,b)\frac{e^{-d(a,b;a'',b'')^2/8T}}{\sinh[d(a,b;a'',b'')]}\right]^{1/2}\right\}\\&\quad+\text{idem with roles of } a',b' \text{ and } a'',b'' \text{ reversed}.\end{aligned}$$

Since

$$D(a,b)\leqslant d(a,b;a'b')+D(a',b')$$

hence

$$-d(a,b;a',b')^2\geqslant-\tfrac{1}{2}D(a,b)^2+D(a',b')^2,$$

the first factor is finite by (4.5). We only need to prove still that, for all $\alpha>0$,

$$\int d\mu(a,b)\frac{e^{-\alpha d(a,b;a',b')^2}}{\sinh[d(a,b;a',b')]}<\infty ,$$

in order to conclude (4.6). Since both the measure $d\mu(a,b)$ and the metric distance d are (left) invariant, it suffices to prove, for all $\alpha>0$,

$$\int d\mu(a,b)\frac{e^{-\alpha D(a,b)^2}}{\sinh D(a,b)}<\infty . \tag{4.8}$$

A careful analysis of the singularities of the integrand in (4.8), using the definition (4.4) of $D(a,b)$, shows that this integral is indeed finite.

Remark: We shall also need the following similar estimate. From (4.7) we obtain

$$\begin{aligned}&\int_0^T dt\int d\mu(a,b)K_{T-t}(a'',b'';a,b)|h(a,b)|K_t(a.b;a',b')\\&\leqslant ¢T^{-5/4}(1+T^{-1/2})\left(\int d\mu(a,b)|h(a,b)|^2\right.\\&\quad\left.\times\exp\left\{\frac{1}{16T}[d(a,b;a',b')^2+d(a,b;a'',b'')^2]\right\}\right)^{1/2}\\&\quad\times\left\{\int d\mu(a,b)[\sinh D(a,b)]^{-1}\right.\\&\quad\left.\times\exp\left[-\frac{1}{16T}D(a,b)^2\right]\right\}^{1/2}.\end{aligned} \tag{4.9}$$

Using the triangle inequality for the metric d one finds that

$$d(a,b;a',b')^2+d(a,b;a'',b'')^2 \geqslant\tfrac{1}{3}D(a,b)^2+\tfrac{1}{3}D(a',b')^2-D(a'',b'')^2.$$

Inserting this into (4.9) we find

$$\int_0^T dt \int d\mu(a,b) K_{T-t}(a'',b'';a,b)\,|h(a,b)|\,K_t(a,b;a',b')$$
$$\leqslant c\!\!/ \exp[(1/8T)D(a'',b'')^2 - (1/40T)D(a',b')^2]\,[k_{(80T)^{-1}}(h)]^{1/2}. \qquad (4.10)$$

We shall impose conditions on the function h other than only (4.3). To formulate them, we first need the following definitions.

For $(a',b')\in M_+$, and $t>0$, we define the following functions on M_+:

$$\phi_{a',b';t}(a,b) = [\exp(-tA)](a,b;a',b'), \qquad (4.11)$$
$$\phi_{a',b';\infty}(a,b) = c_\beta^{-1}\langle a,b;\beta\,|a',b';\beta\rangle = P_\beta(a,b;a',b'). \qquad (4.12)$$

It is clear that

$$\overline{\phi_{a',b';t}(a,b)} = \phi_{a,b;t}(a',b'),$$
$$\overline{\phi_{a',b';\infty}(a,b)} = \phi_{a,b;\infty}(a',b').$$

Some of the calculations in Sec. III can be viewed as estimates on the L^2- and L^∞-norms of these vectors and their difference. We have

$$\|\phi_{a',b';\infty}\| = c_\beta^{-1/2} \quad [\text{by (2.10)}], \qquad (4.13)$$
$$\|\phi_{a',b';t} - \phi_{a',b';\infty}\|^2 = \int d\mu(a,b)\,|[\exp(-tA) - P_\beta](a,b;a',b')|^2$$
$$= [\exp(-2tA) - P_\beta](a',b';a',b')$$
$$= \mathrm{Tr}\{C^{-1/2}[(e^{-2tH_+} - P_0) + e^{-2tH_-}]C^{-1/2}\} \quad [\text{by Lemma 3.6 and (3.31)}]$$
$$\leqslant c\!\!/(1-t^{-1})e^{-2B(\beta)t} \quad [\text{by (3.35), (3.37)}],$$

where

$$B(\beta) = \begin{cases}(\beta-1/2)^2, & \text{if } \beta\leqslant\frac{3}{2},\\ 2(\beta-1), & \text{if } \beta\geqslant\frac{3}{2}.\end{cases} \qquad (4.14)$$

Hence

$$\|\phi_{a',b';t} - \phi_{a',b';\infty}\| \leqslant c\!\!/(1-t^{-1/2})e^{-B(\beta)t}, \qquad (4.15)$$
$$\|\phi_{a',b';t}\| \leqslant c_\beta^{-1/2} + c\!\!/(1+t^{-1/2})e^{-B(\beta)t}. \qquad (4.16)$$

On the other hand, the estimate (3.38) can be rewritten as

$$\|\phi^{a',b';t} - \phi_{a',b';\infty}\|_\infty = \sup_{a,b\in M_+} |(\phi_{a',b';t} - \phi_{a',b';\infty})(a,b)| \leqslant c\!\!/(1+t^{-1})\exp[-B(\beta)t]. \qquad (4.17)$$

In the following three sections we shall consider the multiplication operator h on $L^2(M_+)$ defined by

$$(hf)(a,b) = h(a,b)\,f(a,b).$$

We shall restrict ourselves to real functions h. Then the multiplication operator is self-adjoint, with domain

$$D(h) = \{f\in L^2(M_+);\ hf\in L^2(M_+)\}.$$

In the remainder of this subsection we shall determine sufficient conditions on h ensuring that $\phi_{a',b';\infty}$ and $\phi_{a',b';t}$ are elements of $D(h)$, i.e.,

$$\int d\mu(a,b)\,|h(a,b)|^2|\phi_{a',b';\infty}(a,b)|^2 < \infty, \qquad (4.18)$$
$$\int d\mu(a,b)\,|h(a,b)|^2|\phi_{a',b';t}(a,b)|^2 < \infty. \qquad (4.19)$$

We shall also estimate $\|h(\phi_{a',b';t} - \phi_{a',b';\infty})\|$. We start with (4.18), the easiest one.

Lemma 4.2: If

$$\int d\mu(a,b)\,|h(a,b)|^2\left[\frac{2a}{1+a^2+b^2}\right]^{2\beta} < \infty, \qquad (4.20)$$

then (4.18) is satisfied, and

$$\|h\phi_{a',b';\infty}\| \leqslant c\!\!/\left[\frac{1+a'^2+b'^2}{2a'}\right]^\beta. \qquad (4.21)$$

Proof: By (2.11), we find

$$\|h\phi_{a',b';\infty}\|^2 = \int d\mu(a,b)\,|h(a,b)|^2 c_\beta^{-2}|\langle a,b;\beta\,|a',b';\beta\rangle|^2$$
$$= c_\beta^{-2}\int d\mu(a,b)\,|h(a,b)|^2 \times\left[\frac{1+\cosh d(a,b;a',b')}{2}\right]^{-2\beta}$$
$$\leqslant c\!\!/\int d\mu(a,b)\,|h(a,b)|^2\left[\frac{1+\cosh D(a,b)}{1+\cosh D(a',b')}\right]^{-2\beta}$$
$$[\text{use } D(a,b)\leqslant d(a,b;a',b') + D(a',b')]$$
$$\leqslant c\!\!/\left(\frac{1+a'^2+b'^2}{2a'}\right)^{2\beta}\int d\mu(a,b)\,|h(a,b)|^2 \times\left[\frac{2a}{1+a^2+b^2}\right]^{2\beta} < \infty. \qquad \square$$

The other two estimates involve some additional calculation. We start by estimating weighted L^p-norms of $\phi_{a',b';t}$.

