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Gabor Time-Frequency Lattices and the Wexler–Raz Identity

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ABSTRACT. *Gabor time-frequency lattices are sets of functions of the form* $g_{m\alpha,n\beta}(t) = e^{-2\pi i\alpha mt}g(t-n\beta)$ *generated from a given function* $g(t)$ *by discrete translations in time and frequency. They are potential tools for the decomposition and handling of signals that, like speech or music, seem over short intervals to have well-defined frequencies that, however, change with time. It was recently observed that the behavior of a lattice* $(m\alpha, n\beta)$ *can be connected to that of a dual lattice* $(m/\beta, n/\alpha)$*. Here we establish this interesting relationship and study its properties. We then clarify the results by applying the theory of von Neumann algebras. One outcome is a simple proof that for* $g_{m\alpha,n\beta}$ *to span* L^2*, the lattice* $(m\alpha, n\beta)$ *must have at least unit density. Finally, we exploit the connection between the two lattices to construct expansions having improved convergence and localization properties.*

1. Introduction

This paper concerns expansions of $L^2(\mathbb{R})$-functions $f(x)$ into families $g_{m\alpha,n\beta}(x)$ obtained by translating and modulating a fixed function g in $L^2(\mathbb{R})$,

$$g_{m\alpha,n\beta}(x) = e^{-2\pi im\alpha x}\ g(x-n\beta)\ , \tag{1.1}$$

with $\alpha, \beta > 0$ fixed and m, n ranging over $\mathbb{Z}$. We call such families *Gabor time-frequency lattices* (or *Gabor lattices*, for short), after an expansion of this type proposed by Gabor in [2] (where g was a Gaussian, and $\alpha\beta = 1$). The general problem of identifying coefficients $c_{m,n}(f)$ so that

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$$f(x) = \sum_{m,n} c_{m,n}\; g_{m\alpha,n\beta}(x) \tag{1.2}$$

has been discussed in many places. In [3] this problem was studied in the context of *frames*; the $g_{m\alpha,n\beta}$ are said to constitute a *frame* if there exist $A > 0$ and $B < \infty$ such that, for all $f \in L^2(\mathbb{R})$,

$$A\, \|f\|^2 \leq \sum_{m,n} |\langle f,\; g_{m\alpha,n\beta}\rangle|^2 \leq B\, \|f\|^2\,, \tag{1.3}$$

where $\langle\ ,\ \rangle$ denotes the standard L^2-inner product, $\langle f, g\rangle = \int f(x)\overline{g(x)}\, dx$, and $\|f\| = \langle f, f\rangle^{1/2}$. The condition (1.3) can also be rewritten as

$$A\ \mathrm{Id} \leq S_{g;\alpha,\beta} \leq B\ \mathrm{Id}\,, \tag{1.4}$$

where $S_{g;\alpha,\beta}$ is the *frame operator*, defined as

$$S_{g;\alpha,\beta}\ f = \sum_{m,n} \langle f,\; g_{m\alpha,n\beta}\rangle\; g_{m\alpha,n\beta}\ . \tag{1.5}$$

If the $g_{m\alpha,n\beta}$ satisfy (1.3), then the general theory of frames [4] can be used to show that the least squares choice for the $c_{m,n}$ in (1.2) is given by

$$c_{m,n} = \langle f,\; S^{-1}_{g;\alpha,\beta}\; g_{m\alpha,n\beta}\rangle = \langle f,\; \tilde{g}_{m\alpha,n\beta}\rangle\,, \tag{1.6}$$

where the $\tilde{g}_{m\alpha,n\beta}$ are derived from the single function $\tilde{g} = S^{-1}_{g;\alpha,\beta}\ g$ by the same time-frequency translations as in (1.1). Note that $\tilde{g}$ is well defined, since S^{-1} is bounded by A^{-1} because of (1.4); moreover, one can write a geometrically convergent series for S^{-1} because

$$S = \frac{A+B}{2}\left[\mathrm{Id} - \left(\mathrm{Id} - \frac{2S}{A+B}\right)\right]\,, \tag{1.7}$$

with $\left\|\mathrm{Id} - \frac{2S}{A+B}\right\| < 1$. We will refer to $\tilde{g}$ as the *frame dual function*.

A different approach to finding appropriate $c_{m,n}$ was taken in [5]. By means of a remarkable identity that we shall call the *Wexler–Raz identity*, the problem was linked to the dual Gabor lattice $g_{m/\beta,n/\alpha}(x)$. The authors of [5] show that if h satisfies

$$\langle h,\; g_{m/\beta,n/\alpha}\rangle = \alpha\beta\; \delta_{m,0}\; \delta_{n,0}\,, \tag{1.8}$$

then the coefficients $c_{m,n} = \langle f,\; h_{m\alpha,n\beta}\rangle$ will satisfy (1.2); their method for solving (1.8) involves another operator, S', required to be bounded and invertible. Among all these different "biorthogonal" choices h, they then choose the one with minimal L^2-norm; we will call it the *Wexler–Raz dual function* and denote it by $g^{\#}$. (Strictly speaking, the construction in [5] is for ℓ^2-sequences in "discrete time" rather than for L^2-functions, and the derivation is not rigorous, but the essential idea is as above. Janssen's paper [6] gives a rigorous derivation of the Wexler–Raz result.)

As the functions $\tilde{g}$ and $g^{\#}$ minimize different things (the norms of solutions of (1.2) and (1.8) in ℓ^2 and L^2, respectively), it is not immediately clear how they are related. In this paper, we show in three different ways that they coincide. The most concrete of our proofs is mainly concerned with the Wexler–Raz identity and establishes $\tilde{g} = g^{\#}$ as a byproduct. It occupies §§2 to 4 and is organized as follows. In §2, we introduce our notation and review some properties of the "traditional" frame construction. In §3, we state and prove the Wexler–Raz (WR) identity in operator form and explore its consequences. In §4 we show that the WR approach to solving (1.8) can be carried out—that is, S' is bounded and invertible—if and only if the $g_{m\alpha,n\beta}$ constitute a frame and that one then always has $g^{\#} = \tilde{g}$.

Our second proof sidesteps the Wexler–Raz identity and proves the result directly. This short argument is given in §5.

Section 6 revisits the problem from the point of view of von Neumann algebras. The ad hoc arguments of the first proof are seen to be concrete realizations of a much more elegant construction using von Neumann algebras. This approach proves all the results that were obtained in the first part, that is, the Wexler–Raz identity as well as the equivalence proved at the end of §4. Turning then to the density of the lattice, we show by a simple argument that the condition $\alpha\beta \leq 1$ is necessary for the functions $\{g_{m\alpha,n\beta}\}$ to span L^2. Heretofore this fact was proved by using the coupling constant, a feature of von Neumann algebras that is difficult to establish. Here we reverse the direction of the argument by using the Wexler–Raz identity to derive the existence of the coupling constant in an elementary way.

Finally, part of the motivation of Wexler and Raz in [5] was that, in practice, the $g^{\#}$ of minimal L^2-norm may *not* be the preferred choice. They present several particular constructions for examples where, for instance, a different "biorthogonal" function leads to better time concentration. We conclude the present paper with a discussion in §7 of a more systematic approach to find biorthogonal functions that optimize other than L^2-norms.

Some of our results overlap with those of [7] by A. J. E. M Janssen and [8] by A. Ron and Z. Shen, of which we became aware while writing the present paper, but our techniques are different.

2. The Frame Dual Function — A Quick Review

In this section we introduce our notation and give a geometric argument showing that the frame approach gives the least squares solution for the $c_{m,n}$ in (1.2).

Given $g \in L^2(\mathbb{R})$ and $\alpha, \beta > 0$, we shall denote by $T_{g;\alpha,\beta}$ the operator mapping $f \in L^2(\mathbb{R})$ to the sequence $\langle f,\ g_{m\alpha,n\beta}\rangle$:

$$T_{g;\alpha,\beta}\ f = (\langle f,\ g_{m\alpha,n\beta}\rangle)_{m,n\in\mathbb{Z}}\ . \tag{2.1}$$

We shall typically consider $T_{g;\alpha,\beta}$ as an operator from $L^2(\mathbb{R})$ to $\ell^2(\mathbb{Z}^2)$. While every component of $T_{g;\alpha,\beta}$ defines a bounded linear functional on $L^2(\mathbb{R})$, putting them together does not always lead to a bounded operator to $\ell^2(\mathbb{Z}^2)$. A counterexample is given by, for example, $\alpha = \beta = 1$ and g piecewise constant on the intervals $[k, k+1)$, with $g\,|_{[k,k+1)} = (1 + |k|)^{-5/8}$; one then easily checks that $\sum_{m,n} |\langle g, g_{m,n}\rangle|^2$ diverges. Similar counterexamples can be constructed for other choices of

α, β. Note, however, that $T_{g;\alpha,\beta}$ is always densely defined (since $T_{g;\alpha,\beta} f \in \ell^2$ when $f \in C_0^\infty(\mathbb{R})$, regardless of the choice of $g \in \ell^2$) and closable (since $f_n \to 0$ and $T_{g;\alpha,\beta} f_n \to c \in \ell^2$ implies $c_{k,\ell} = \lim_{n\to\infty}\langle f_n, g_{k\alpha,\ell\beta}\rangle = 0$). We shall generally restrict ourselves to the case where $T_{g;\alpha,\beta}$ is bounded; all exceptions will be clearly indicated. In order to guarantee that $T_{g;\alpha,\beta}$ is bounded, for any choice of α, β, it is sufficient to require some mild decay for $g(x)$ or $\hat{g}(\xi)$.

2.1. Proposition

Suppose that g satisfies either $|g(x)| \leq C(1+|x|)^{-1-\epsilon}$ for all $x \in \mathbb{R}$ or $|\hat{g}(\xi)| \leq C(1+|\xi|)^{-1-\epsilon}$ for all $\xi \in \mathbb{R}$, with $\epsilon > 0$. Then, for all $\alpha, \beta > 0$, $T_{g;\alpha,\beta}$ is bounded from $L^2(\mathbb{R})$ to $\ell^2(\mathbb{Z}^2)$.

Proof.

1. With the normalization $\hat{f}(\xi) = \int e^{-2\pi i x\xi} f(x)\,dx$ for the Fourier transform, one easily checks that

$$\langle f,\ g_{m\alpha,n\beta}\rangle = e^{2\pi i n m\alpha\beta}\langle \hat{f},\ \hat{g}_{n\beta,-m\alpha}\rangle\ .$$

It therefore suffices to discuss only the case where $g(x)$ decays.

2. Now, for any $f \in L^2(\mathbb{R})$ with compact support

$$\|T_{g;\alpha,\beta}\, f\|^2 = \sum_{m,n} |\, \langle f,\ g_{m\alpha,n\beta}\rangle\, |^2$$

$$= \sum_{m,n} \left| \int_{-\infty}^{\infty} e^{2\pi i m\alpha x}\ f(x)\ \overline{g(x-n\beta)}\, dx \right|^2$$

$$= \sum_{m,n} \left| \int_0^{1/\alpha} e^{2\pi i m\alpha x} \sum_k f\left(x - \frac{k}{\alpha}\right)\ \overline{g\left(x - \frac{k}{\alpha} - n\beta\right)}\, dx \right|^2$$

$$= \frac{1}{\alpha} \sum_n \int_0^{1/\alpha} \left| \sum_k f\left(x - \frac{k}{\alpha}\right)\ \overline{g\left(x - \frac{k}{\alpha} - n\beta\right)} \right|^2 dx$$

$$\leq \frac{1}{\alpha} \sum_{n,k,\ell} \int_0^{1/\alpha} \left| f\left(x - \frac{k}{\alpha}\right)\right| \left| f\left(x - \frac{\ell}{\alpha}\right)\right| \left| g\left(x - \frac{k}{\alpha} - n\beta\right)\right| \left| g\left(x - \frac{\ell}{\alpha} - n\beta\right)\right| dx$$

$$= \frac{1}{\alpha} \sum_{m,n} \int_{-\infty}^{\infty} |f(x)| \left| f\left(x - \frac{m}{\alpha}\right)\right| \left| g\left(x - \frac{m}{\alpha} - n\beta\right)\right| |g(x-n\beta)|\, dx$$

$$\leq \frac{1}{\alpha} \left\{ \sum_m \int_{-\infty}^{\infty} |f(x)|^2 \sum_n |g(x-n\beta)| \left| g\left(x - n\beta - \frac{m}{\alpha}\right)\right| dx \right\}^{1/2} \times$$

$$\left\{ \sum_m \int_{-\infty}^{\infty} \left| f\left(x - \frac{m}{\alpha}\right) \right|^2 \sum_n |g(x - n\beta)| \; \left| g\left(x - n\beta - \frac{m}{\alpha}\right) \right| \, dx \right\}^{1/2}$$

$$= \frac{1}{\alpha} \left\{ \int_{-\infty}^{\infty} |f(x)|^2 \sum_n |g(x - n\beta)| \sum_m \left| g\left(x - n\beta - \frac{m}{\alpha}\right) \right| \, dx \right\}^{1/2} \times$$

$$\left\{ \int_{-\infty}^{\infty} |f(y)|^2 \sum_n |g(y - n\beta)| \sum_m \left| g\left(y - n\beta + \frac{m}{\alpha}\right) \right| \, dy \right\}^{1/2}$$

$$\leq C \|f\|^2 ,$$

where we have used that $\sum_k |g(z - k\gamma)| \leq C(1 + |\gamma|^{-1})$ because of the decay condition on g.

3. Since the functions with compact support are dense in $L^2(\mathbb{R})$, the same bound then holds for all f in $L^2(\mathbb{R})$. □

If $T_{g;\alpha,\beta}$ is bounded, then so is its adjoint $T^*_{g;\alpha,\beta}$ from $\ell^2(\mathbb{Z}^2)$ to $L^2(\mathbb{R})$. The action of this adjoint on sequences $c \in \ell^2(\mathbb{Z}^2)$ is given by

$$T^*_{g;\alpha,\beta}\, c = \sum_{m,n} c_{m,n}\; g_{m\alpha,n\beta} \;, \tag{2.2}$$

where the truncated sums for $|m|, |n| \leq K$ converge in norm to the limit $T^*_{g;\alpha,\beta}\, c$ as $K \to \infty$.

The $g_{m\alpha,n\beta}$ constitute a frame if and only if $T_{g;\alpha,\beta}$ is bounded and $\|T_{g;\alpha,\beta}\, f\| \geq A^{1/2}\|f\|$ for all f. This last statement is equivalent to requiring that Ker $T_{g;\alpha,\beta} = \{0\}$ and that Ran $T_{g;\alpha,\beta}$ be a closed subspace of $\ell^2(\mathbb{Z}^2)$.

For a given $f \in L^2(\mathbb{R})$, finding the possible choices $c \in \ell^2(\mathbb{Z}^2)$ such that (2.2) holds amounts to finding the sequences c for which

$$T^*_{g;\alpha,\beta}\, c = f \;. \tag{2.3}$$

Clearly, adding any element of Ker $T^*_{g;\alpha,\beta}$ to an arbitrary solution c of (2.3) gives another solution. We can "mod out" by Ker $T^*_{g;\alpha,\beta}$ by taking the orthogonal projection of c onto Ran $T_{g;\alpha,\beta} = (\text{Ker } T^*_{g;\alpha,\beta})^{\perp}$. This means that we have to identify $h \in L^2(\mathbb{R})$ so that

$$c - T_{g;\alpha,\beta}\, h \perp \text{Ran } T_{g;\alpha,\beta}, \quad \text{or} \quad T^*_{g;\alpha,\beta}(c - T_{g;\alpha,\beta}\, h) = 0 \;.$$

This immediately gives $f = T^*_{g;\alpha,\beta}\, T_{g;\alpha,\beta}\, h$; the sequence of coefficients $c = T_{g;\,\alpha,\beta}\, h$ with minimum ℓ^2-norm in (2.2) is thus given by

$$(T_{g;\,\alpha,\beta}\,h)_{m,n} = \langle (T^*_{g;\,\alpha,\beta}\,T_{g;\,\alpha,\beta})^{-1}\,f,\ g_{m\alpha,n\beta}\rangle$$

$$= \langle f,\ (T^*_{g;\,\alpha,\beta}\,T_{g;\,\alpha,\beta})^{-1}\,g_{m\alpha,n\beta}\rangle\ .$$

Let us denote the time-frequency shifts in (1.1) by $W(m\alpha, n\beta)$, that is,

$$[W(p,q)f](x) = e^{-2\pi i p x}\ f(x-q)\ . \tag{2.4}$$

One easily checks that $T^*_{g;\,\alpha,\beta}\,T_{g;\,\alpha,\beta}$ commutes with all the $W(m\alpha, n\beta)$; it follows that its inverse commutes with them as well, so that

$$(T^*_{g;\,\alpha,\beta}\,T_{g;\,\alpha,\beta})^{-1}\,W(m\alpha,n\beta)g = W(m\alpha,n\beta)\tilde{g}\ , \tag{2.5}$$

with

$$\tilde{g} = (T^*_{g;\,\alpha,\beta}\,T_{g;\,\alpha,\beta})^{-1}\,g\ . \tag{2.6}$$

We shall call this $\tilde{g}$ the *frame dual function* for the $g_{m\alpha,n\beta}$. The following proposition summarizes our findings (which can also be found in [3], [4]).

2.2. Proposition

If the $g_{m\alpha,n\beta}$ constitute a frame, then the sequences $c \in \ell^2(\mathbb{Z}^2)$ for which (2.3) *holds are exactly the elements of $T_{\tilde{g};\,\alpha,\beta}\ f + \mathrm{Ker}\ T^*_{g;\,\alpha,\beta}$, with $T_{\tilde{g};\,\alpha,\beta}\ f$ the minimum ℓ^2-norm solution.*

The argument above is essentially the standard argument for a dual frame construction, going back to [4]. What makes Gabor lattices special is that the commutation of $W(m\alpha, n\beta)$ with $T^*_{g;\,\alpha,\beta}\,T_{g;\,\alpha,\beta}$ allows us to write the dual frame as consisting of time-frequency translates of a single function again, as in (1.1). This does not happen for wavelet frames, for example [3]. We shall come back to the significance of these commutation rules.

3. The Wexler–Raz Identity

The observation on which the analysis by Wexler and Raz in [5] rests is essentially an identity that links operators $T_{g;\,\alpha,\beta}$ with their counterparts $T_{f;1/\beta,1/\alpha}$ for the "dual" lattice parameters $\frac{1}{\beta}$, $\frac{1}{\alpha}$. (We shall come back to how this "duality" is to be understood.) The derivation in [5], by means of the Poisson summation formula, is justified only for some functions; in [6] Janssen has given a different, rigorous proof for some consequences of the original WR identity. We wish here to stick to the identity itself; our first task is to state and prove it.

3.1. Theorem

Suppose $f,\ g, h \in L^2(\mathbb{R})$ and $\alpha, \beta > 0$ are such that $T_{g;\,\alpha,\beta}$, $T_{f;\,\alpha,\beta}$, $T_{h;\,1/\beta,1/\alpha}$ are bounded. Then

$$T^*_{f;\,\alpha,\beta}\ T_{g;\,\alpha,\beta}\ h = \frac{1}{\alpha\beta}\ T^*_{h;\,1/\beta,1/\alpha}\ T_{g;\,1/\beta,1/\alpha}\ f\ . \tag{3.1}$$

Remark. Note that we did not include any assumptions on the boundedness of $T_{g;\,1/\beta,1/\alpha}$. This is because the boundedness of $T_{g;\,\alpha,\beta}$ already implies that $T_{g;\,1/\beta,1/\alpha}$ is bounded.

We shall prove this theorem in several steps. We start by restricting ourselves to nice h, f.

3.2. Lemma

Assume that $T_{g;\,\alpha,\beta}$ and $T_{g;\,1/\beta,1/\alpha}$ are both bounded, and let h, f be compactly supported and bounded. Then (3.1) *holds.*

Proof.

1. Take any $\varphi \in L^2(\mathbb{R})$. Mimicking the computation in part 2 of the proof of Proposition 2.1, we find

$$\langle T^*_{f;\,\alpha,\beta}\ T_{g;\,\alpha,\beta}\ h, \varphi\rangle = \langle T_{g;\,\alpha,\beta}\ h,\ T_{f;\,\alpha,\beta}\ \varphi\rangle$$

$$= \frac{1}{\alpha}\sum_{m,n}\int_{-\infty}^{\infty} h\left(x - \frac{m}{\alpha}\right)\ \overline{g\left(x - \frac{m}{\alpha} - n\beta\right)}\ f(x - n\beta)\ \overline{\varphi(x)}\, dx\ ,$$

where the integral and series converge absolutely.

2. Similarly

$$\langle T^*_{h;\,1/\beta,1/\alpha}\ T_{g;\,1/\beta,1/\alpha}\ f, \varphi\rangle$$

$$= \beta\sum_{k,\ell}\int_{-\infty}^{\infty} f(x - k\beta)\ \overline{g\left(x - k\beta - \frac{\ell}{\alpha}\right)}\ h\left(x - \frac{\ell}{\alpha}\right)\ \overline{\varphi(x)}\, dx\ ,$$

and (3.1) follows immediately. □

Next we introduce an auxiliary operator to rewrite (3.1) in a different form. Define the operator R_γ on $\ell^2(\mathbb{Z}^2)$ by

$$(R_\gamma c)_{k,\ell} = e^{2\pi i\gamma k\ell} c_{k,-\ell}\ ;$$

note that R_γ is unitary and symmetric, $R_\gamma^2 = \mathrm{Id}$, $R_\gamma^* = R_\gamma$. It is easy to check that if $T_{f;\,\alpha,\beta}$ and $T_{g;\,\alpha,\beta}$ are bounded, then

$$T_{f;\,\alpha,\beta}\; g = R_{\alpha\beta}\; T_{\overline{g};\alpha,\beta}\; \overline{f}\;. \tag{3.2}$$

We can then use the results of Lemma 3.2 to prove the following lemma.

3.3. Lemma

Assume that $T_{g;\,\alpha,\beta}$ and $T_{g;\,1/\beta,1/\alpha}$ are bounded, and choose φ in $L^2(\mathbb{R})$ so that $|\varphi(x)| \le c(1+|x|)^{-1-\epsilon}$ for some $\epsilon > 0$. Then

$$T^*_{\varphi;\alpha,\beta}\; R_{\alpha\beta}\; T_{g;\,\alpha,\beta} = \frac{1}{\alpha\beta}\; T^*_{\overline{g};1/\beta,1/\alpha}\; R_{1/\alpha\beta}\; T_{\varphi;1/\beta,1/\alpha}\;. \tag{3.3}$$

Proof.

1. If f, h are bounded and compactly supported, then

$$\begin{aligned}\langle T_{g;\,\alpha,\beta}\; h,\; T_{f;\,\alpha,\beta}\; \overline{\varphi}\rangle &= \frac{1}{\alpha\beta}\;\langle T_{g;\,1/\beta,1/\alpha}\; f,\; T_{h;\,1/\beta,1/\alpha}\; \overline{\varphi}\rangle\;. \\ &= \frac{1}{\alpha\beta}\;\langle T_{\overline{h};1/\beta,1/\alpha}\; \varphi,\; T_{\overline{g};1/\beta,1/\alpha}\; \overline{f}\rangle\;,\end{aligned} \tag{3.4}$$

where we have used that

$$(T_{f;\,\alpha,\beta}\, g)^*_{m,n} = (T_{\overline{f};\alpha,\beta}\; \overline{g})_{-m,n}\;. \tag{3.5}$$

2. By (3.2) we can rewrite this as

$$\langle T_{g;\,\alpha,\beta}\; h,\; R_{\alpha,\beta}\; T_{\varphi;\alpha,\beta}\; \overline{f}\rangle = \frac{1}{\alpha\beta}\;\langle R_{1/\alpha,\beta}\; T_{\overline{\varphi};1/\beta,1/\alpha}\; h,\; T_{\overline{g};1/\beta,1/\alpha}\; \overline{f}\rangle\;.$$

Since this now holds for all bounded and compactly supported h and $\overline{f}$, (3.3) follows. □

We now immediately have

3.4. Lemma

If $T_{f;\,\alpha,\beta}$, $T_{g;\,\alpha,\beta}$, $T_{h;\,1/\beta,1/\alpha}$, and $T_{g;\,1/\beta,1/\alpha}$ are bounded, then (3.1) *holds.*

Proof. Apply both sides of (3.3) to h, and take the inner product with $\overline{f}$. Since $T_{f;\,\alpha,\beta}$ and $T_{h;\,1/\beta,1/\alpha}$ are bounded, we can use (3.2) and (3.5) to bring this back to the form (3.4). Since this is now valid for all φ in the dense set of functions decaying at least as fast as $(1+|x|)^{-1-\epsilon}$, we conclude (3.1). □

This proves the Wexler–Raz identity, except that we still have to get rid of the separate condition that $T_{g;\,1/\beta,1/\alpha}$ be bounded. We first prove the following lemma.

3.5. Lemma

For arbitrary $\alpha, \beta > 0$*, define* $J = J(\alpha, \beta) := \lceil \alpha\beta \rceil$*, that is,* $J \in \mathbb{N}$ *and* $J - 1 < \alpha\beta \le J$*. For* $j = 0, \ldots, J-1$*, define*

$$a_j = j/\alpha;$$

$$b_j = \min\left(\frac{j+1}{\alpha}, \beta\right) \quad \left(\textit{that is, } b_j = \frac{j+1}{\alpha} \ \textit{ if } \ j \le J-2,\ b_{J-1} = \beta\right);$$

$$\varphi_j(x) = \chi_{[a_j,b_j)}(x) = \begin{matrix} 1 & \textit{if } a_j \le x < b_j, \\ 0 & \textit{otherwise}. \end{matrix}$$

Then

$$\sum_{j=0}^{J-1} T^*_{\varphi_j;\alpha,\beta}\, T_{\varphi_j;\alpha,\beta} = \frac{1}{\alpha}\, \mathrm{Id}_{L^2(\mathbb{R})} \tag{3.6}$$

and

$$\sum_{j=0}^{J-1} T_{\varphi_j;1/\beta,1/\alpha}\, T^*_{\varphi_j;1/\beta,1/\alpha} = \beta\, \mathrm{Id}_{\ell^2(\mathbb{Z}^2)}\ . \tag{3.7}$$

Proof.