Lemma 4.3: For $\lambda>1$, $\mu>0$, one has

$$I_{\lambda,\mu}(t) \equiv \int d\mu(a,b)\,|[\exp(-tA)](a,b;1,0)|^\lambda \times\left[\frac{1+a^2+b^2}{2a}\right]^\mu$$
$$\leqslant c\!\!/\ (1/\sqrt{\lambda-1})\,t^{1-\lambda}\exp[\epsilon(\lambda;\mu)t], \qquad (4.22)$$

with

$$\epsilon(\lambda,\mu) = \lambda\left(\beta-\frac{1}{4}\right) + \max\left[\frac{(1-\lambda)^2}{4} + \frac{(\mu+1/2)^2}{\lambda-1},\ M\left(\mu+\frac{1-\lambda}{2}\right) + \frac{1}{4(\lambda-1)}\right],$$

where, for all $\alpha\in\mathbb{R}$, $M(\alpha) = \max(\alpha,\alpha^2)$.

Proof: We first estimate $|[\exp(-tA)](a,b;1,0)|$, using the same technique as in the proof of Lemma 3.5. By Lemma 3.6

$$\overline{[\exp(-tA)](a,b;1,0)} = [\exp(-tA)](1,0;a,b) = \sum_\epsilon \mathrm{Tr}\,[U_\epsilon(a,b)C^{-1/2}e^{-H_\epsilon t}C^{-1/2}].$$

Using again the unitary transform (3.12) we can rewrite this as (using the Feynman–Kac formula)

$$[\exp(-TA)](1,0;a,b)$$
$$=\sum_\epsilon \int_{-\infty}^{\infty} ds\, e^{i\epsilon b e^s} \frac{1}{2\pi} e^{s+(1/2)\ln a}$$
$$\times \exp\left[-T\left(-\frac{d^2}{ds^2}+V_\epsilon\right)\right](s+\ln a, s)$$
$$=\not c \sum_\epsilon \int_{-\infty}^{\infty} ds\, e^{i\epsilon b e^s} e^{s+(1/2)\ln a} \int d\rho_{W,T;0,\ln a}$$
$$\times \exp\left\{-\int_0^T dt\, V_\epsilon[s+\omega(t)]\right\}$$
$$=\sum_\epsilon \int_0^\infty dx\, e^{i\epsilon bx}\sqrt{a}\int d\rho_{W,T;0,\ln a}$$
$$\times \exp\left\{-\int_0^T dt\, V_\epsilon[\omega(t)+\ln x]\right\}$$
$$=\not c \int_{-\infty}^{\infty} dx\, e^{ibx}\sqrt{a}\int d\rho_{W,T;0,\ln a}$$
$$\times \exp\left\{-\int_0^T dt\, V_{\epsilon(x)}\{\omega(t)+\ln|x|\}\right\}, \qquad (4.23)$$

where $\epsilon(x)=x/|x|$ for $x\neq 0$. Since

$$V_{\epsilon(x)}[\omega(t)+\ln|x|]$$
$$=x^2e^{2\omega(t)}-2\beta\epsilon(x)|x|e^{\omega(t)}+(\beta-\tfrac{1}{2})^2$$
$$=x^2e^{2\omega(t)}-2\beta x e^{\omega(t)}+(\beta-\tfrac{1}{2})^2,$$

we have

$$[\exp(-TA)](1,0;a,b)$$
$$=\not c\sqrt{a}\, e^{(\beta-1/4)T}\int d\rho_{W,T;0,\ln a}$$
$$\times\int_{-\infty}^{\infty} dx\, e^{ibx}\exp[-(a_0x^2-2a_1x+a_2)],$$

where

$$a_0=\int_0^T dt\, e^{2\omega(t)},\quad a_1=\beta\int_0^T dt\, e^{\omega(t)},$$
$$a_2=\beta^2T.$$

Hence

$$[\exp(-TA)](1,0;a,b)$$
$$=\not c\sqrt{a}\, e^{(\beta-1/4)T}\int d\rho_{W,T;0,\ln a}\, a_0^{-1/2} e^{iba_1/a_0}$$
$$\times e^{-b^2/4a_0}e^{-a_2+a_1^2/a_0}. \qquad (4.24)$$

From the Cauchy–Schwarz inequality

$$a_1^2=\beta^2T^2\left[\int_0^T\frac{dt}{T}e^{\omega(t)}\right]^2\leqslant\beta^2T^2\int_0^T\frac{dT}{T}e^{2\omega(t)},$$

hence $-a_2+a_1^2/a_0\leqslant 0$. Hence

$$|[\exp(-TA)](a,b;1,0)|$$
$$\leqslant\not c\sqrt{a}\, e^{(\beta-1/4)T}\int d\rho_{W,T;0,\ln a}[a_0(\omega)]^{-1/2}$$
$$\times\exp\left[-\frac{b^2}{4a_0(\omega)}\right],$$

with

$$a_0(\omega)=\int_0^T dt\, e^{2\omega(t)}.$$

Since $\lambda>1$, we find (use either Jensen's or Young's inequality)

$$\left[\int d\rho_{W,T;0,\ln a}[a_0(\omega)]^{-1/2}e^{-b^2/4a_0(\omega)}\right]^\lambda$$
$$\leqslant\not c\, T^{-(\lambda-1)/2}\exp\left[-\frac{\lambda-1}{4T}(\ln a)^2\right]$$
$$\times\int d\rho_{W,T;0,\ln a}[a_0(\omega)]^{-\lambda/2}e^{-\lambda b^2/4a_0(\omega)}.$$

Hence [see (4.22)],

$$I_{\lambda,\mu}(T)\leqslant\not c\, e^{\lambda T(\beta-1/4)}T^{-(\lambda-1)/2}\int_0^\infty da\int_{-\infty}^{\infty}db\, a^{\lambda/2-2-\mu}$$
$$\times(1+a^{2\mu}+b^{2\mu})\exp\left[-\frac{\lambda-1}{4T}(\ln a)^2\right]\int d\rho_{W,T;0,\ln a}[a_0(\omega)]^{-\lambda/2}e^{-\lambda b^2/4a_0(\omega)}$$
$$\leqslant\not c\, e^{\lambda T(\beta-1/4)}T^{-(\lambda-1)/2}\int_0^\infty da\int d\rho_{W,T;0,\ln a}\, a^{\lambda/2-2-\mu}$$
$$\times\exp\left[-\frac{\lambda-1}{4T}(\ln a)^2\right]\{(1+a^{2\mu})[a_0(\omega)]^{(1-\lambda)/2}+[a_0(\omega)]^{\mu+(1-\lambda)/2}\}. \qquad (4.25)$$

Let us estimate

$$J_{\delta,T}(x)\equiv\int d\rho_{W,T;0,x}\left[\int_0^T dt\, e^{2\omega(t)}\right]^\delta.$$

If either $\delta\leqslant 0$ or $\delta>1$ we can apply Jensen's inequality and obtain

$$J_{\delta,T}(x)\leqslant\int d\rho_{W,T;0,x}T^{\delta-1}\int_0^T dt\, e^{2\delta\omega(t)}$$
$$\leqslant\not c\, T^{\delta-1}\int_0^T dt\frac{1}{\sqrt{t(T-t)}}\int_{-\infty}^{\infty}dy\exp\left[-\frac{y^2}{4t}-\frac{(y-x)^2}{4(T-t)}+2\delta y\right]$$
$$\leqslant\not c\, T^{\delta-1}\int dy\exp\left[-\frac{(y-x/2)^2}{T}+2\delta y\right]\int_0^T dt\,[t(T-t)]^{-1/2}\leqslant\not c\, T^{\delta-1/2}e^{T\delta^2}e^{\delta x}. \qquad (4.26)$$