1. To prove (3.6), it suffices to show that, for all $f \in L^2(\mathbb{R})$,

$$\sum_{j=0}^{J-1} \|T_{\varphi_j;\alpha,\beta}\, f\|^2 = \frac{1}{\alpha}\|f\|^2\ .$$

Now

$$\|T_{\varphi_j;\alpha,\beta}\, f\|^2 = \sum_{m,n} \left| \int_{n\beta+a_j}^{n\beta+b_j} e^{-2\pi i m\alpha x} f(x)\, dx \right|^2$$

$$= \frac{1}{\alpha} \sum_n \int |f(x)|^2 \, \chi_{[a_j,b_j)}(x-n\beta)\, dx \;.$$

Since

$$\sum_{j=0}^{J-1} \chi_{[a_j,b_j)}(y) = \chi_{[0,\beta)}(y)\;, \tag{3.8}$$

we have

$$\sum_{j=0}^{J-1} \|T_{\varphi_j;\alpha,\beta}\, f\|^2 = \frac{1}{\alpha} \sum_n \int |f(x)|^2 \chi_{[n\beta,(n+1)\beta)}(x)\, dx = \frac{1}{\alpha} \|f\|^2\;.$$

2. Similarly, to prove (3.7), it is sufficient to establish that, for all $c \in \ell^2(\mathbb{Z}^2)$, $\beta\|c\|^2 = \sum_{j=0}^{J-1} \|T_{\varphi_j;1/\beta,1/\alpha}\, c\|^2$. If we introduce the β-periodic function $c_n(x) = \sum_m c_{m,n} e^{-2\pi i m x/\beta}$, then we have

$$\|T^*_{\varphi_j;1/\beta,1/\alpha}\, c\|^2 = \int \left| \sum_n c_n(x) \chi_{[a_j;b_j)} \left(x - \frac{n}{\alpha}\right) \right|^2 dx$$

$$= \sum_n \int |c_n(x)|^2 \chi_{[a_j,b_j)} \left(x - \frac{n}{\alpha}\right) dx\;,$$

where we have used that, for fixed j, the intervals $\left[a_j + \frac{n}{\alpha}, b_j + \frac{n}{\alpha}\right)$, $n \in \mathbb{Z}$, are all disjoint. It follows then from (3.8) that

$$\sum_{j=0}^{J-1} \|T^*_{\varphi_j;1/\beta,1/\alpha}\, c\|^2 = \sum_n \int |c_n(x)|^2 \chi_{[0,\beta)} \left(x - \frac{n}{\alpha}\right) dx$$

$$= \sum_n \int_0^\beta \left| \sum_m c_{m,n}\, e^{-2\pi i m (x+n/\alpha)/\beta} \right|^2 dx$$

$$= \beta \sum_{n,m} |c_{m,n}\, e^{-2\pi i m n/\alpha\beta}|^2 = \beta \|c\|^2\;. \qquad \square$$

This now implies the following lemma.

3.6. Lemma

Assume that $T_{g;\,\alpha,\beta}$ and $T_{\overline{g};1/\beta,1/\alpha}$ are bounded. Let $J \geq 1$ and φ_j, $j = 0, \ldots, J-1$, be defined as in Lemma 3.5. Then

$$T^*_{\overline{g};1/\beta,1/\alpha} = \alpha \sum_{j=0}^{J-1} T^*_{\varphi_j;\alpha,\beta}\; R_{\alpha\beta}\; T_{g;\,\alpha,\beta}\; T^*_{\varphi_j;1/\beta,1/\alpha}\; R_{1/\alpha\beta}\;. \tag{3.9}$$

Proof. Apply (3.3) (with $\varphi = \varphi_j$) to $\alpha\; T^*_{\varphi_j;1/\beta,1/\alpha}\; R_{1/\alpha\beta}\; c$, and sum over j. Then (3.9) immediately follows from (3.7). □

Formula (3.9) now suggests that $T_{\overline{g};1/\beta,1/\alpha}$ is defined as a bounded operator whenever $T_{g;\,\alpha,\beta}$ is. This is in fact the case.

3.7. Lemma

If $T_{g;\,\alpha,\beta}$ is bounded, then so is $T_{\overline{g};1/\beta,1/\alpha}$. Moreover, $T^*_{\overline{g};1/\beta,1/\alpha}$ is given by formula (3.9).

Proof.

1. The right-hand side of (3.9) certainly defines a bounded operator. Let us check how this operator acts on the "elementary" sequences $e_{k,\ell} \in \ell^2(\mathbb{Z}^2)$ defined by $(e_{k,\ell})_{m,n} = \delta_{k,m}\,\delta_{\ell,n}$. One immediately has

$$T^*_{\varphi_j;1/\beta,1/\alpha}\; R_{1/\alpha\beta}\; e_{k,\ell} = e^{-2\pi i\; k\ell/\alpha\beta}\; T^*_{\varphi_j;1/\beta,1/\alpha}\; e_{k,-\ell}$$

$$= e^{-2\pi i\; k\ell/\alpha\beta}\; W\left(\frac{k}{\beta}, \frac{-\ell}{\alpha}\right)\varphi_j\;,$$

where the W-operators are as defined in (2.4). Applying (3.2) we then have

$$R_{\alpha\beta}\; T_{g;\,\alpha,\beta}\; T^*_{\varphi_j;1/\beta,1/\alpha}\; R_{1/\alpha\beta}\; e_{k,\ell} = e^{-2\pi i\; k\ell/\alpha\beta}\; T_{W(-k/\beta,-\ell/\alpha)\;\varphi_j;\alpha,\beta}\; \overline{g}\;.$$

2. Next, observe that $W(m\alpha, n\beta)$ commutes with $W\left(\frac{k}{\beta}, \frac{\ell}{\alpha}\right)$, for any $m, n, k, \ell \in \mathbb{Z}$. It follows that

$$\left(T_{W(k/\beta,\ell/\alpha)\varphi_j;\alpha,\beta}\;\overline{g}\right)_{m,n} = \left\langle W\left(\frac{-k}{\beta}, \frac{-\ell}{\alpha}\right)^*\;\overline{g},\; W(m\alpha, n\beta)\varphi_j\right\rangle$$

$$= e^{2\pi i\; k\ell/\alpha\beta}\left\langle W\left(\frac{k}{\beta}, \frac{\ell}{\alpha}\right)\;\overline{g},\; W(m\alpha, n\beta)\varphi_j\right\rangle$$

$$= e^{2\pi i\; k\ell/\alpha\beta}\; T_{\varphi_j;\alpha,\beta}\;\overline{g}_{k/\beta,\ell/\alpha}\;.$$

Putting it all together, we have now

$$\alpha\, T^*_{\varphi_j;\alpha,\beta}\, R_{\alpha\beta}\, T_{g;\,\alpha,\beta}\, T^*_{\varphi_j;1/\beta,1/\alpha}\, R_{1/\alpha\beta}\, e_{k,\ell} = \alpha\, T^*_{\varphi_j;\alpha,\beta}\, T_{\varphi_j;\alpha,\beta}\, \overline{g}_{k/\beta,\ell/\alpha}\ .$$

3. Summing over j leads to (use (3.6))

$$\alpha \sum_{j=0}^{J-1} T^*_{\varphi_j;\alpha,\beta}\, R_{\alpha\beta}\, T_{g;\alpha,\beta}\, T^*_{\varphi_j;1/\beta,1/\alpha}\, R_{1/\alpha\beta}\, e_{k,\ell} = \bar{g}_{k/\beta,\ell/\alpha} = T^*_{\bar{g};1/\beta,1/\alpha}\, e_{k,\ell}\ ,$$

which proves our claim. □

Since $T_{g;1/\beta,1/\alpha}$ is obviously bounded if and only if $T_{\overline{g};1/\beta,1/\alpha}$ is, this shows that we can indeed drop the additional condition that $T_{g;1/\beta,1/\alpha}$ be bounded in the statement of Lemma 3.4. This completes the proof of Theorem 3.1.

Now that we have established (3.1) rigorously, with minimal assumptions on the functions involved, we can exploit it as in the original paper [5]. Suppose that g^b is any "dual function" for the frame $g_{m\alpha,n\beta}$ in the sense that $T_{g^b;\alpha,\beta}$ is bounded, and $T^*_{g;\,\alpha,\beta}\, T_{g^b;\alpha,\beta} = \mathrm{Id}$, or, for all f in $L^2(\mathbb{R})$,

$$f = \sum_{m,n} \langle f,\ g^b_{m\alpha,n\beta}\rangle g_{m\alpha,n\beta}\ .$$

(The dual frame function $\tilde{g}$ constructed above is an example of such a g^b, but it is by no means the only one in general.) Taking adjoints, we obtain $T^*_{g^b;\alpha,\beta}\, T_{g;\,\alpha,\beta} = \mathrm{Id}$; applying this to h, which decays like $(1+|x|)^{-1-\epsilon}$, and using the WR identity (3.1) we conclude that

$$T^*_{h;1/\beta,1/\alpha}\, T_{g;1/\beta,1/\alpha}\, g^b = \alpha\beta\, h\ . \tag{3.10}$$

This means that the sequence $c = T_{g;1/\beta,1/\alpha}\, g^b$ should satisfy

$$\sum_{m,n} c_{m,n}\, e^{-2\pi i\ (m/\beta)x} h\left(x - \frac{n}{\alpha}\right) = \alpha\beta\ h(x)\ ,$$

for all h in a dense set. It suffices to choose $h(x) = \chi_{[0,\beta]}(x)$ to see that this is possible only if $c_{m,n} = \alpha\beta\ \delta_{m,0}\ \delta_{n,0}$. (Note that $\alpha\beta \le 1$, since we are assuming that the $g_{m\alpha,n\beta}$ constitute a frame.) This argument has proved part of the following result.

3.8. Proposition

Assume that the $g_{m\alpha,n\beta}$ constitute a frame. Then a function g^b such that $T_{g^b;\alpha,\beta}$ is bounded satisfies

$$T^*_{g;\alpha,\beta}\, T_{g^b;\alpha,\beta} = \mathrm{Id} = T^*_{g^b;\alpha,\beta}\, T_{g;\alpha,\beta} \tag{3.11}$$

if and only if

$$T_{g;1/\beta,1/\alpha}\ g^b = \alpha\beta\ e_{0,0} = T_{g^b;1/\beta,1/\alpha}\ g\ . \tag{3.12}$$

Proof. We already saw that (3.11) implies (3.12). To prove the converse, observe that (3.12) immediately implies (3.10) for all h with decay like $(1+|x|)^{-1-\epsilon}$. By the WR identity, this implies $T^*_{g^b;\alpha,\beta}\ T_{g;\alpha,\beta}h = h$ for these h, which proves (3.11). □

Note that (3.12) also implies

$$\begin{aligned}\left(T_{g;1/\beta,1/\alpha}\ g^b_{k/\beta,\ell/\alpha}\right)_{m,n} &= \left\langle g^b, W\left(\frac{k}{\beta},\frac{\ell}{\alpha}\right)^* W\left(\frac{m}{\beta},\frac{n}{\alpha}\right) g\right\rangle \\ &= e^{-2\pi i\ (k/\alpha\beta)(\ell-m)} \left\langle g^b, W\left(\frac{m-k}{\beta},\frac{n-\ell}{\alpha}\right) g\right\rangle \\ &= \alpha\beta\ \delta_{k,m}\ \delta_{\ell,n} = \alpha\beta(e_{k,\ell})_{m,n}\ .\end{aligned}$$

It follows that $T_{g;1/\beta,1/\alpha}\ T^*_{g^b;1/\beta,1/\alpha}\ c = \alpha\beta\ c$ for the dense set of c in ℓ^2 with finitely many nonzero entries, or

$$T_{g;1/\beta,1/\alpha}\ T^*_{g^b;1/\beta,1/\alpha} = \alpha\beta\ \mathrm{Id} = T_{g^b;1/\beta,1/\alpha}\ T^*_{g;1/\beta,1/\alpha}\ . \tag{3.13}$$

This way of rewriting (3.12) enhances the symmetry between (3.11) and (3.12). Note also that for $\alpha\beta \le 1$, the integer J in Lemmas 3.5 to 3.7 is exactly 1, and the corresponding special function φ_0 is "self-dual", in the sense that

$$T_{\varphi_0;1/\beta,1/\alpha}\ T^*_{\varphi_0;1/\beta,1/\alpha} = \beta\ \mathrm{Id}; \qquad T^*_{\varphi_0;\alpha,\beta}\ T_{\varphi_0;\alpha,\beta} = \frac{1}{\alpha}\ \mathrm{Id}\ .$$

In fact, (3.9) is equally valid if φ_0 is replaced by a general function h for which the $h_{m\alpha,n\beta}$ constitute a frame; if h^b is dual to h in the sense of (3.11), one has

$$T^*_{g;1/\beta,1/\alpha} = T^*_{h^b;\alpha,\beta}\ R_{\alpha\beta}\ T_{g;\,\alpha,\beta}\ T^*_{h;1/\beta,1/\alpha}\ R_{1/\alpha\beta}\ . \tag{3.14}$$

4. The Wexler–Raz Dual Function Is Identical To the Frame Dual Function

Proposition 3.8 characterizes all the functions g^b "dual" to the $g_{m\alpha,n\beta}$ in the sense of (3.11) as the pre-images of $\alpha\beta\ e_{0,0}$ under the map $T_{g;1/\beta,1/\alpha}$. Among all these dual functions g^b we can pick the one with minimal norm; we shall call this the *Wexler–Raz dual function* (as opposed to the frame dual function $\tilde{g}$ at the end of §2) and denote it by $g^{\#}$. We can find an explicit expression for $g^{\#}$ by an argument similar to what led to (2.6) for $\tilde{g}$. Again, the minimal norm solution g^b to $T_{g;1/\beta,1/\alpha}\, g^b = \alpha\beta\, e_{0,0}$ will result if we "mod out" by Ker $T_{g;1/\beta,1/\alpha}$ by taking the orthogonal projection of an arbitrary solution g^b onto (Ker $T_{g;1/\beta,1/\alpha})^{\perp}$ = Ran $T^*_{g;1/\beta,1/\alpha}$. We have used here that the range of $T^*_{g;1/\beta,1/\alpha}$ is a closed set, which follows from (3.13), since, for any dual function g^b,

$$\|c\| = \frac{1}{\alpha\beta}\, \|T_{g^b;1/\beta,1/\alpha}\, T^*_{g;1/\beta,1/\alpha}\, c\| \le \frac{1}{\alpha\beta}\, \|T_{g^b;1/\beta,1/\alpha}\|\, \|T^*_{g;1/\beta,1/\alpha}\, c\| \,. \tag{4.1}$$

To find the projection of g^b onto Ran $T^*_{g;1/\beta,1/\alpha}$, we have to identify c in $\ell^2(\mathbb{Z}^2)$ so that $g^b - T^*_{g;1/\beta,1/\alpha}\, c \perp$ Ran $T^*_{g;1/\beta,1/\alpha}$ or $T_{g;1/\beta,1/\alpha}(g^b - T^*_{g;1/\beta,1/\alpha}\, c) = 0$. This is immediately equivalent with $T_{g;1/\beta,1/\alpha}\, T^*_{g;1/\beta,1/\alpha}\, c = \alpha\beta\, e_{0,0}$ or, since $T_{g;1/\beta,1/\alpha}\, T^*_{g;1/\beta,1/\alpha}$ has a bounded inverse by (4.1), $c = \alpha\beta(T_{g;1/\beta,1/\alpha}\, T^*_{g;1/\beta,1/\alpha})^{-1}\, e_{0,0}$. The projection of g^b onto (Ker $T_{g;1/\beta,1/\alpha})^{\perp}$ is therefore $T^*_{g;1/\beta,1/\alpha}\, c = \alpha\beta\, T^*_{g;1/\beta,1/\alpha}\, (T_{g;1/\beta,1/\alpha}\, T^*_{g;1/\beta,1/\alpha})^{-1}\, e_{0,0}$. The following proposition summarizes these findings.

4.1. Proposition

Assume that the $g_{m\alpha,n\beta}$ constitute a frame. Then the functions g^b for which $T_{g;1/\beta,1/\alpha}\, g^b = \alpha\beta\, e_{0,0}$ are exactly the elements of $g^{\#}$ + Ker $T_{g;1/\beta,1/\alpha}$, with

$$g^{\#} = \alpha\beta\, T^*_{g;1/\beta,1/\alpha}\, (T_{g;1/\beta,1/\alpha}\, T^*_{g;1/\beta,1/\alpha})^{-1}\, e_{0,0}\,; \tag{4.2}$$

$g^{\#}$ itself is the solution with minimum L^2-norm.

Even though $\tilde{g}$ and $g^{\#}$ are obtained as the solutions of different minimization problems, their geometric interpretation as the result of projecting arbitrary c solving (2.3) or arbitrary g^b solving (3.12) now easily leads to the following result.

4.2. Proposition

Assume that the $g_{m\alpha,n\beta}$ constitute a frame. Then the frame dual function $\tilde{g}$ defined by (2.6) *and the Wexler–Raz dual function $g^{\#}$ defined by* (4.2) *are identical.*

Proof.

1. We start by rederiving in a few lines that the $\tilde{g}_{m\alpha,n\beta}$ are a frame with frame dual function g. By the definition of $\tilde{g}$, we have

$$\tilde{g} = (T^*_{g;\,\alpha,\beta}\, T_{g;\,\alpha,\beta})^{-1}\, g \tag{4.3}$$

or

$$g = T^*_{g;\,\alpha,\beta}\, T_{g;\,\alpha,\beta}\, \tilde{g}\,.$$

It follows easily that

$$W(m\alpha, n\beta)g = (T^*_{g;\,\alpha,\beta}\, T_{g;\,\alpha,\beta})\, W(m\alpha, n\beta)\, \tilde{g}\,;$$

hence

$$T^*_{g;\,\alpha,\beta} = T^*_{g;\,\alpha,\beta}\, T_{g;\,\alpha,\beta}\, T^*_{\tilde{g};\alpha,\beta}\,. \tag{4.4}$$

Consequently,

$$T^*_{g;\,\alpha,\beta}\, T_{g;\,\alpha,\beta} = T^*_{g;\,\alpha,\beta}\, T_{g;\,\alpha,\beta}\, T^*_{\tilde{g};\alpha,\beta}\, T_{\tilde{g};\alpha,\beta}\, T^*_{g;\,\alpha,\beta}\, T_{g;\,\alpha,\beta}$$

or

$$(T^*_{g;\,\alpha,\beta}\, T_{g;\,\alpha,\beta})^{-1} = T^*_{\tilde{g};\alpha,\beta}\, T_{\tilde{g};\alpha,\beta}\,. \tag{4.5}$$

This implies, by (4.3),

$$\tilde{g} = T^*_{\tilde{g};\alpha,\beta}\, T_{\tilde{g};\alpha,\beta}\, g\,. \tag{4.6}$$

2. Since $T_{\tilde{g};\alpha,\beta}$ is bounded by (4.5), we can apply the WR identity to (4.6) and find

$$\tilde{g} = \frac{1}{\alpha\beta}\, T^*_{g;1/\beta,1/\alpha}\, T_{\tilde{g},1/\beta,1/\alpha}\, \tilde{g}\,. \tag{4.7}$$

This implies that $\tilde{g}$ is an element of Ran $T^*_{g;1/\beta,1/\alpha}$. Its projection onto Ran $T^*_{g;1/\beta,1/\alpha}$ is therefore again $\tilde{g}$. On the other hand, $\tilde{g}$ is a dual function for the $g_{m\alpha,n\beta}$ in the sense of (3.11), implying that it solves (3.12). By Proposition 4.1, its projection onto Ran $T^*_{g;1/\beta,1/\alpha}$ is therefore equal to $g^{\#}$. It follows that $g^{\#} = \tilde{g}$, as claimed. □

Our analysis so far has shown that if the $g_{m\alpha,n\beta}$ constitute a frame, then

1. $T_{g;1/\beta,1/\alpha}$ is bounded. (4.8)
2. $T_{g;1/\beta,1/\alpha}\, T^*_{g;1/\beta,1/\alpha}$ is invertible and has a bounded inverse. (4.9)
3. $\tilde{g} = \alpha\beta\, T^*_{g;1/\beta,1/\alpha}(T_{g;1/\beta,1/\alpha}\, T^*_{g;1/\beta,1/\alpha})^{-1}\, e_{0,0}$. (4.10)

Note that this WR construction of the dual function $g^{\#}$ works whenever (4.8)–(4.10) hold; no a priori assumption that the $g_{m\alpha,n\beta}$ constitute a frame needs to be made. In order to establish complete equivalence between the frame dual function and the WR dual function, it remains to show that (4.8)–(4.10) also imply that the $g_{m\alpha,n\beta}$ constitute a frame. This essentially follows from the results proved in §3. We already know that $T_{g;\alpha,\beta}$ is bounded if and only if $T_{g;1/\beta,1/\alpha}$ is (by Lemma 3.7). We therefore only need to prove that if $T_{g;1/\beta,1/\alpha}\, T^*_{g;1/\beta,1/\alpha}$ has a bounded inverse (necessary to make (4.10) work), then $T^*_{g;\alpha,\beta}\, T_{g;\alpha,\beta}$ likewise has a bounded inverse. By interchanging the roles of (α, β) and $(\frac{1}{\beta}, \frac{1}{\alpha})$, in Lemma 3.5, we construct $K \geq 1$ and functions ψ_k, $k = 0, \ldots, K-1$, such that

$$\sum_{k=0}^{K-1} T^*_{\psi_k;1/\beta,1/\alpha}\, T_{\psi_k;1/\beta,1/\alpha} = \beta\, \mathrm{Id}_{L^2(\mathbb{R})},$$

$$\sum_{k=0}^{K-1} T_{\psi_k;\alpha,\beta}\, T^*_{\psi_k;\alpha,\beta} = \frac{1}{\alpha}\, \mathrm{Id}_{\ell^2(\mathbb{Z}^2)}\, . \tag{4.11}$$

By (3.3) we have

$$T^*_{\psi_k;\alpha,\beta}\, R_{\alpha\beta}\, T_{g;\alpha,\beta} = \frac{1}{\alpha\beta}\, T^*_{\bar{g};1/\beta,1/\alpha}\, R_{1/\alpha\beta}\, T_{\psi_k;1/\beta,1/\alpha}\, .$$

Multiplying each side on the left with its adjoint and summing over k leads to

$$T^*_{g;\alpha,\beta}\, T_{g;\alpha,\beta} = \frac{1}{\alpha\beta^2} \sum_{k=0}^{K-1} T^*_{\psi_k;1/\beta,1/\alpha}\, R_{1/\alpha\beta}\, T_{\bar{g};1/\beta,1/\alpha}\, T^*_{\bar{g};1/\beta,1/\alpha}\, R_{1/\alpha\beta}\, T_{\psi_k;1/\beta,1/\alpha}\, .$$

For all $f \in L^2(\mathbb{R})$ we therefore have

$$\|T_{g;\alpha,\beta}\, f\|^2 = \frac{1}{\alpha\beta^2} \sum_{k=0}^{K-1} \left\| T^*_{\bar{g};1/\beta,1/\alpha}\, R_{1/\alpha\beta}\, T_{\psi_k;1/\beta,1/\alpha}\, f \right\|^2\, . \tag{4.12}$$

If now, for all $c \in \ell^2(\mathbb{Z}^2)$,

$$A\|c\|^2 \leq \|T^*_{\bar{g};1/\beta,1/\alpha}\, c\|^2 \leq B\|c\|^2\, ,$$

with $A > 0$, $B < \infty$ (that is, if $T_{g;1/\beta,1/\alpha}$ is bounded and $T_{g;1/\beta,1/\alpha}\, T^*_{g;1/\beta,1/\alpha}$ has a bounded inverse), then (4.12) implies

$$\frac{A}{\alpha\beta}\, \|f\|^2 \leq \|T_{g;\alpha\beta}\, f\|^2 \leq \frac{B}{\alpha\beta}\|f\|^2\, ,$$

where we have used that $\sum_{k=0}^{K-1} \|T_{\psi_k;1/\beta,1/\alpha}\, f\|^2 = \beta\|f\|^2$, by (4.11). The following theorem summarizes all our findings.

4.3. Theorem

For $g \in L^2(\mathbb{R})$, $\alpha, \beta > 0$, the operator $T_{g;\alpha,\beta}$ defined by (2.1) *is bounded from $L^2(\mathbb{R})$ to $\ell^2(\mathbb{Z}^2)$ if and only if $T_{g;1/\beta,1/\alpha}$ is. Moreover, $T^*_{g;\alpha,\beta}\, T_{g;\alpha,\beta}$ has a bounded inverse if and only if $T_{g;1/\beta,1/\alpha}\, T^*_{g;1/\beta,1/\alpha}$ has a bounded inverse, and then the "frame dual function" $\tilde{g}$ and the "Wexler–Raz dual function" $g^\#$ coincide, that is,*

$$(T^*_{g;\alpha,\beta}\, T_{g;\alpha,\beta})^{-1}\, g = \alpha\beta\, T^*_{g;1/\beta,1/\alpha}(T_{g;1/\beta,1/\alpha}\, T^*_{g;1/\beta,1/\alpha})^{-1}\, e_{0,0}\ ,$$

where $e_{0,0} \in \ell^2(\mathbb{Z}^2)$ is the sequence $(e_{0,0})_{k,\ell} = \delta_{k,0}\,\delta_{\ell,0}$.

The main practical interest of this result is that it leads to a simpler construction of the dual function $\tilde{g}$ than the recursion proposed in [3]. For the case $g(x) = \pi^{-1/4}e^{-x^2/2}$, $\alpha = 0.25$, $\beta = 2.0$, for instance, one finds $A \simeq 1.600$, $B \simeq 2.425$ (see [3]; more accurate values can be found in [9]), and the frame dual function $\tilde{g}$ is then computed as

$$\tilde{g} = \frac{2}{A+B} \sum_{k=0}^{\infty} \left(\mathrm{Id} - \frac{2}{A+B}\, T^*_{g;1/4,2}\, T_{g;1/4,2}\right)^k g = \lim_{K\to\infty} g^K\ ,$$

where $g^K = \frac{2}{A+B}\sum_{k=0}^{K}(\mathrm{Id} - \frac{2}{A+B}T^*_{g;1/4,2}\, T_{g;1/4,2})^k g$ can also be defined recursively by

$$g^{K+1} = \frac{2}{A+B}\, g + g^K - \frac{2}{A+B} \sum_{m,n} \langle g^K, g_{m/4,2n}\rangle\, g_{m/4,2n}$$

and

$$g^0 = \frac{2}{A+B}\, g\ .$$

If one writes

$$g^K = \sum_{m,n} \lambda^K_{m,n}\, g_{m/4,2n}\ ,$$

then this leads to a recursive computation of the $\lambda^K_{m,n}$, which converges exponentially fast in K, $\|\tilde{g} - g^K\| \le C\left(\frac{B-A}{B+A}\right)^K \simeq C(0.2)^K$. The computation of $g^\#$ can be carried out by exactly the same trick, with

$$\begin{aligned}&(T_{g;1/2,4}\, T^*_{g;1/2,4})^{-1}\, e_{0,0}\\ &\quad= \frac{4}{A+B} \sum_{k=0}^{\infty}\left(\mathrm{Id} - \frac{4}{A+B}\, T_{g;1/2,4}\, T^*_{g;1/2,4}\right)^k e_{0,0} = \lim_{K\to\infty} e^K\ ,\end{aligned}$$

where $e^0 = \frac{4}{A+B}\, e_{0,0}$ and $e^K = \frac{4}{A+B}\, e_{0,0} + e^K - \frac{4}{A+B}\, T_{g;1/2,4}\, T^*_{g;1/2,4}\, e^K$. This amounts to writing $g^\#$ as a limit of $\sum_{m,n} \mu^K_{m,n}\, g_{m/2,4n}$, where the $\mu^K_{m,n}$ obey a recursion similar to that of the $\lambda^K_{m,n}$ above. For the same precision, we need however to compute fewer $\mu^K_{m,n}$ (about $\frac{1}{4}$ as many) than $\lambda^K_{m,n}$, and the recursion will also use smaller matrices because $\langle g_{m/2,4n},\ g\rangle$ decays faster in m, n than $\langle g_{m/4,2n},\ g\rangle$.