If $0\leqslant\delta\leqslant 1$, then, by Young's inequality,

$$J_{\delta,T}(x)\leqslant\left[\int d\rho_{W,T;0,x}\int_0^T dt\, e^{2\omega(t)}\right]^{\delta}\left[\int d\rho_{W,T;0,x}\right]^{1-\delta}\leqslant ¢\, T^{\delta-1/2}e^{\delta T}e^{\delta x}. \tag{4.27}$$

Combining (4.26) with (4.27) we obtain, with $M(\delta)=\max(\delta,\delta^2)$,

$$J_{\delta,T}(x)\leqslant ¢\, T^{\delta-1/2}e^{\delta x}e^{M(\delta)T}.$$

Substituting this into (4.25) we find

$$I_{\lambda,\mu}(T)\leqslant ¢\, e^{\lambda T(\beta-1/4)}\,T^{-\lambda+1/2}\int_{-\infty}^{\infty}dx\exp\left[-\frac{\lambda-1}{4T}x^2+\frac{x}{2}\right]\left[e^{T(1-\lambda)^2/4}e^{-\mu x}(1+e^{2\mu x})+T^{\mu}e^{TM(\mu+(1-\lambda)/2)}\right]$$

$$\leqslant ¢\, T^{1-\lambda}e^{\lambda T(\beta-1/4)}\frac{1}{\sqrt{\lambda-1}}\left[\exp\left\{T\left[\frac{(1-\lambda)^2}{4}+\frac{(\mu+1/2)^2}{\lambda-1}\right]\right\}\right.$$

$$\left.+T^{\mu}\exp\left\{T\left[M\left(\mu+\frac{1-\lambda}{2}\right)+\frac{1}{4(\lambda-1)}\right]\right\}\right].$$

It is easy to see that this leads to (4.22). □

With the help of Lemma 4.3 we can now estimate

$$\|h(\phi_{a',b';t}-\phi_{a',b';\infty})\|.$$

Lemma 4.4: Let h be a function satisfying

$$C_{\mu,r}(h)\equiv\int d\mu(a,b)\,|h(a,b)|^{2+r}\left[\frac{2a}{1+a^2+b^2}\right]^{\mu}<\infty, \tag{4.28}$$

for positive parametes r,μ satisfying the following conditions:

$$\mu<r(\beta-\tfrac{1}{2})+2\beta, \tag{4.29a}$$

$$\sup_{\alpha\in(m,1)}\left[2(1-\alpha)B(\beta)-\frac{r}{r+2}\,\epsilon\left(\alpha\,\frac{2(\pi+2)}{r},\frac{2\mu}{r}\right)\right]>0, \tag{4.29b}$$

where

$$m=\frac{r}{2(\pi+2)}\max\left(1,\frac{1+2\mu/r}{\beta}\right).$$

Here

$$\epsilon(\lambda,\gamma)=\lambda\left(\beta-\frac{1}{4}\right)+\max\left[\frac{(1-\lambda)^2}{4}+\frac{(\gamma+1/2)^2}{\lambda-1},\quad M\left(\gamma+\frac{1-\lambda}{2}\right)+\frac{1}{4(\lambda-1)}\right],$$

with $M(\delta)=\max(\delta,\delta^2)$, and $B(\beta)=(\beta-\frac{1}{2})^2$ if $\beta\leqslant\frac{3}{2}$, $B(\beta)=2(\beta-1)$ if $\beta\geqslant\frac{3}{2}$.

Then there exist constants $¢_1$, $¢_2>0$ such that

$$\|h(\phi_{a',b';t}-\phi_{a',b';\infty})\|\leqslant ¢_1[1+t^{-1+r/[2(r+2)]}]e^{-¢_2 t}\left(\frac{1+a'^2+b'^2}{2a'}\right)^{\mu/(r+2)}. \tag{4.30}$$

Proof:

$$\|h(\phi_{a',b';t}-\phi_{a',b';\infty})\|^2\leqslant\left[\int d\mu(a,b)\,|h(a,b)|^{2+r}\left(\frac{2a}{1+a^2+b^2}\right)^{\mu}\right]^{2/(r+2)}$$

$$\times\left[\int d\mu(a,b)\,|[\phi_{a',b';t}-\phi_{a',b';\infty}](a,b)|^{2(r+2)/r}\left(\frac{1+a^2+b^2}{2a}\right)^{2\mu/r}\right]^{r/(r+2)}$$

$$\leqslant[C_{\mu,r}(h)]^{2/(r+2)}\quad\sup_{a,b}|[\phi_{a',b';t}-\phi_{a',b';\infty}](a,b)|^{2(1-\alpha)}$$

$$\times\left\{\int d\mu(a,b)\,|[\phi_{a',b';t}-\phi_{a',b';\infty}](a,b)|^{2\alpha(r+2)/r}[\cosh D(a',b')\cosh d(a,b;a',b')]^{2\mu/r}\right\}^{r/(r+2)}$$

$$\leqslant ¢[1+t^{-2(1-\alpha)}]\,e^{-2(1-\alpha)B(\beta)t}\left(\frac{1+a'^2+b'^2}{2a'}\right)^{2\mu/(r+2)}$$

$$\times\left\{\int d\mu(a,b)\left[|\phi_{1,0;t}(a,b)|^{2\alpha(r+2)/r}+|\phi_{1,0;\infty}(a,b)|^{2\alpha(r+2)/r}\right]\left(\frac{1+a^2+b^2}{2a}\right)^{2\mu/r}\right\}^{r/(r+2)}$$

[use (4.17) and the left invariance of the measure $d\mu$]. This holds for all $\alpha\in[0,1]$. If we choose α such that $\alpha>m$, with m as defined above, then $2\alpha\beta(r+2)/r-2\mu/r>1$, hence

$$\int d\mu|\phi_{1,0;\infty}(a,b)|^{2\alpha(r+2)/r}\left(\frac{1+a^2+b^2}{2a}\right)^{2\mu/r}$$
$$=\not c\int d\mu\left(\frac{2a}{1+a^2+b^2}\right)^{2\alpha\beta(r+2)/r-2\mu/r}<\infty\ .$$

On the other hand $\alpha>m$ also implies $2\alpha(r+2)/r>1$. Using Lemma 4.3 leads then to

$$\|h(\phi_{a',b';t}-\phi_{a',b';\infty})\|$$
$$\leqslant\not c[1+t^{-(1-\alpha)}][1+t^{r/[2(r+2)]-\alpha}]$$
$$\times e^{-2(1-\alpha)B(\beta)t}\left(\frac{1+a'^2+b'^2}{2a'}\right)^{2\mu/(r+2)}$$
$$\times[1+I_{2\alpha(r+2)/r,2\mu/r}]^{r/(r+2)}$$
$$\leqslant\not c[1+t^{-1+r/[2(r+2)]}]\left(\frac{1+a'^2+b'^2}{2a'}\right)^{2\mu/(r+2)}$$
$$\times\exp\left[-(1-\alpha)B(\beta)t+\frac{r}{2(r+2)}t\epsilon\right.$$
$$\left.\times\left(\frac{2\alpha(r+2)}{r},\frac{2\mu}{r}\right)\right].$$

This holds for all $\alpha\in(m,1]$. It is clear from this that (4.30) follows if the conditions (4.29) are satisfied. ■