Remark. The dual function $\tilde{g} = g^\#$ can be used to expand arbitrary functions into the $g_{m\alpha,n\beta}$, as in

$$f = \sum_{m,n} \langle f,\ \tilde{g}_{m\alpha,n\beta}\rangle\, g_{m\alpha,n\beta}\ .$$

In the case $\alpha\beta = \frac{1}{2}$, as in the example above, it is also possible to regroup the $g_{m\alpha,n\beta}$ and $g_{-m\alpha,n\beta}$ into new functions $E_{|m|\alpha,n\beta}$, that are then independent. The expansion of f into this basis $E_{|m|\alpha,n\beta}$ uses coefficients that can again be computed by means of inner products of f with the $\tilde{g}_{m\alpha,n\beta}$. This gives another use for dual frame functions, allowing, for example, expansions in Gaussian basis functions, without redundancy. For details, see [10].

5. An Independent Proof

Here we return to §4, proving the identity of the two minimizing dual functions, $\tilde{g}$ and $g^\#$, directly from basic considerations, without recourse to the Wexler–Raz formula. Accordingly, we assume that $g_{m\alpha,n\beta}$ constitute a frame and, as in (2.6), define

$$\tilde{g} = (T^*_{g;\alpha,\beta}\, T_{g;\alpha,\beta})^{-1} g\ .$$

The extremal property of $\tilde{g}$ is that (1.6) generates the expansion coefficients in (1.2) of least ℓ^2 norm. On combining (1.6) and (1.2) we see that $\tilde{g}$ satisfies

$$T^*_{g;\alpha,\beta}\, T_{\tilde{g};\alpha,\beta} = I\ .$$

However, the equation

$$T^*_{g;\alpha,\beta}\, T_{g^b;\alpha,\beta} = I \tag{5.1}$$

can also have other solutions g^b, and we denote by $g^\#$ that solution having least L^2 norm.

Any two g^b in (5.1) differ by u for which

$$T^*_{g;\alpha,\beta}\, T_{u;\alpha,\beta} = 0$$

or, equivalently, one for which

$$\langle T_{u;\alpha,\beta}\ p,\ T_{g;\alpha,\beta}\ q\rangle = 0\ ,$$

for all $p, q \in L^2$. Since, as we have seen, there is a unitary map between $T_{u;\alpha,\beta}\ p$ and $T_{\bar{p};\alpha,\beta}\ \bar{u}$, it follows that

$$\langle T_{p;\alpha,\beta}\ u,\ T_{q;\alpha,\beta}\ g\rangle = 0$$

or, finally,

$$\langle u,\ T^*_{p;\alpha,\beta}\ T_{q;\alpha,\beta}\ g\rangle = 0\ .$$

We conclude that $g^\#$ is the (unique) projection of g^b onto the subspace to which u is orthogonal, that is, the unique solution of (5.1) that lies in the subspace $\mathcal{G}$ generated by functions $T^*_{p;\alpha,\beta}\ T_{q;\alpha,\beta}\ g$. But as in (4.5),

$$(T^*_{g;\alpha,\beta}\ T_{g;\alpha,\beta})^{-1} = T^*_{\tilde{g};\alpha,\beta}\ T_{\tilde{g};\alpha,\beta}\ ,$$

whence

$$\tilde{g} = T^*_{\tilde{g};\alpha,\beta}\ T_{\tilde{g};\alpha,\beta}\ g \in \mathcal{G}\ .$$

Consequently $g^\# = \tilde{g}$, as was to be proved.

6. von Neumann Algebras

The operator families $W(j\alpha, k\beta)$ and $W(\frac{m}{\beta}, \frac{n}{\alpha})$ linked by the Wexler–Raz formula are distinguished by commuting with one another. This and some of our earlier proofs point to commutativity as an important factor here and thereby suggest a possible connection with von Neumann algebras, which have commutativity at their core. Indeed, these algebras have already been successfully applied to time-frequency lattices, for until the recent proof [11] they gave the only means to show that for the span of $W(j\alpha, k\beta)g$ to be dense in $L^2(\mathbb{R})$, it was necessary that $\alpha\beta \leq 1$. However, this argument had relied on the *coupling constant*, a feature of these algebras that is hard to establish.

We will sketch the basic facts of the subject—an excellent reference is [12]—and show that it offers a clear view of the Wexler–Raz identity and of its properties. Further, we will establish the above density condition in a simple and direct way. Finally, we will also use the Wexler–Raz formula to give an elementary proof of the existence of the coupling constant for von Neumann algebras associated with time-frequency lattices.

Definition. Let $\mathcal{B}(H)$ denote the space of bounded linear operators on a Hilbert space H. Suppose that a given set of operators $\mathcal{T} \subset \mathcal{B}(H)$ includes the identity as well as the adjoint T^* of every $T \in \mathcal{T}$. Let the *commutant algebra* $\mathcal{A}' \subset \mathcal{B}(H)$ be the collection of bounded operators that commute with every $T \in \mathcal{T}$, and denote by $\{\{\mathcal{T}\}\}$ the linear combinations of a finite number of finite products of $T_i \in \mathcal{T}$. The aim is to add to $\{\{\mathcal{T}\}\}$ all the bounded operators that commute with $\mathcal{A}'$. By the *double commutant theorem* [12], this can be done by forming the closure $\mathcal{A}$ of $\{\{\mathcal{T}\}\}$ with respect to either of:

- *strong convergence*: $A \in \mathcal{A}$ if there is a sequence (or net) $T_i \in \{\{\mathcal{T}\}\}$ with $T_i(h) \to A(h)$ for each $h \in H$;
- *weak convergence*: $A \in \mathcal{A}$ if there is a sequence (or net) $T_i \in \{\{\mathcal{T}\}\}$ with $(T_i h_1, h_2) \to (Ah_1, h_2)$ for each $h_1, h_2 \in H$.

$\mathcal{A}$ is termed a *von Neumann algebra*. Clearly, $\mathcal{A}'$ commutes also with $\mathcal{A}$.

Here, let $\mathcal{T}$ consist of the frequency and time translations $W(j\alpha, k\beta)$, $-\infty < j, k < \infty$, acting on $L^2(\mathbb{R})$, as defined in (2.4),

$$W(j\alpha, k\beta) f = e^{-2\pi i \alpha j t} f(t - k\beta) \ . \tag{6.1}$$

The commutation relation

$$W(\alpha, 0) W(0, \beta) = e^{2\pi i \alpha\beta} \, W(0, \beta) W(\alpha, 0) \tag{6.2}$$

reduces all finite products of the generators to the form $e^{2\pi i \alpha\beta M} W(j\alpha, k\beta)$ for suitable integers M, j, k. Since $W(j\alpha, k\beta)$ is unitary,

$$W(j\alpha, k\beta)^* = [W(j\alpha, k\beta)]^{-1} = e^{-2\pi i \alpha\beta jk} W(-j\alpha, -k\beta) \ , \tag{6.3}$$

so $\mathcal{T}$ contains the adjoints of its elements as well as $I = W(0, 0)$, as required.

Commutant. The elements $W(\frac{m}{\beta}, \frac{n}{\alpha})$ clearly commute with $\mathcal{A}$ and so belong to $\mathcal{A}'$. A straightforward argument (Appendix 6.1) shows that $\mathcal{A}'$ is generated by $\{\{W(\frac{m}{\beta}, \frac{n}{\alpha})\}\}$.

Trace. A *faithful trace* is a linear functional on $\mathcal{B}(H)$ with the properties

$$\begin{gathered} \text{tr}\,(I) = 1 \ , \\ \text{tr}\,(AB) = \text{tr}\,(BA) \ , \\ \text{tr}\,(A^* A) > 0 \ \text{ except when } A = 0. \end{gathered} \tag{6.4}$$

In the algebra $\{\{W(j\alpha, k\beta)\}\}$, the elements consist of finite linear combinations

$$T = \sum_{|j,k|<N} c_{jk} W(j\alpha, k\beta) \ .$$

Using (6.3), we can verify by explicit calculation that the choice

$$\text{tr}\,(T) \equiv c_{00} \tag{6.5}$$

has the required properties.

Let χ_i, $i = 1, \ldots, K$, denote the characteristic functions of disjoint contiguous intervals, none longer than $\min(\frac{1}{\alpha}, \beta)$, that together make up the interval $[0, \frac{1}{\alpha}]$. Then we have

$$\text{tr}_{\mathcal{A}}(T) = \alpha \sum_{i=1}^{K} (T\chi_i, \chi_i) \ . \tag{6.6}$$

For it is sufficient to check (6.6) for $T = W(j\alpha, k\beta)$. Since the intervals are no longer than β, any two translates of one of them by multiples of β are disjoint. Thus the only nonzero contribution to (6.6) comes from $k = 0$, whereupon the sum in (6.6) becomes

$$\alpha \int_0^{1/\alpha} e^{2\pi i\alpha jt}\, dt = 0, \qquad j \neq 0 \ .$$

The right-hand side of (6.6) allows the trace to be extended to $\mathcal{A}$ by continuity.

When $\alpha\beta$ is irrational, the trace is unique. For if $j \neq 0$, the requirement $\text{tr}\, AB = \text{tr}\, BA$, together with the commutation relation (6.2), yield

$$\begin{aligned} \text{tr}\, W(j\alpha, k\beta) &= \text{tr}\, W(j\alpha, (k-1)\beta) W(0, \beta) \\ &= \text{tr}\, W(0, \beta) W(j\alpha, (k-1)\beta) = \text{tr}\, e^{-2\pi i\alpha\beta j} W(j\alpha, k\beta) \ , \end{aligned}$$

implying that $\text{tr} W(j\alpha, k\beta) = 0$; while if $j = 0$, the same argument applies to

$$\text{tr}\, W(0, k\beta) = \text{tr}\, W(-\alpha, 0) W(0, k\beta) W(\alpha, 0) = \text{tr}\, e^{-2\pi i\alpha\beta k} W(0, k\beta) \ .$$

A New Hilbert Space. We can now use the trace to define a scalar product on the algebra $\{\{W(j\alpha, k\beta)\}\}$, viewed as a linear space, by

$$[A, B] \equiv \text{tr}_{\mathcal{A}} B^* A \ . \tag{6.7}$$

Since each $A \in \mathcal{A}$ is a weak limit of operators from the algebra, the expression (6.6) shows that it is likewise a limit in the norm defined by (6.7). Thus the completion of $\{\{W(j\alpha, k\beta)\}\}$ in the scalar product (6.7) yields a Hilbert space (of operators) that includes $\mathcal{A}$ and which we therefore denote by $L^2(\mathcal{A})$. There is an isomorphism between $\mathcal{A}$ acting on the original $\mathcal{H}$, and its acting on $L^2(\mathcal{A})$ by left multiplication; this identification is known as the G–N–S (Gelfand–Naimark–Segal) construction. An important fact for us is that, by (6.5), the elements $W(j\alpha, k\beta)$ constitute an orthonormal basis for $L^2(\mathcal{A})$. Of course, the identical considerations apply to $\mathcal{A}'$ and give $W(\frac{m}{\beta}, \frac{n}{\alpha})$ as an orthonormal basis for $L^2(\mathcal{A}')$.

As heretofore, we assume $f, g \in L^2$, and suppose initially that $T_{f;\alpha,\beta}$ and $T_{g;\alpha,\beta}$ are bounded operators. We have already seen that $T^*_{f;\alpha,\beta}\, T_{g;\alpha,\beta}$ commutes with $W(j\alpha, k\beta)$, hence

$$T^*_{f;\alpha,\beta}\, T_{g;\alpha,\beta} \in \mathcal{A}' \, .$$

To find its trace in $\mathcal{A}'$ we use (6.6) with α replaced by $\frac{1}{\beta}$ to reflect the change of algebra, obtaining

$$\mathrm{tr}_{\mathcal{A}'}(T^*_{f;\alpha,\beta}\, T_{g;\alpha,\beta}) = \frac{1}{\beta}\sum_j \langle T^*_{f;\alpha,\beta}\, T_{g;\alpha,\beta}\, \chi_j, \chi_j\rangle = \frac{1}{\beta}\sum_j \langle T_{g;\alpha,\beta}\, \chi_j,\ T_{f;\alpha,\beta}\, \chi_j\rangle \, , \tag{6.8}$$

with χ_j characteristic functions of intervals smaller than $\frac{1}{\alpha}$ that decompose $[0, \beta]$. For each k, the components $c_{j,k}$ of $T_{f;\alpha,\beta}\, \chi_j$ are the Fourier coefficients in the basis $\{e^{-2\pi i\alpha jt}\}$ of $\overline{f(t)}\chi_j(t - k\beta)$, a function supported on an interval no longer than $\frac{1}{\alpha}$. By Parseval's theorem,

$$\langle T_{g;\alpha,\beta}\, \chi_j, T_{f;\alpha,\beta}\, \chi_j\rangle = \frac{1}{\alpha}\sum_k \langle \overline{g(t)}\, \chi_j(t + k\beta), \overline{f(t)}\rangle,$$

and on summing these components over j we find

$$\mathrm{tr}_{\mathcal{A}'}(T^*_{f;\alpha,\beta} T_{g;\alpha,\beta}) = \frac{1}{\alpha\beta}\sum_k \langle \overline{g}\, \chi_{[0,\beta]}(t + k\beta), \overline{f}\rangle = \frac{1}{\alpha\beta}\langle f, g\rangle \, . \tag{6.9}$$