Remark: The conditions (4.29) are sufficient conditions on the pair (r,μ), given β, ensuring that (4.30) holds. The conditions (4.29) are however rather complicated, and may not be easy to check. It is possible, of course, to only consider one value for α, instead of the whole interval $(m,1)$. This considerably simplifies the condition on r,μ, but may be too restrictive. One possibility of choosing such a fixed value for α is, e.g., $\alpha=r/(r+2)$. It is then sufficient that

$$\mu<r(\beta-\tfrac{1}{2})\ ,$$
$$r<\frac{4B(\beta)}{\epsilon(2,2\beta-1)}\ ,$$

to ensure that the conditions (4.29) are satisfied. This allows only a finite range for the parameter r, however, and is thus very restrictive. It turns out that it is easier to proceed in the inverse direction, i.e., to start from the pair (r,μ) and to determine for which values of β the conditions (4.29) are satisfied. One finds that the following conditions imply (4.29):

$$\beta\geqslant r(1+2\mu/r)/2(r+2)]\ ,$$
$$\beta>\tfrac{3}{2}, \tag{4.31a}$$
$$\beta>\frac{1}{2(2-3\alpha)}\left[4-\tfrac{9}{2}\alpha+\frac{r}{r+2}\tilde{\epsilon}\left(2\alpha\frac{r+2}{r},\frac{2\mu}{r}\right)\right],$$

for some α satisfying

$$\frac{r}{2(r+2)}<\alpha<\tfrac{2}{3}\text{ and }\alpha<\frac{r(1+2\mu/r)}{2\beta(r+2)}. \tag{4.31b}$$

Here $\tilde{\epsilon}$ is defined by

$$\tilde{\epsilon}(\lambda,\gamma)=\epsilon(\lambda,\gamma)-\lambda(\beta-\tfrac{1}{4})$$
$$=\max\left[\frac{(1-\lambda)^2}{4}+\frac{(\gamma+1/2)^2}{\lambda-1},\right.$$
$$\left.M\left(\gamma+\frac{1-\lambda}{2}\right)+\frac{1}{4(\lambda-1)}\right], \tag{4.31c}$$

with $M(x)=\max(x,x^2)$.

Note that the second condition on α in (4.31b) is an implicit condition, since it contains β again, and β is bounded below by a function depending on α. In the explicit examples below (see Remark 2 at the end of Sec. IV) we shall first disregard this extra condition on α, compute a lower bound on β, and then verify that the condition is satisfied.

Our last estimate involves $\|h\phi_{a',b';t}\|$. From Lemma 4.2 and 4.4 one immediately has

$$|h\phi_{a',b';t}|<\not c\ [1+t^{-1+r/2(r+2)}]\left(\frac{1+a'^2+b'^2}{2a'}\right)^{2\mu/r} \tag{4.32}$$

if h satisfies (4.28), where μ, r, β fulfill either the conditions (4.29) or the conditions (4.31).

All in all we have three different technical conditions on h. The first one, (4.5), ensures that $\mathscr{P}^h_\nu$ is well defined. The second one, (4.20), ensures that $\phi_{a',b';\infty}\in D(h)$ for all $(a',b')\in M_+$. The third one, (4.28), ensures that $\phi_{a',b';t}\in D(h)$ for all $(a',b')\in M_+$, and all $t>0$. Note that $(4.28)\rightarrow(4.20)\rightarrow(4.5)$.

In what follows we shall always assume that (4.28) is satisfied.

B. The path as integral kernel of a contraction semigroup

Since h satisfied condition (4.28), hence condition (4.5), we know by Lemma 4.1 that $\mathscr{P}^h_\nu$ is well-defined. Copying the argument in Ref. 4 the following proposition can be proved.

Proposition 4.5: Let h be a real function satisfying condition (4.28). Then there exists a strongly continuous semigroup of contractions $E(\nu,h;t)$ on $L^2(M_+;d\mu)$ such that

$$[E(\nu,h;t)](a'',b'';a',b')=c_\beta^{-1}\,\mathscr{P}^h_\nu(a'',b'';a',b';t)\ . \tag{4.33}$$

These contraction operators are related to $\exp(-\nu AT)$ by the integral equation

$$\langle f_2,E(\nu,h;T)f_1\rangle=\langle f_2,e^{-\nu AT}f_1\rangle-i\int_0^T dt$$
$$\times\langle f_2,E(\nu,h;T-t)he^{-\nu AT}f_1\rangle\ . \tag{4.34}$$

This integral equation holds if f_1, $f_2\in C_0^\infty(M_+)$ or if $f_1\in D$ and $f_2\in C_0^\infty(M_+)\cup D$. Here D is the finite linear span of the vectors $\phi_{a,b;\infty}$ defined by (4.12).

Proof: This proposition is completely analogous to proposition 2.1 in Ref. 24, and the proof runs along exactly the same lines. We shall therefore only outline the main arguments, and fill in the technical details only where the present situation is different from that in Ref. 24.

Equation (4.33) is proved in three steps: for $h\in C_0^\infty(M_+)$, for $h\in L^\infty(M_+)$, and finally for all h satisfying (4.28).

For $h\in C_0^\infty(M_+)$ one uses the Trotter product formula to show that

$$\mathscr{P}^h_\nu(a'',b'',a',b';T)$$
$$=c_\beta\{\exp[-(\nu A+ih)T]\}(a'',b'';a',b')\ . \tag{4.35}$$

Since h is bounded, the operator $\nu A+ih$ is well defined, and generates a semigroup. Since $A\geqslant0$, and h is a real function, this is a semigroup of contractions.

Using the dominated convergence theorem for $\mathscr{P}^h_\nu$, and strong resolvent convergence for $\exp[-(\nu A + ih)T]$, one can extend (4.35) to all bounded functions h.

In a next step one uses again dominated convergence arguments to show that, for all functions h satisfying (4.5), there exists a strongly continuous semigroup of contractions $E(\nu, h; t)$ satisfying (4.33). These operators are constructed as s-$\lim_{n\to\infty} \exp[-(\nu A + ih_n)t]$, where $h_n(a,b) = h(a,b)$ if $|h(a,b)| < n$, $h_n(a,b) = n \operatorname{sgn} h(a,b)$ otherwise. (See Ref. 4; the arguments given there carry over without problems.)

To prove (4.31), we use the fact that the integral kernel of $E(\nu, h; t)$ is given by a path integral, i.e., (4.33). We have, for all (a', b'), $(a'', b'')\in M_+$ for all $T>0$ (see Ref. 4),

$$\begin{aligned}\mathscr{P}^h_\nu(a'',b'';a',b';T) &= \mathscr{P}^0_\nu(a'',b'';a',b';T) - i c_\beta^{-1}\int_0^T dt \int d\mu(a,b) \\ &\quad\times \mathscr{P}^h_\nu(a'',b'';a,b;T-t)h(a,b)\mathscr{P}^0_\nu(a,b;a',b';t)\,. \end{aligned} \tag{4.36}$$

Take now f_1, $f_2\in C_0^\infty(M_+)$. We multiply (4.35) by $\overline{f_2(a'',b'')}$ $f_1(a',b')$ and integrate over $d\mu(a',b')\times d\mu(a'',b'')$. Using the upper bound (valid for all h [this follows from (4.1)])

$$|\mathscr{P}^h_\nu(a'',b'';a',b';t)| \leqslant c_\beta e^{\nu t\beta} K_{\nu t}(a'',b'';a',b')\,, \tag{4.37}$$

and the estimate (4.6), one sees that the resulting integral converges absolutely. This allows us to change the order of the integrations, and leads to (4.34), for all $f_1, f_2\in C_0^\infty(M_+)$.

We can extend this to the case where $f_1\in D$. To do this, we use (4.10). Take $f_1\in D, f_2\in C_0^\infty(M_+)$. Again we multiply (4.36) by $f_2(a'',b'')f_1(a',b')$ and integrate over $d\mu(a',b')\times d\mu(a'',b'')$. Since the resulting integral is absolutely convergent by (4.37) and (4.10), we may again reverse the order of the integrations. We thus obtain

$$\begin{aligned}\langle f_2, E(\nu,h;T)f_1\rangle &= \langle f_2, e^{-\nu AT}f_1\rangle - i c_\beta^{-2}\int_0^T dt \int d\mu(a'',b'')f_2(a'',b'') \\ &\quad\times\int d\mu(a,b)\mathscr{P}^h_\nu(a'',b'';a,b;T-t)h(a,b) \\ &\quad\times\int d\mu(a',b')\mathscr{P}^0_\nu(a,b;a',b';t)f_1(a',b')\,. \end{aligned} \tag{4.38}$$

We know however that

$$f_1 = \sum_{j=1}^N c_j \phi_{a_j,b_j;\infty} \in \mathscr{H}_\beta\,,$$

hence $e^{-\nu At}f_1 = f_1$ for all t. This means in particular that $e^{-\nu At}f_1\in D(h)$ for all t, so that we may rewrite (4.38) in the form (4.34).

Once (4.34) is obtained for $f_1\in D, f_2\in C_0^\infty(M_+)$, one uses a straightforward approximation argument, using again that $e^{-\nu At}f_1 = f_1$, together with the fact that $C_0^\infty(M_+)$ is dense, to conclude (4.34) for $f_1, f_2\in D$.

Remark: By exactly the same arguments one can also prove that for all $f_1, f_2\in C_0^\infty(M_+)$

$$\begin{aligned}\langle f_2, E(\nu,h;T)(1-P_\beta)f_1\rangle &= \langle f_2, e^{-\nu AT}(1-P_\beta)f_1\rangle \\ &\quad + i\int_0^T dt\,\langle f_2, E(\nu,h;T-t)he^{-\nu At}(1-P_\beta)f_1\rangle\,. \end{aligned} \tag{4.39}$$

C. Operator convergence of $E(\nu,h;T)$ for $\nu\to\infty$

The proof of the strong operator convergence of $E(\nu,h;T)$ hinges on Eq. (4.34). Again the proof in Ref. 4 can essentially be taken over, without major problems. The only difference is that we have to be a little more careful, because the operator A had a purely discrete spectrum in the Weyl–Heisenberg case, and we could therefore conveniently use an orthonormal basis consisting of eigenvectors of A. This is not possible here. We shall therefore, in our proof of Proposition 4.6 below (the analog of Proposition 2.2 in Ref. 4) pay attention only to those technical details where our argument differs from that in Ref. 4.

Proposition 4.6: Let h be a real function on M_+ satisfying (4.28). Define the operator $P_\beta h P_\beta$ on the domain $\{f;\ P_\beta f\in D(h)\}$. Clearly D, the finite linear span of the $\phi_{a,b;\infty}$, satisfies $D\subset D(P_\beta h P_\beta)$. If $P_\beta h P_\beta$ is essentially self-adjoint on $D\oplus\mathscr{H}_\beta^\perp$, then, for all $T>0$,

$$\operatorname*{s-lim}_{\nu\to\infty} E(\nu,h;T) = P_\beta \exp[-iP_\beta h P_\beta T]P_\beta\,. \tag{4.40}$$

Proof: To prove (4.40), the operator $E(\nu,h;T)$ is split into three parts,

$$\begin{aligned}E(\nu,h;T) &= E(\nu,h;T)(\mathbb{1}-P_\beta) + P_\beta E(\nu,h;T)P_\beta \\ &\quad + (\mathbb{1}-P_\beta)E(\nu,h;T)P_\beta\,. \end{aligned}$$

The treatment of the last two terms is completely analogous to the proof of Proposition 2.2 in Ref. 4. We shall therefore restrict ourselves here to a discussion of the first term and an estimate related to it.

From (4.39) we obtain, for all $f_1, f_2\in C_0^\infty(M_+)$,

$$|\langle f_2, E(\nu,h;T)(\mathbb{1}-P_\beta)f_1\rangle| \leqslant \|f_2\|\cdot\|e^{-\nu AT}(\mathbb{1}-P_\beta)\|\cdot\|f_1\| + \|f_2\|\cdot\int_0^T dt\,\|he^{-\nu At}(\mathbb{1}-P_\beta)f_1\|\,. \tag{4.41}$$

We have $\|e^{-\nu AT}(\mathbb{1}-P_\beta)\|\leqslant e^{-\nu TB(\beta)}$, with $B(\beta)$ as defined by (4.14), and

$$\begin{aligned}\|he^{-\nu At}(\mathbb{1}-P_\beta)f_1\|^2 &= \|h(e^{-\nu At}-P_\beta)f_1\|^2 \leqslant \int d\mu(a',b')\int d\mu(a'',b'')|f_1((a',b')|\,|f_1(a'',b'')| \\ &\quad\times\left[\int d\mu(a,b)|h(a,b)|^2|(\phi_{a',b';\nu t}-\phi_{a',b';\infty})(a,b)|^2\right]^{1/2} \\ &\quad\times\left[\int d\mu(a,b)|h(a,b)|^2|\phi_{a'',b'';\nu t}-\phi_{a'',b'';\infty}(a,b)|^2\right]^{1/2} \\ &< \mathscr{C}[1+(\nu t)^{-1+r/[2(r+2)]}]^2 e^{-2k\nu t}\left[\int d\mu(a,b)|f_1(a,b)|\left[\frac{1+a^2+b^2}{2a}\right]^\beta\right]^2, \end{aligned}$$

by Lemma 4.4. Substituting this into (4.41) leads to

$$\|E(\nu,h;T)(1-P_\beta)f_1\| \leqslant e^{-\nu TB(\beta)}\|f_1\| + \frac{k}{\nu}\left\| \left| f_1 \cdot \left[\frac{1+a^2+b^2}{2a}\right]^\beta \right| \right\|_1 \times \int_0^\infty dt\,[1+t^{-1+r/2(r+2)}]e^{-kt}. \qquad (4.42)$$

This holds for all $f_1 \in C_0^\infty(M_+)$. Since C_0^∞ is a dense subspace of $L^2(M_+;d\mu)$ and since the operators $E(\nu,h;T)$ are contractions, this implies, for all $T>0$,

$$\operatorname*{s-lim}_{\nu\to\infty} E(\nu,h;T)(\mathbb{1}-P_\beta)=0.$$

From (4.42) we can clearly also conclude that

$$\lim_{\nu\to\infty}\int_0^T dt\,|\langle f_2,E(\nu,h;T-t)(\mathbb{1}-P_\beta)f_1\rangle|=0,$$

for all $f_1,f_2 \in C_0^\infty(M_+)$, and hence (by the same density arguments as above) for all $f_1,f_2 \in L^2(M_+;d\mu)$. This estimate is needed in the discussion of $P_\beta E(\nu,h;T)P_\beta$ (see Ref. 4).