Expanding $T^*_{f;\alpha,\beta}\, T_{g;\alpha,\beta} \in \mathcal{A}' \subset L^2(\mathcal{A}')$ in the above basis, we obtain

$$T^*_{f;\alpha,\beta} T_{g;\alpha,\beta} = \sum \left[T^*_{f;\alpha,\beta} T_{g;\alpha,\beta}, W\left(\frac{j}{\beta}\, ,\, \frac{k}{\alpha}\right)\right]_{\mathcal{A}'} W\left(\frac{j}{\beta}\, ,\, \frac{k}{\alpha}\right) \, , \tag{6.10}$$

the sum converging in the norm of $L^2(\mathcal{A}')$. Denote the coefficients in (6.10) by $\gamma_{j,k}$; to evaluate them, we have by (6.7) and (6.4),

$$\gamma_{j;k} \equiv \left[T^*_{f;\alpha,\beta}\, T_{g;\alpha,\beta}, W\left(\frac{j}{\beta}\, ,\, \frac{k}{\alpha}\right)\right]_{\mathcal{A}'} = \mathrm{tr}_{\mathcal{A}'}\, T^*_{f;\alpha,\beta}\, T_{g;\alpha,\beta}\, W^*\left(\frac{j}{\beta}\, ,\, \frac{k}{\alpha}\right) \, . \tag{6.11}$$

For an operator $S \in \mathcal{A}'$, and any $u \in L^2$, by definition

$$T_{\varphi;\alpha,\beta} S u = \{\langle Su, W(m\alpha, n\beta)\varphi\rangle\} = \left\{\langle u, S^* W(m\alpha, n\beta)\varphi\rangle\right\} = \left\{\langle u, W(m\alpha, n\beta) S^*\varphi\rangle\right\} \, .$$

This means that

$$T_{\varphi;\alpha,\beta} S \equiv T_{S^*\varphi;\alpha,\beta} \tag{6.12}$$

are merely different ways of writing the same operator. Applying this to (6.11) with $S = W^*(\frac{j}{\beta}, \frac{k}{\alpha})$ and using (6.9) we find

$$\gamma_{j,k} = \mathrm{tr}_{\mathcal{A}'} T^*_{f;\alpha,\beta} T_{W(j/\beta,k/\alpha)g;\alpha,\beta} = (\alpha\beta)^{-1} \left\langle f, W\left(\frac{j}{\beta}\, ,\, \frac{k}{\alpha}\right) g \right\rangle .$$

Consequently,

$$\{\gamma_{j,k}\} = (\alpha\beta)^{-1} T_{g;1/\beta,1/\alpha} f \ , \tag{6.13}$$

that is, f is in the domain of the operator $(\alpha\beta)^{-1} T_{g;1/\beta,1/\alpha}$ and is taken by it to $\{\gamma_{j,k}\}$; being the coefficients of an orthonormal expansion, this sequence is square-summable. We note that this holds as soon as the product $T^*_{f;\alpha,\beta} T_{g;\alpha,\beta}$ is bounded, without requiring boundedness of the factors separately.

A Unitary Equivalence. We can use the unitary map between the Hilbert spaces $L^2(\mathcal{A}')$ and ℓ^2, given by the coefficients of an element in $L^2(\mathcal{A}')$ in the basis $\{W(\frac{m}{\beta}, \frac{n}{\alpha})\}$, to carry an operator on $L^2(\mathcal{A}')$ into its correspondent on ℓ^2. In Appendix 6.2 we show that this map leads to a conjugate-linear algebra isomorphism carrying $T^*_{f;\alpha,\beta}\, T_{g;\alpha,\beta}$ acting on $L^2(\mathcal{A}')$ to $(\alpha\beta)^{-1} T_{f;1/\beta,1/\alpha}\, T^*_{g;1/\beta,1/\alpha}$ acting on ℓ^2. When composed with the G–N–S construction, this produces a conjugate-linear isomorphism taking $T^*_{f;\alpha,\beta}\, T_{g;\alpha,\beta}$ acting on L^2 to $(\alpha\beta)^{-1} T_{f;1/\beta,1/\alpha}\, T^*_{g;1/\beta,1/\alpha}$ acting on ℓ^2. As any algebra isomorphism preserves the spectrum, and as the spectrum of an element is independent of the C^* algebra that contains it [1, p. 241], it follows that the spectrum of one of these operators is the complex conjugate of that of the other. When $f = g$, this shows that

$$\text{spectrum } \left\{(\alpha\beta) T^*_{g;\alpha,\beta}\, T_{g;\alpha,\beta}\right\} = \text{spectrum } \left\{T_{g;1/\beta,1/\alpha}\, T^*_{g;1/\beta,1/\alpha}\right\} \ ,$$

implying that

$$(\alpha\beta)\|T^*_{g;\alpha,\beta}\, T_{g;\alpha,\beta}\| = \|T_{g;1/\beta,1/\alpha}\, T^*_{g;1/\beta,1/\alpha}\| = \|T^*_{g;1/\beta,1/\alpha}\, T_{g;1/\beta,1/\alpha}\| \ , \tag{6.14}$$

whence

$$\sqrt{\alpha\beta}\ \|T_{g;\alpha,\beta}\| = \|T_{g;1/\beta,1/\alpha}\| \ .$$

This recovers Theorem 4.3.

The Wexler–Raz Identity. Formally, the Wexler–Raz formula comes from applying to $h \in L^2$ the orthonormal expansion (6.10) and (6.13), rewritten here for convenience

$$T^*_{f;\alpha,\beta}\, T_{g;\alpha,\beta} = \sum \gamma_{j,k}\, W\left(\frac{j}{\beta}\,,\ \frac{k}{\alpha}\right),$$

$$\{\gamma_{j,k}\} = (\alpha\beta)^{-1}\, T_{g;1/\beta,1/\alpha}\, f.$$

For the left-hand side of (6.10) is then $T^*_{f;\alpha,\beta}\, T_{g;\alpha,\beta}\, h$, while the right-hand side is $T^*_{h;1/\beta,1/\alpha}\,\{\gamma_{jk}\} = (\alpha\beta)^{-1}\, T^*_{h;1/\beta,1/\alpha}\, T_{g;1/\beta,1/\alpha}\, f$. The equality of these quantities, suggested by (6.10), gives the Wexler–Raz formula. However, the technical point to be established is that (6.10), which refers to convergence in trace norm, implies the corresponding equality

$$T^*_{f;\alpha,\beta}\, T_{g;\alpha,\beta}\, h = (\alpha\beta)^{-1}\, T^*_{h;1/\beta,1/\alpha}\, T_{g;1/\beta,1/\alpha}\, f \tag{6.15}$$

in L^2. We prove this in Appendix 6.3.

Alternatively, let V denote the unitary map sending $T^*_{f;\alpha,\beta}\, T_{g;\alpha,\beta} \in L^2(\mathcal{A}')$ onto its coefficients $\{\gamma_{j,k}\}$. Suppose to start that u and φ have compact support, and consider

$$\begin{aligned}\langle V\, T^*_{f;\alpha,\beta}\, T_{g;\alpha,\beta},\, V\, T^*_{\varphi;\alpha,\beta}\, T_{u;\alpha,\beta}\rangle &= \frac{1}{(\alpha\beta)^2}\langle T_{g;1/\beta,1/\alpha} f,\ T_{u;1/\beta,1/\alpha}\,\varphi\rangle \\ &= \frac{1}{(\alpha\beta)^2}\langle T^*_{u;1/\beta,1/\alpha}\, T_{g;1/\beta,1/\alpha}\, f, \varphi\rangle\ ,\end{aligned} \tag{6.16}$$

the first equality by (6.13). At the same time, since V is unitary,

$$\begin{aligned}\langle V\, T^*_{f;\alpha,\beta}\, T_{g;\alpha,\beta},\, V\, T^*_{\varphi;\alpha,\beta}\, T_{u;\alpha,\beta}\rangle &= [T^*_{f;\alpha,\beta}\, T_{g;\alpha,\beta},\ T^*_{\varphi;\alpha,\beta}\, T_{u;\alpha,\beta}]_{L^2(\mathcal{A}')} \\ &\equiv \mathrm{tr}_{\mathcal{A}'}\, T^*_{u;\alpha,\beta}\, T_{\varphi;\alpha,\beta}\, T^*_{f;\alpha,\beta}\, T_{g;\alpha,\beta}\ .\end{aligned} \tag{6.17}$$

By (6.12) with $S = T^*_{f;\alpha,\beta}\, T_{g;\alpha,\beta}$ and (6.9),

$$\mathrm{tr}_{\mathcal{A}'}\, T^*_{u;\alpha,\beta}\, T_{\varphi;\alpha,\beta}\, T^*_{f;\alpha,\beta}\, T_{g;\alpha,\beta} = \frac{1}{\alpha\beta}\langle u,\ T^*_{g;\alpha,\beta}\, T_{f;\alpha,\beta}\,\varphi\rangle = \frac{1}{\alpha\beta}\langle T^*_{f;\alpha,\beta}\, T_{g;\alpha,\beta}\, u, \varphi\rangle\ , \tag{6.18}$$

and by comparing (6.17) and (6.18) with (6.16) we find that, as elements of L^2,

$$T^*_{f;\alpha,\beta} T_{g;\alpha,\beta} u = (\alpha\beta)^{-1} T^*_{u;1/\beta,1/\alpha} T_{g;1/\beta,1/\alpha} f\ . \tag{6.19}$$

The argument of Appendix 6.3, beginning at (A3.3), extends this equality to all $u \in L^2$, thereby again proving the Wexler–Raz identity (6.15).

In (6.15) we have a slight generalization of Theorem 3.1, for $T_{h;1/\beta,1/\alpha}$ is not required to be bounded; the formula asserts that if $T^*_{f;\alpha,\beta}\, T_{g;\alpha,\beta}$ is bounded, the sequence $T_{g;1/\beta,1/\alpha} f$ is in the domain of $T^*_{h;1/\beta,1/\alpha}$ and is taken by it into $(\alpha\beta) T^*_{f;\alpha,\beta}\, T_{g;\alpha,\beta} h$.

An Extension. The preceding arguments apply without modification when only $T^*_{f;\alpha,\beta}\,T_{g;\alpha,\beta}$ is bounded, $f, g \in L^2$, without requiring boundedness for $T_{f;\alpha,\beta}$ and $T_{g;\alpha,\beta}$ individually. Here we mean that for some K and each compactly supported $u \in L^2$, the sequence $T_{g;\alpha,\beta}\,u$, known to be square-summable, lies in the domain of $T^*_{f;\alpha,\beta}$ and that $\|T^*_{f;\alpha,\beta}(T_{g;\alpha,\beta}\,u)\| \le K\|u\|$. The operator $T^*_{f;\alpha,\beta}\,T_{g;\alpha,\beta}$ is then defined and bounded on a dense subset of L^2 and can be extended to all of L^2 by continuity. Then for $v \in L^2$ of compact support,

$$\langle T^*_{f;\alpha,\beta}\,T_{g;\alpha,\beta}\,u, v\rangle = \langle T_{g;\alpha,\beta}\,u,\ T_{f;\alpha,\beta}\,v\rangle\ , \tag{6.20}$$

both sides being well defined, so that the trace formula (6.9) continues to hold. Moreover, since from (6.20),

$$|\langle T_{g;\alpha,\beta}\,u,\ T_{f;\alpha,\beta}\,v\rangle| \le (K\|v\|)\ \|u\|\ ,$$

the scalar product on the left is a bounded linear functional of u; hence $T_{f;\alpha,\beta}\,v$ lies in the domain of $T^*_{g;\alpha,\beta}$, and

$$\|T^*_{g;\alpha,\beta}\ T_{f;\alpha,\beta}\,v\| \le K\|v\|\ .$$

Consequently $T^*_{g;\alpha,\beta}\ T_{f;\alpha,\beta}$ is likewise bounded, and again from (6.20)

$$(T^*_{f;\alpha,\beta}\ T_{g;\alpha,\beta})^* = T^*_{g;\alpha,\beta}\ T_{f;\alpha,\beta}\ ,$$

so that (6.19) follows as before. We conclude that the Wexler–Raz identity (6.15) remains valid whenever the operator on either side is bounded.

Lattice Density. We now give an elementary argument for the heretofore difficult result than if $g_{m\alpha,n\beta}$ span L^2 the lattice $(m\alpha, n\beta)$ must be sufficiently dense.

6.1. Theorem

If for some $g \in L^2$ the functions $W(j\alpha, k\beta)g$ span L^2, then $\alpha\beta \le 1$.