As already mentioned above, the remainder of the proof is a transcription of the proof of Proposition 2.2 in Ref. 4. □

Our ultimate goal is to link $\mathscr{P}_\nu^h$, at least in the limit for $\nu\to\infty$, to the unitary group $\exp(-iTH)$ generated by a Hamiltonian H on $L^2(\mathbb{R}_+)$. This is in fact achieved by Proposition 4.6. To see this, write the integral kernel of $P_\beta h P_\beta$,

$$(P_\beta h P_\beta)(a'',b'';a',b') = c_\beta^{-2}\int d\mu(a,b) \langle a'',b'';\beta|a,b;\beta\rangle h(a,b)\langle a,b;\beta|a',b';\beta\rangle.$$

One easily checks from (2.13) that this is exactly the integral kernel of $U_\beta H U_\beta^*$, with

$$H=c_\beta^{-1}\int d\mu(a,b)\,|a,b;\beta\rangle h(a,b)\langle a,b;\beta|.$$

Thus $P_\beta h P_\beta = U_\beta H U_\beta^*$. The condition that $P_\beta h P_\beta$ be essentially self-adjoint on $D\oplus\mathscr{H}_\beta$ is exactly equivalent to the condition that H be essentially self-adjoint on D_c, the finite linear span of the (affine) coherent states $|a,b;\beta\rangle$.

The conclusion (4.40) can now be rewritten in terms of H. One finds (see also Ref. 4)

$$[P_\beta \exp(-iP_\beta h P_\beta T)P_\beta](a'',b'';a',b') = c_\beta^{-1}\langle a''b'';\beta|\exp(-iHT)|a',b';\beta\rangle.$$

The strong convergence (4.40) implies, in particular, convergence of the corresponding integral kernels, in a distributional sense (i.e., when evaluated on test functions). We have therefore, at least in a distributional sense,

$$\lim_{\nu\to\infty}\mathscr{P}_\nu^h(a'',b'';a',b';T)=\langle a'',b'';\beta|e^{-iHT}|a'b';\beta\rangle. \qquad (4.43)$$

This result will be sharpened to pointwise convergence in the next subsection.

D. Pointwise convergence of $\mathscr{P}_\nu^h$ for $\nu\to\infty$

To prove (4.43) for all points (a'',b''), $(a',b')\in M_+$, rather than in a distributional sense, we again use an integral equation relating $\mathscr{P}_\nu^h$ and $\mathscr{P}_\nu^0$, obtained by combining (4.36) with the complex conjugate version of (4.36) for $-h$.

$$\begin{aligned}\mathscr{P}_\nu^h(a'',b'';a',b';T) &= \mathscr{P}_\nu^0(a'',b'';a',b';T) - i c_\beta^{-1}\int_0^T dt\int d\mu(a,b)\,\mathscr{P}_\nu^0(a'',b'';a,b;T-t)h(a,b)\mathscr{P}_\nu^0(a,b;a',b';t)\\ &\quad - c_\beta^{-2}\int_0^T dt_2\int_0^{t_2}dt_1\int d\mu(a_1,b_1)\int d\mu(a_2,b_2)\,\mathscr{P}_\nu^0(a'',b'';a_2,b_2;T-t_2)h(a_2,b_2)\\ &\quad \times \mathscr{P}_\nu^h(a_2,b_2;a_1b_1;t_2-t_1)h(a_1,b_1)\mathscr{P}_\nu^0(a_1,b_1;a',b';t_1).\end{aligned}$$

Rewriting this in terms of $\phi_{a,b;t}$ and $\phi_{a,b;\infty}$, and combining it with an analogous integral equation for the coherent state matrix elements of $\exp(-iTH)$ leads to (see Ref. 4)

$$\begin{aligned} c_\beta^{-1}&[\mathscr{P}_\nu^h(a'',b'';a',b';T)-\langle a'',b'';\beta|e^{-iTH}|a',b';\beta\rangle]\\ &= (\phi_{a'',b'';\nu T}-\phi_{a'',b'';\infty})(a',b')\\ &\quad -i\int_0^T dt\,\langle \phi_{a'',b'';\nu(T-t)},h(\phi_{a',b';\nu t}-\phi_{a',b';\infty})\rangle - i\int_0^T dt\,\langle \phi_{a'',b'';\nu(T-t)}-\phi_{a'',b'';\infty},h\phi_{a',b';\infty}\rangle\\ &\quad -\int_0^T dt_2\int_0^{t_2}dt_1\langle h\phi_{a'',b'';\nu(T-t_2)},E(\nu,h;t_2-t_1)h\,[\phi_{a',b';\nu t_1}-\phi_{a',b';\infty}]\rangle\\ &\quad -\int_0^T dt_2\int_0^{t_2}dt_1\langle h\,[\phi_{a'',b'';\nu(T-t_2)}-\phi_{a'',b'';\infty}],E(\nu,h;t_2-t_1)h\phi_{a',b';\infty}\rangle\\ &\quad -\int_0^T dt_2\int_0^{t_2}dt_1\langle h\phi_{a'',b'';\infty},[E(\nu,h;t_2-t_1)-P_\beta e^{-iP_\beta hP_\beta(t_2-t_1)}P_\beta]h\phi_{a',b';\infty}\rangle. \end{aligned} \qquad (4.44)$$

Denote the six terms in the right-hand side of (4.45) by $\Delta_1,...,\Delta_6$. We show that $\Delta_j \to_{\nu\to\infty} 0$ for $j = 1,...,6$.

The estimates (4.15) and (4.30) can be rewritten as

$$\|\phi_{a,b;t} - \phi_{a,b;\infty}\| \leqslant f(t) ,$$

$$\|h(\phi_{a,b;t} - \phi_{a,b;\infty})\| \leqslant g(a,b;t) ,$$

where the functions $f(\cdot)$ and $g(a,b;\cdot)$ [(a,b) fixed] are monotonically decreasing in t, and integrable,

$$\int_0^\infty dt\, f(t) \leqslant \infty ,$$

$$\int_0^\infty dt\, g(a,b;t) \leqslant \infty .$$

On the other hand, (4.13) and Lemma 4.2 tell us that

$$\|\phi_{a,b;\infty}\| = c_\beta^{-1/2} \quad [\text{for all } (a,b)] ,$$

and $[(a',b'), (a'',b'')$ fixed$]$

$$\|h\phi_{a'',b'';\infty}\| \leqslant ¢, \quad \|h\phi_{a',b';\infty}\| \leqslant ¢ .$$

We now discuss the terms $\Delta_1,...,\Delta_6$ one by one.

Using (3.38) we have immediately

$$\Delta_1 \leqslant ¢[1 + (\nu T)^{-1}]e^{-B(\beta)\nu T} \underset{\nu\to\infty}{\to} 0 .$$

The next four terms can be estimated in terms of f, g,

$$\begin{aligned}
\Delta_2 &\leqslant ¢ \int_0^T dt\, [1 + f(\nu(T-t))]g(a',b';\nu t) \\
&\leqslant ¢\frac{1}{\nu}\int_0^\infty dt\, g(a',b';t) \\
&\quad + ¢f\left(\frac{\nu T}{2}\right)\frac{1}{\nu}\int_0^\infty dt\, g(a',b';t) \\
&\quad + ¢g\left(a',b';\frac{\nu T}{2}\right)\frac{1}{\nu}\int_0^\infty dt\, f(t) \\
&\leqslant \left(\frac{1}{\nu}\right)¢ \underset{\nu\to\infty}{\to} 0 ,
\end{aligned}$$

$$\Delta_3 \leqslant ¢\int_0^T dt\, f(\nu(T-t)) \leqslant \left(\frac{1}{\nu}\right)¢ \underset{\nu\to\infty}{\to} 0 ,$$

$$\begin{aligned}
\Delta_4 &\leqslant ¢\int_0^T dt_2 \int_0^{t_2} dt_1 [1 + g(a'',b'';\nu(T-t_2))]g(a',b';\nu t_1) \\
&\leqslant ¢\int_0^T dt_1\, g(a',b';\nu t_1)\cdot(T-t_1) \\
&\quad \times\frac{1}{\nu^2}¢\left[\int_0^\infty dt_2 g(a'',b'';t_2)\right]\cdot\left[\int_0^\infty dt_1 g(a',b';t_1)\right] \\
&\leqslant ¢\left(\frac{1}{\nu^2}\right) + ¢T\frac{1}{\nu}\int_0^\infty dt_1\, g(a',b';t_1) \underset{\nu\to\infty}{\to} 0 ,
\end{aligned}$$

$$\begin{aligned}
\Delta_5 &\leqslant \int_0^T dt_2 \int_0^{t_2} dt_1\, g(a'',b'';\nu(T-t_2)) \\
&\leqslant T\frac{1}{\nu}\int_0^\infty dt\, g(a'',b'';t) \underset{\nu\to\infty}{\to} 0 .
\end{aligned}$$

Finally, $\Delta_6 \underset{\nu\to\infty}{\to} 0$ follows from Proposition 4.5 and the dominated convergence theorem. This completes the proof of our main result.

Theorem 4.6: Let h be a real function on M_+. Suppose that (1) h satisfies condition (4.28), (2) the operator

$$H = c_\beta^{-1}\int d\mu(a,b)\,|a,b;\beta\rangle h(a,b)\langle a,b;\beta| \qquad (4.45)$$

is essentially self-adjoint on D_c, the finite linear span of the affine coherent states. Then, for all (a',b'), $(a'',b'')\in M_+$ and for all $T>0$

$$\lim_{\nu\to\infty} c_\beta e^{\nu T\beta}\int \exp\left[-i\beta\int a^{-1}db - i\int h(a,b)dt\right] \times d\mu_W^\nu(a,b) = \langle a'',b'';\beta|e^{-iTH}|a',b';\beta\rangle .$$

E. Remarks

1. The main result in the pq-notation

We define $\tilde{\Delta}_\beta = \beta^{-1}\partial_p p^2\partial_p + \beta p^{-2}\partial_q^2$.

Let $\tilde{K}_t$ be the associated heat kernel, in $L^2(M_+;[(1-1/2\beta)/2\pi]dp\,dq)$,

$$\begin{aligned}
\tilde{K}_t(p'',q'';\, p',q') &\equiv [\exp(t\tilde{\Delta}_\beta)](p'',q'';p',q') \\
&= \frac{e^{-t/4\beta}\beta^{3/2}}{2\sqrt{2\pi}(\beta-\frac{1}{2})t^{3/2}} \\
&\quad\times\int_\delta^\infty \frac{xe^{-\beta x^2/4t}}{\sqrt{\cosh x - \cosh\delta}}dx,
\end{aligned}$$

where

$$\begin{aligned}
\delta &= d(p'',q'';p',q') \\
&= \cosh^{-1}\left\{1 + \frac{p'p''}{2}[(p'^{-1} - p''^{-1})^2 + \beta^2(q'-q'')^2]\right\}.
\end{aligned}$$

Define $d\tilde{\mu}^{\nu,T}_{W;p'',q'';p',q'}$ to be the associated Wiener process with diffusion constant ν, pinned at p',q' for $t=0$, at p'',q'' for $t=T$. In particular $d\tilde{\mu}^\nu_W$ satisfies

$$\int d\tilde{\mu}^{\nu,T}_{W;p'',q'';p',q'} = \tilde{K}_{\nu T}(p'',q'';p',q'),$$

$$\frac{1-1/2\beta}{2\pi}\int dp\,dq\;\, d\tilde{\mu}^{\nu,T-t}_{W;p'',q'';p,q}\, d\tilde{\mu}^{\nu,t}_{W;p,q;p',q'} = d\tilde{\mu}^{\nu,T}_{W;p'',q'';p',q'} .$$

Let h be a function on M_+ satisfying

$$\int dp\,dq|h(p,q)|^{2+r}\left[\frac{p}{1+p^2(q^2+1)}\right]^\mu < \infty, \qquad (4.46)$$

for some μ,r satisfying condition (4.29).

Let H be the operator on $L^2(\mathbb{R}_+)$ defined by

$$H = \frac{1-1/2\beta}{2\pi}\int dp\,dq|p,q;\beta> h(p,q) <p,q;\beta| ,$$

where, for $\psi\in L^2(\mathbb{R}_+)$,

$$\langle p,q;\beta|\psi\rangle = (2\beta)^\beta[\Gamma(2\beta)]^{-1/2}p^{-\beta} \times\int_0^\infty dk\, k^\beta e^{-k(\beta p^{-1} - iq)}\psi(k)$$

[see (2.14)].

Define the path integral

$$\mathscr{P}^h_\nu(p'',q'';p',q';T) = e^{\nu T/2}\int \exp\left[i\int pdq - i\int h(p,q)dt\right]d\tilde{\mu}^{\nu,T}_{W;p'',q'';p',q'}$$

[this differs by a factor c_β from (4.1); this factor is absorbed in the measure in the *pq*-notation].

Translated into the *pq*-notation, the main theorem now reads (1) if h satisfies (4.45), and (2) if H is essentially self-adjoint on $\tilde{D}_c$, the finite linear span of the $|p,q;\beta\rangle$, then, for all (p'',q''), $(p',q')\in M_+$, for all $T>0$,

$$\lim_{\nu\to\infty} \tilde{\mathscr{P}}^h_\nu (p'',q'';p',q';T) = \langle p'',q'';\beta|\exp(-iTH)|p',q';\beta\rangle.$$

2. Examples

(a) The simplest example is, of course, provided by bounded functions $h(a,b)$,

$$|h(a,b)|\leqslant M.$$

In this case the operator H defined by (4.45) is also bounded by M; H is thus clearly essentially self-adjoint on D_c. Moreover the condition (4.28) is satisfied for arbitrary $r>0$ and for all $\mu>1$. Let us now determine from (4.29) or (4.31) the restrictions imposed on β by the condition $\mu>1$. Two possibilities have to be distinguished: $\frac{1}{2}<\beta\leqslant\frac{3}{2}$ or $\beta\geqslant\frac{3}{2}$. In the first case we have $B(\beta)=(\beta-\frac{1}{2})^2$ in (4.29b), leading to the condition

$$2(1-\alpha)\beta^2-2\beta+\tfrac{1}{2}>\frac{r}{r+2}\,\tilde{\epsilon}\left(2\alpha\frac{r+2}{r},\frac{2\mu}{r}\right), \quad (4.47)$$

with $\tilde{\epsilon}$ as defined by (4.31c). It turns out there is no set of values (α,r,μ) with $r/2(r+2)<\alpha<1$, and $\mu>1$, such that (4.47) is satisfied for $\beta\in(\frac{1}{2},\frac{3}{2}]$.

For $\beta\geqslant\frac{3}{2}$ we have to determine β satisfying the conditions (4.31). One has then to choose (α,r,μ) so as to produce the smallest possible lower bound on β consistent with the other conditions. For $\mu>1$, $r=\frac{1}{2}$, and $\alpha=\frac{1}{3}$ one finds that (4.31a) reduces to $\beta>2.06$, while all the other conditions are fulfilled also.

This means that Theorem 4.6 allows us to conclude that, for bounded Hamiltonians H associated to bounded functions $h(a,b)$,

$$\lim_{\nu\to\infty} c_\beta e^{\nu T\beta}\int \exp\left[-i\beta\int a^{-1}\,db - i\int h(a,b)dt\right] \times d\mu^\nu_W(a,b) = \langle a'',b'';\beta|e^{iTH}|a',b';\beta\rangle, \quad (4.48)$$

for all $\beta>2.06$.

We believe that, for bounded functions h, (4.48) should hold for all $\beta>\frac{1}{2}$, since it holds for $h=\text{const}$ whenever $\beta>\frac{1}{2}$. The 2.06-bound found here is probably an artifact of our method of proof, which uses Young's and Jensen's inequalities several times (in the proof of Lemma 4.3).