Proof. We want to consider arbitrary $g \in L^2$ so that $T_{g;\alpha,\beta}$ may be unbounded. We have seen in §2, however, that this operator is defined and closable in $\mathcal{C}^\infty_0(\mathbb{R})$, and we denote its closure by $T_{g;\alpha,\beta}$ as well. With $\epsilon > 0$, the operator $\epsilon I + T^*_{g;\alpha,\beta}\ T_{g;\alpha,\beta}$ is selfadjoint and bounded below away from zero and, hence, invertible. Let

$$p_\epsilon = (\epsilon I + T^*_{g;\alpha,\beta}\ T_{g;\alpha,\beta})^{-1} g\ .$$

Clearly, $T^*_{g;\alpha,\beta}\ T_{g;\alpha,\beta}\ (\epsilon I + T^*_{g;\alpha,\beta}\ T_{g;\alpha,\beta})^{-1}$ is bounded and, by [13, p. 307], so is $T_{g;\alpha,\beta}(\epsilon I + T^*_{g;\alpha,\beta}\ T_{g;\alpha,\beta})^{-1}$. Consequently, p_ϵ is in the domain of $T^*_{g;\alpha,\beta}\ T_{g;\alpha,\beta}$ and of $T_{g;\alpha,\beta}$. We have

$$g = \epsilon p_\epsilon + T^*_{g;\alpha,\beta}\, T_{g;\alpha,\beta}\, p_\epsilon\ ,$$

and on forming the scalar product with p_ϵ we obtain

$$\epsilon \| p_\epsilon \|^2 + \| T_{g;\alpha,\beta}\, p_\epsilon \|^2 = \langle g,\, p_\epsilon \rangle\ . \tag{6.21}$$

Setting $\nu = \langle g,\, p_\epsilon \rangle$, it follows from (6.21) that

$$\nu \geq \| T_{g;\alpha,\beta}\, p_\epsilon \|^2\ ,$$

while $\nu = \langle p_\epsilon, T^*_{g;\alpha,\beta}\, e_{00} \rangle = \langle T_{g;\alpha,\beta}\, p_\epsilon, e_{00} \rangle$, so that

$$\| T_{g;\alpha,\beta}\, p_\epsilon \|^2 \geq \nu^2\ ;$$

it follows that $\nu \leq 1$. (By way of intuition, $T_{g;\alpha,\beta}\, p_\epsilon$ is constructed to approximate the projection of e_{00} onto the range of $T_{g;\alpha,\beta}$.) Now for the bounded operator $T^*_{g;\alpha,\beta}\, T_{g;\alpha,\beta}(\epsilon I + T^*_{g;\alpha,\beta}\, T_{g;\alpha,\beta})^{-1}$ we find from (6.12) and the definition of p_ϵ that

$$T^*_{g;\alpha,\beta}\, T_{g;\alpha,\beta}\, (\epsilon I + T^*_{g;\alpha,\beta}\, T_{g;\alpha,\beta})^{-1} = T^*_{g;\alpha,\beta}\, T_{p_\epsilon;\alpha,\beta}\ ,$$

whence by using (6.9), extended as in the preceding section, to express the trace of the right-hand side,

$$\alpha\beta\, tr_{\mathcal{A}'}\, T^*_{g;\alpha,\beta}\, T_{g;\alpha,\beta}\, (\epsilon I + T^*_{g;\alpha,\beta}\, T_{g;\alpha,\beta})^{-1} = \langle g,\, p_\epsilon \rangle = \nu \leq 1\ . \tag{6.22}$$

By the functional calculus for selfadjoint operators [13, p. 341], according to which an operator is represented as multiplication by x on its spectrum, as $\epsilon \to 0$ the operator on the left-hand side of (6.22) approaches the projection onto the range of $T^*_{g;\alpha,\beta}\, T_{g;\alpha,\beta}$, which is dense in L^2 whenever that of $T^*_{g;\alpha,\beta}$ is. Thus the limit of the trace in (6.22) is tr $I = 1$, whence $\alpha\beta \leq 1$.

This completes the proof of Theorem 6.1. □

The Coupling Constant. The argument in Theorem 6.1 makes no use of the coupling constant, on which an earlier proof had depended. The existence of this constant is a feature of certain von Neumann algebras that is difficult to establish. Here we proceed in the other direction and use some of our earlier considerations to give a simple proof of the existence of the coupling constant for the algebra $\mathcal{A}$.

6.2. Theorem

With $g \in L^2(\mathbb{R})$, let $\mathcal{Y}$ denote the closure of the subspace of $L^2(\mathbb{R})$ generated by the elements of $\mathcal{A}g$, and let P be the orthogonal projection of $L^2(\mathbb{R})$ onto $\mathcal{Y}$. Analogously, let $\mathcal{Y}'$ be the closure of $\mathcal{A}'g$ and P' be the projection onto $\mathcal{Y}'$. Then $P \in \mathcal{A}'$, $P' \in \mathcal{A}$, and

$$\frac{\mathrm{tr}_{\mathcal{A}'}(P)}{\mathrm{tr}_{\mathcal{A}}(P')} = \frac{1}{\alpha\beta} ,$$

independently of the choice of g. The above quotient is called the coupling constant for $\mathcal{A}'$.

Proof. We first verify that $P \in \mathcal{A}'$, as claimed. Let $f = u + v$ be a decomposition of $f \in L^2$ into components in $\mathcal{Y}$ and in $\mathcal{S}$, its orthogonal complement. Since each $W(m\alpha, n\beta)$ maps $\mathcal{A}g$ onto itself, it does the same with $\mathcal{Y}$ and $\mathcal{S}$. It follows that $Wf = Wu + Wv$ is the corresponding decomposition of Wf; hence $PWf = Wu = WPf$. Thus P commutes with $\mathcal{A}$; hence $P \in \mathcal{A}'$. Analogously, $P' \in \mathcal{A}$.

Suppose first that $T_{g;\alpha,\beta}$ is bounded. As the ranges of $T^*_{g;\alpha,\beta}$ and of $T^*_{g;\alpha,\beta}\, T_{g;\alpha,\beta}$ have the same orthogonal complement, their closures coincide and consist of $\mathcal{Y}$. To simplify notation, set $S = T^*_{g;\alpha,\beta}\, T_{g;\alpha,\beta}$ and $Q = T^*_{g;1/\beta,1/\alpha}\, T_{g;1/\beta,1/\alpha}$; we recall that $S \in \mathcal{A}'$ while $Q \in \mathcal{A}$. Again by the functional calculus, the projection P onto $\mathcal{Y} = \text{range } S$ is given by

$$P = \lim_{k\to\infty} \left\{ I - \left(I - \frac{S}{\|S\|} \right)^k \right\}$$

since $1 - (1-x)^k$ for $x \in [0, 1]$ has limit 0 at $x = 0$ and 1 otherwise, while $x = 0$ and $x \neq 0$ correspond in the spectral decomposition to the null space and range of S, respectively. Thus

$$\mathrm{tr}_{\mathcal{A}'} P = \lim_{k\to\infty} \sum_{j=1}^{k} (-1)^{j+1} \binom{k}{j} \frac{tr_{\mathcal{A}'} S^j}{\|S\|^j} . \tag{6.23}$$

The analogous formula applies to $\mathrm{tr}_{\mathcal{A}} P'$. Now our preceding results allow us to compare traces and norms of S and Q. For by the Wexler–Raz identity

$$Sg = \frac{1}{\alpha\beta}\, Qg ;$$

whence for $k > 1$, because S and Q commute,

$$S^k g = S^{k-1} Sg = \frac{1}{\alpha\beta}\, S^{k-1}\, Qg = \frac{1}{\alpha\beta}\, QS^{k-1} g ,$$

so by induction

$$S^k g = \frac{1}{(\alpha\beta)^k}\, Q^k g . \tag{6.24}$$

Thereupon by (6.12) and (6.9)

$$\begin{aligned}\operatorname{tr}_{\mathcal{A}'} S^k &= \operatorname{tr}_{\mathcal{A}'} S(S^{k-1}) = \operatorname{tr}_{\mathcal{A}'} T^*_{g;\alpha,\beta}\; T_{g;\alpha,\beta}(S^{k-1}) = \operatorname{tr}_{\mathcal{A}'} T^*_{g;\alpha,\beta}\; T_{S^{k-1}g;\alpha,\beta} \\ &= \frac{1}{\alpha\beta}\;\langle g, S^{k-1}g\rangle = \frac{1}{(\alpha\beta)^k}\;\langle g, Q^{k-1}g\rangle\;, \end{aligned} \tag{6.25}$$

the last equality by (6.24). By the analogous argument applied to Q (in which α, β are replaced by $1/\beta$, $1/\alpha$), stopping at the next-to-last of the above equalities yields

$$\operatorname{tr}_{\mathcal{A}} Q^k = \alpha\beta(g, Q^{k-1}g)\;. \tag{6.26}$$

Finally by (6.14)

$$\|S\| = \frac{1}{\alpha\beta}\;\|Q\|\;;$$

hence from (6.25) and (6.26)

$$\operatorname{tr}_{\mathcal{A}'} S^j/\|S\|^j = \frac{1}{\alpha\beta}\;\operatorname{tr}_{\mathcal{A}} Q^j/\|Q\|^j$$

for each j. Consequently

$$\frac{\sum_{j=1}^k \binom{k}{j}\operatorname{tr}_{\mathcal{A}'} S^j/\|S\|^j}{\sum_{j=1}^k \binom{k}{j}\operatorname{tr}_{\mathcal{A}} Q^j/\|Q\|^j} = \frac{1}{\alpha\beta}\;,$$

which is then also the value of the limit as $k \to \infty$, which equals the quotient of traces.

To complete the argument, we show next that for any $g \in L^2$ the traces $\operatorname{tr}_{\mathcal{A}'} P$ and $\operatorname{tr}_{\mathcal{A}} P'$ can be approximated arbitrarily well by corresponding traces of projections P_h and P'_h onto subspaces $\mathcal{A}h$ and $\mathcal{A}'h$, with some h for which $T_{h;\alpha,\beta}$ is bounded. For by (6.6), to approximate the traces it is sufficient to approximate $P\chi$ and $P'\chi$ for a finite number of suitable characteristic functions χ. Now $P\chi$ is the closest element to χ from $\mathcal{Y}$, hence is approximable by $A_0 g$, with A_0 some linear combination of the generators $W(j\alpha, k\beta)$ of $\mathcal{A}$, and $A_0 g$ is in turn approximable by $A_0 h$ with any $h \in L^2$ sufficiently close to g. We conclude that, given $\epsilon > 0$, there exists δ such that if $\|g-h\| < \delta$, then $\overline{\mathcal{A}h}$, the closure of the subspace $\mathcal{A}h$, is not more than ϵ further from χ than is $\mathcal{Y}$. If, moreover, $h \in \mathcal{Y}$, then $\overline{\mathcal{A}h} \subset \mathcal{Y}$ and so cannot be nearer to χ than is $\mathcal{Y}$. Thus $\|P_h\chi\| \le \|P\chi\| \le \|P_h\chi\| + \epsilon$. By choosing δ sufficiently small, we can similarly approximate each $P\chi$ that figures in the trace, thereby approximating $\operatorname{tr}_{\mathcal{A}'} P$ arbitrarily well by $\operatorname{tr}_{\mathcal{A}'} P_h$. If $h \in \mathcal{Y}'$, the same argument applies to $\operatorname{tr}_{\mathcal{A}} P'$.

Let u be a smooth function of compact support with $\|g-u\|_{L^2} < \delta$; the operator $T_{u;1/\beta,1/\alpha}$ is then always bounded. Let $h = P(P'u)$, which exhibits that $h \in \mathcal{Y}$; and since P and P' commute, then also $h = P'(Pu) \in \mathcal{Y}'$. Moreover, since $g \in \mathcal{Y} \cap \mathcal{Y}'$, $g = PP'g$, and

$$\|g-h\| = \|PP'(g-u)\| \le \|g-u\| < \delta\;.$$

It remains to show that $T_{h;\alpha,\beta}$ is bounded. Invoking (6.14) and using the fact that $P \in \mathcal{A}'$, $P' \in \mathcal{A}$ in (6.12), we have

$$\infty > \|T_{u;1/\beta,1/\alpha}\| \geq \|T_{u;1/\beta,1/\alpha}\ P\| = \|T_{Pu;1/\beta,1/\alpha}\|$$

$$= \sqrt{\alpha\beta}\ \|T_{Pu;\alpha,\beta}\| \geq \sqrt{\alpha\beta}\ \|T_{Pu;\alpha,\beta}\ P'\| = \sqrt{\alpha\beta}\ \|T_{h;\alpha\beta}\|\ ,$$

showing that $T_{h;\alpha,\beta}$ is bounded. This completes the proof of Theorem 6.2. □

Appendix 6.1

Here we show that $\mathcal{A}'$ is generated by $W(\frac{m}{\beta}, \frac{n}{\alpha})$. $\mathcal{A}$ contains the algebra generated by $W(j\alpha, 0)$, which includes multiplication by all bounded functions of period $1/\alpha$; an operator $T \in \mathcal{A}'$ then commutes with multiplication by such functions. Choosing as an instance $\sum_k \chi_\tau(t + \frac{k}{\alpha})$, with $\chi_\tau(t)$ the characteristic function of an arbitrarily small interval centered at τ, shows that for smooth functions $f(t)$ of compact support, the value of Tf at a point $t = \tau$ is a bounded linear functional of $\{f(\tau - \frac{k}{\alpha})\}$, $-\infty < k < \infty$. Thus

$$Tf = \sum_k m_k(t) f\left(t - \frac{k}{\alpha}\right)\ , \tag{A1.1}$$

and $\sum_k |m_k(t)|^2$ is bounded uniformly in t since T is bounded in L^2. Now the commutativity of T and $W(0, \beta)$ means that $[Tf](t + \beta) = T[f(t + \beta)]$; whence by (A1.1) each $m_k(t)$ has period β. Analogously, letting $\mathcal{C}$ denote the von Neumann algebra generated by $W(\frac{m}{\beta}, \frac{n}{\alpha})$, each $S \in \mathcal{C}'$ has the form

$$Sf = \sum_j n_j(t) f(t - j\beta)\ , \tag{A1.2}$$

with $n_j(t)$ of period $\frac{1}{\alpha}$, and $\sum |n_j(t)|^2$ uniformly bounded in t.