(b) We next turn to examples of the form

$$H=-\frac{d^2}{dx^2}+V(x)$$

on $L^2(\mathbb{R}_+)$.

In order for this operator to be essentially self-adjoint on D_c, V must have a singularity at the origin. More precisely, H will be essentially self-adjoint on D_c (regardless of β), e.g., for $V(x)$ of the form

$$V(x)=C_1x^{-\alpha_1}+C_2x^{\alpha_2},$$

where either $\alpha_1>2$, $C_1>0$ or $\alpha_1=2$, $C_1\geqslant\frac{3}{4}$, and either $0\leqslant\alpha_2\leqslant 2$, C_2 arbitrary, or $\alpha_2>2$, $C_2\geqslant 0$. In all these cases V has a strong singularity at $x=0$; for $x\to\infty$, V may tend to ∞, a constant, or $-\infty$, depending on the values chosen for the different parameters.

Let us now construct the corresponding functions $h(a,b)$, and determine the values of β for which Theorem 4.6 applies. The function $h_0(a,b)$ corresponding to $-d^2/dx^2$ is given by

$$h_0(a,b)=b^2-(1/2\beta)a^2$$

[one easily checks that substitution of h_0 into (4.45) leads to $-d^2/dx^2$]. Similarly, the function $h_\alpha(a,b)$ associated with $x^{-\alpha}$ is given by

$$h_\alpha(a,b)=\frac{2^\alpha\Gamma(2\beta-1)}{\Gamma(2\beta+\alpha-1)}a^\alpha.$$

Hence the function $h(a,b)$ corresponding to the Hamiltonian $-(d^2/dx^2)+V$, with V as above, is given by

$$h(a,b)=b^2-\frac{1}{2\beta}a^2+C_1\frac{2^{\alpha_1}\Gamma(2\beta-1)}{\Gamma(2\beta+\alpha_1-1)}a^{\alpha_1} + C_2\frac{2^{-\alpha_2}\Gamma(2\beta-1)}{\Gamma(2\beta-\alpha_2-1)}a^{-\alpha_2}.$$

If $C_2\neq 0$ we have to impose the additional restriction $2\beta-\alpha_2-1\notin-\mathbb{N}$.

We shall restrict ourselves to one particular case now. We take $C_2=0$, $\alpha_1=2$, and $C_1\geqslant\frac{3}{4}$. The Hamiltonian H is essentially self-adjoint, and

$$h(a,b)=b^2+\frac{1}{\beta}\left(\frac{C_1}{\beta-\frac{1}{2}}-\frac{1}{2}\right)a^2.$$

The pairs (r,μ) for which this function satisfies the condition (4.28) are restricted by the condition $\mu>2(r+2)$. We have thus to find (r,α,μ) satisfying this condition as well as the conditions (4.31b); this then enables us, from (4.31a) to compute a β_0 such that Theorem 4.6 applies, for this Hamiltonian, for all $\beta>\beta_0$. For $\alpha=\frac{1}{3}$, $r=1$, and $\mu>6$, one finds that (4.31a) becomes $\beta>27.33$. It is easy to check that all the other conditions are satisfied as well. Hence Theorem 4.6 applies to $H=-d^2/dx^2+Cx^{-2}$, $C\geqslant\frac{3}{4}$, if $\beta>27.33$. Again we believe that this is not optimal. The true lower bound β_0 on β for which (4.48) would hold, whenever $\beta>\beta_0$, is probably much smaller than the here computed value 27.33, though possibly larger than $\frac{1}{2}$.

3. A formula giving the function h from the operator H

Formula (4.45) defines the operator H for a given function h. If we define the function $\mathrm{H}(a,b)$ to be the diagonal coherent state matrix elements of H,

$$\mathrm{H}(a,b)=\langle a,b;\beta|H|a,b;\beta\rangle,$$

then (4.45) leads to

$$\mathrm{H}(a,b)=c_\beta^{-1}\int\frac{d\mu(a',b')}{a'^2}h(a',b')|\langle a,b;\beta|a',b';\beta\rangle|^2 = c_\beta^{-1}\int\frac{d\mu(a',b')}{a'^2}h(a',b') \times\left[\frac{2}{1+\cosh d(a,b;a',b')}\right]^{2\beta}.$$

This formula can be inverted. Using results in Ref. 17 one finds

$$h(a,b) = (\mathrm{TH})(a,b), \tag{4.49}$$

where the operator T, acting on the function H, is given by

$$\mathrm{T} = \prod_{n=0}^{\infty} \left[1 - \frac{\Delta}{(2\beta + n + 1)(2\beta + n + 2)} \right], \tag{4.50}$$

with $\Delta = a^2(\partial_a^2 + \partial_b^2)$, the Laplace–Beltrami operator on the Lobachevsky plane.

It turns out that this infinite product can be rewritten in terms of Γ-functions. One way to see this is to use the correspondence (4.49) for a family of special cases. For $H = x^{-\alpha}$ we know already that

$$h(a,b) = \frac{2^\alpha \Gamma(2\beta - 1)}{\Gamma(2\beta + \alpha - 1)} a^\alpha.$$

On the other hand, the corresponding function $\mathrm{H}(a,b)$ is

$$\mathrm{H}(a,b) = \langle a,b;\beta | x^{-\alpha} | a,b;\beta \rangle = \frac{2^\alpha \Gamma(2\beta - \alpha)}{\Gamma(2\beta)} a^\alpha.$$

This implies that

$$\prod_{n=0}^{\infty} \left[1 + \frac{-\alpha(\alpha - 1)}{(2\beta + n + 1)(2\beta + n + 2)} \right] = \frac{\Gamma(2\beta)\Gamma(2\beta - 1)}{\Gamma(2\beta - \alpha)\Gamma(2\beta + \alpha - 1)}.$$

By analytic continuation one finds that, for all $t > 0$,

$$\prod_{n=0}^{\infty} \left[1 + \frac{t^2 + 1/4}{(2\beta + n + 1)(2\beta + bn + 2)} \right] = \frac{\Gamma(2\beta)\Gamma(2\beta - 1)}{\Gamma(2\beta - it - \frac{1}{2})\Gamma(2\beta + it - \frac{1}{2})} = \frac{B(2\beta, 2\beta - 1)}{B(2\beta - it - \frac{1}{2}, 2\beta + it - \frac{1}{2})}. \tag{4.51}$$

Since the spectrum of $-\Delta = -a^2(\partial_a^2 + \partial_b^2)$ on the Lobachevsky plane is $[\frac{1}{4}, \infty)$, (4.51) determines (4.50) completely. For particular values of β, (4.51) and hence (4.50) can be further simplified. For $\beta = 1$, e.g., we find

$$\frac{B(2,1)}{B(\frac{3}{2} - it, \frac{3}{2} + it)} = \frac{\pi}{(\frac{9}{4} + t^2)\cosh(\pi t)}.$$

This can then be used to give an integral representation for T. We have, e.g.,

$$\int_0^\pi dx \frac{\cos tx}{\cosh x/2} = \frac{\pi}{\cosh t\pi},$$

hence

$$\prod_{n=0}^{\infty} \left[1 + \frac{-\Delta}{(n+3)(n+4)} \right] = (-\Delta + 2)^{-1} \int_0^\infty dt \frac{\cos[t\sqrt{-\Delta + \frac{1}{4}}]}{\cosh t/2},$$

with

$$\cos[t\sqrt{-\Delta + \tfrac{1}{4}}] = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} t^{2n} \left(-\Delta + \frac{1}{4} \right)^n.$$

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