We want to show that S and T commute on a dense set in $L^2(\mathbb{R})$. To that end, suppose that the support of f is smaller than $\min[\beta, \frac{1}{\alpha}]$, whereupon the components in (A1.1) or (A1.2) do not overlap. Further, let $n_j^{(M)}$ be smooth, have period $\frac{1}{\alpha}$, and approximate n_j in $L^2[0, \frac{1}{\alpha}]$ well enough so that

$$\lim_{M\to\infty} \sum_j \|n_j - n_j^{(M)}\|^2_{L^2[0,1/\alpha]} = 0\ .$$

Then

$$Sf = \lim_{M,N\to\infty} \sum_{|j|\leq N} n_j^{(M)}(t) f(t - j\beta)$$

in $L^2(\mathbb{R})$, with the functions on the right smooth and of compact support and, hence, in the domain of (A1.1). Since T is bounded,

$$\begin{aligned} TSf &= \lim_{M,N\to\infty} \sum_{|j|\le N} Tn_j^{(M)}(t) f(t - j\beta) \\ &= \lim_{M,N\to\infty} \sum_{|j|\le N} \sum_k m_k(t) n_j^{(M)}\left(t - \frac{k}{\alpha}\right) f\left(t - j\beta - \frac{k}{\alpha}\right) \\ &= \sum_j \sum_k m_k(t) n_j(t) f\left(t - j\beta - \frac{k}{\alpha}\right) , \end{aligned}$$

the sum here likewise consisting of nonoverlapping components. By the analogous argument with the roles of S and T reversed, STf has the same value. Thus $TS = ST$ on the space of all linear combinations of smooth functions of small support, which is dense in $L^2(R)$; whence the operators, being bounded, commute everywhere. We conclude that $\mathcal{C}'$ commutes with $\mathcal{A}'$, that is, $\mathcal{C}' \subset (\mathcal{A}')'$. However, $\mathcal{C} \subset \mathcal{A}'$; whence $(\mathcal{A}')' \subset \mathcal{C}'$. Thus by the double commutant theorem

$$\mathcal{C}' = (\mathcal{A}')' = \mathcal{A}$$

or equivalently

$$\mathcal{C} = \mathcal{A}' ,$$

as was to be shown.

Appendix 6.2

The G–N–S construction realizes $\mathcal{A}'$ on a Hilbert space $L^2(\mathcal{A}')$ that is naturally identifiable with ℓ^2; moreover, the operators of $\mathcal{A}'$ act simply on $L^2(\mathcal{A}')$ and can be carried easily to their counterparts on ℓ^2. Specifically, let V denote the unitary map from an element in $L^2(\mathcal{A}')$ to its coefficients in the basis (6.10). We have seen in (6.13) that

$$V T^*_{f;\alpha,\beta} T_{g;\alpha,\beta} = (\alpha\beta)^{-1} T_{g;1/\beta,1/\alpha} f \, . \tag{A2.1}$$

We can use (A2.1) to carry an operator on $L^2(\mathcal{A}')$ into its correspondent on ℓ^2. To this end, we note that if $X \in L^2(\mathcal{A}')$ has the expansion $\sum \gamma_{j,k} \, W(\frac{j}{\beta}, \frac{k}{\alpha})$, then X^* corresponds to $\sum \overline{\gamma}_{j,k} \, W^*(\frac{j}{\beta}, \frac{k}{\alpha}) = \sum \overline{\gamma_{j,k}} \, e^{2\pi i \, jk/\alpha\beta} \, W(-\frac{j}{\alpha}, -\frac{k}{\beta})$. Thus

$$V \, X^* = U \, V \, X \, , \tag{A2.2}$$

with

$$U\gamma_{j,k} \equiv \overline{\gamma_{-j,-k}\, e^{-2\pi i\, jk/\alpha\beta}}\ ;$$

hence

$$U^* = U^{-1} = U\ .$$

Moreover, as an identity of sequences, and regardless of their square-summability,

$$U\,T_{p;1/\beta,1/\alpha}\,q = U\left\{\left\langle q,\ W\left(\frac{j}{\beta}\,,\ \frac{k}{\alpha}\right)\ p\right\rangle\right\} = \left\{\overline{\left\langle q,\ W\left(-\frac{j}{\beta},-\frac{k}{\alpha}\right)\ p\right\rangle\, e^{-2\pi i\, jk/\alpha\beta}}\right\}$$

$$= \left\{\left\langle p,\ W\left(\frac{j}{\beta}\,,\ \frac{k}{\alpha}\right)\ q\right\rangle\right\} = T_{q;1/\beta,1/\alpha}\ p\ . \tag{A2.3}$$

(U is the R of (3.2) combined with the complex conjugation there.) Thus if $X \in \mathcal{A}'$ is a linear combination of a finite number of $W(\frac{j}{\beta}, \frac{k}{\alpha})$, then by (6.12), (A2.1), (A2.3), and the identity $Xg \equiv T^*_{g;1/\beta,1/\alpha}\,V\,X$,

$$V\ T^*_{f;\alpha,\beta}\ T_{g;\alpha,\beta}X = V\ T^*_{f;\alpha,\beta}\ T_{X^*g;\alpha,\beta} = (\alpha\beta)^{-1}\ T_{X^*g;1/\beta,1/\alpha}\,f$$

$$= (\alpha\beta)^{-1}\ U\,T_{f;1/\beta,1/\alpha}X^*g$$

$$= (\alpha\beta)^{-1}\ U\,T_{f;1/\beta,1/\alpha}\ T^*_{g;1/\beta,1/\alpha}V\ X^*$$

$$= (\alpha\beta)^{-1}\ U\,T_{f;1/\beta,1/\alpha}\ T^*_{g;1/\beta,1/\alpha}U\,V\ X\ . \tag{A2.4}$$

Since $T^*_{f;\alpha,\beta}\ T_{g;\alpha,\beta}$ is bounded (on L^2, hence by the G–N–S correspondence also on $L^2(\mathcal{A}')$), (A2.4) can be extended by continuity to all $X \in L^2(\mathcal{A}')$; whence

$$T^*_{f;\alpha,\beta}\ T_{g;\alpha,\beta} = V^*U(\alpha\beta)^{-1}\ T_{f;1/\beta,1/\alpha}\ T^*_{g;1/\beta,1/\alpha}U\,V \quad \text{on}\ \ L^2(\mathcal{A}')\ , \tag{A2.5}$$

$$U\,V\ T^*_{f;\alpha,\beta}\ T_{g;\alpha,\beta}V^*U = (\alpha\beta)^{-1}\ T_{f;1/\beta,1/\alpha}\ T^*_{g;1/\beta,1/\alpha} \quad \text{on}\ \ \ell^2\ .$$

Thus $T^*_{f;\alpha,\beta}\ T_{g;\alpha,\beta}$, viewed as acting on $L^2(\mathcal{A}')$, is unitarily equivalent to $(\alpha\beta)^{-1}\ T_{f;1/\beta,1/\alpha}\ T^*_{g;1/\beta,1/\alpha}$ on ℓ^2, except for a complex conjugation, U being conjugate-linear.

Appendix 6.3

Here we prove that the convergence of $\sum \gamma_{j,k}\, W(\frac{j}{\beta}, \frac{k}{\alpha})$ to $T^*_{f;\alpha,\beta}\, T_{g;\alpha,\beta}$ in the norm of $L^2(\mathcal{A}')$, asserted by (6.10), implies that, for $h \in L^2$, $\sum \gamma_{j,k}\, W(\frac{j}{\beta}, \frac{k}{\alpha})\, h$ converges to $T^*_{f;\alpha,\beta}\, T_{g;\alpha,\beta}\, h$ in L^2. We begin by showing that $\sum \gamma_{j,k}\, W(\frac{j}{\beta}, \frac{k}{\alpha})$ converges strongly on functions u of compact support. For since $T^*_{f;\alpha,\beta} T_{g;\alpha,\beta} \in \mathcal{A}'$, there exist linear combinations of the generators of $\mathcal{A}'$,

$$S_n \equiv \sum_{|j,k|\leq n} a^{(n)}_{j,k}\, W\left(\frac{j}{\beta}\,,\ \frac{k}{\alpha}\right)$$

which converge to it strongly and, hence, as we have seen, also in $L^2(\mathcal{A}')$. Consequently,

$$\sum |a^{(n)}_{j,k} - \gamma_{j,k}|^2 = \|S_n - T^*_{f;\alpha,\beta} T_{g;\alpha,\beta}\|^2_{L^2(\mathcal{A}')} = \epsilon_n^2 \to 0\ , \tag{A3.1}$$

with $a^{(n)}_{j,k} = 0$ outside $|j,k| \leq n$. Set $T_n = \sum_{|j,k|\leq n} \gamma_{j,k}\, W(\frac{j}{\beta}, \frac{k}{\alpha})$. Then

$$S_n u = T_n u + \sum_{|j,k|\leq n} (a^{(n)}_{j,k} - \gamma_{j,k})\, W\left(\frac{j}{\beta}\,,\ \frac{k}{\alpha}\right) u = T_n u + T^*_{u;1/\beta,1/\alpha}\{a^{(n)}_{j,k} - \gamma_{j,k}\}_{|j,k|\leq n}\ . \tag{A3.2}$$

As u is compactly supported, $T^*_{u;1/\beta,1/\alpha}$ is bounded by Proposition 2.1, and since by (A3.1)

$$\sum_{|j,k|\leq n} |a^{(n)}_{j,k} - \gamma_{j,k}|^2 \leq \sum |a^{(n)}_{j,k} - \gamma_{j,k}|^2 = \epsilon_n^2\ ,$$

the last term in (A3.2) converges to 0. Thus from (A3.2)

$$\lim_{n\to\infty} T_n u = \lim_{n\to\infty} S_n u = T^*_{f;\alpha,\beta} T_{g;\alpha,\beta} u\ ,$$

the last equality by strong convergence of S_n. By (6.13), we recognize the left-hand limit to be $(\alpha\beta)^{-1} T^*_{u;1/\beta,1/\alpha} T_{g;1/\beta,1/\alpha} f$, and so find that, as elements of L^2,

$$T^*_{f;\alpha,\beta}\ T_{g;\alpha,\beta}\, u = (\alpha\beta)^{-1} T^*_{u;1/\beta,1/\alpha}\ T_{g;1/\beta,1/\alpha}\ f. \tag{A3.3}$$

For arbitrary $h \in L^2$, let $u_n \to h$ in L^2, with each u_n of compact support. Applying (A3.3), since $T^*_{f;\alpha,\beta} T_{g;\alpha,\beta}$ is bounded,

$$T^*_{f;\alpha,\beta} T_{g;\alpha,\beta} h = \lim_{n\to\infty} (\alpha\beta)^{-1} T^*_{u_n;1/\beta,1/\alpha} T_{g;1/\beta,1/\alpha} f\ . \tag{A3.4}$$

We now observe that the right-hand operator is closed as a map of u_n. For if $\gamma \in \ell^2$ is a fixed sequence, ρ, σ are constants, and $T^*_{u_n;\rho,\sigma}\gamma \to q \in L^2$, then with φ a function of compact support,

$$(q, \varphi) = \lim_{n\to\infty}(T^*_{u_n;\rho,\sigma}\gamma, \varphi) = \lim_{n\to\infty}(\gamma, T_{u_n;\rho,\sigma}\varphi) . \tag{A3.5}$$

Again by Proposition 2.1, $T_{\bar{\varphi};\rho,\sigma}$ is bounded, and therefore $T_{\overline{\varphi};\rho,\sigma}\overline{u}_n \to T_{\overline{\varphi};\rho\sigma}\overline{h}$. But since $T_{\overline{\varphi};\rho,\sigma}\overline{u}_n$ and $T_{u_n;\rho,\sigma}\varphi$ are unitarily equivalent, there is corresponding convergence in (A3.5). Consequently,

$$(q, \varphi) = (\gamma, T_{h;\rho,\sigma}\varphi) ;$$

whence γ is in the domain of $T^*_{h;\rho,\sigma}$, with $q = T^*_{h;\rho,\sigma}\gamma$. From (A3.4) we thus obtain

$$T^*_{f;\alpha,\beta}T_{g;\alpha,\beta}h = (\alpha\beta)^{-1}T^*_{h;1/\beta,1/\alpha}T_{g;1/\beta,1/\alpha}f ,$$

as was to be shown.

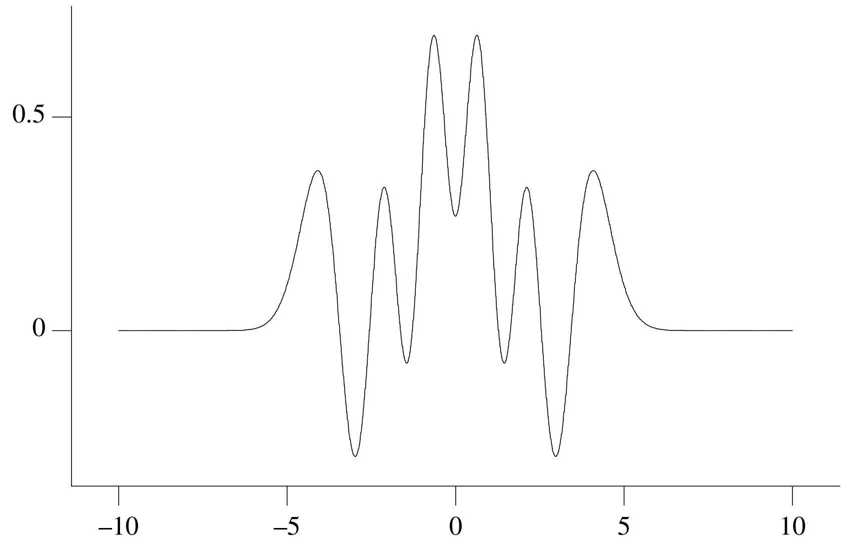
7. Dual Functions That Optimize Other Norms

As remarked by Wexler and Raz in [5], there are cases when other functions than the minimal L^2-dual function $g^\# = \tilde{g}$ for the frame $(g_{m\alpha,n\beta})_{m,n\in\mathbb{Z}}$ lead to better concentration and/or convergence. This can easily be illustrated by the following example. If one takes the limit in which both α and β tend to zero, then the discrete family $g_{m\alpha,n\beta}$ becomes the family $g_{p,q} = W(p,q)g$, where p, q range continuously over $\mathbb{R}$. It is then well known (see, for example, [10]; a short review is also in [3]) that, for any f, h in $L^2(\mathbb{R})$,

$$\iint_{\mathbb{R}^2} \langle f, h_{p,q}\rangle g_{p,q}\, dp\, dq = \langle g, h\rangle f .$$

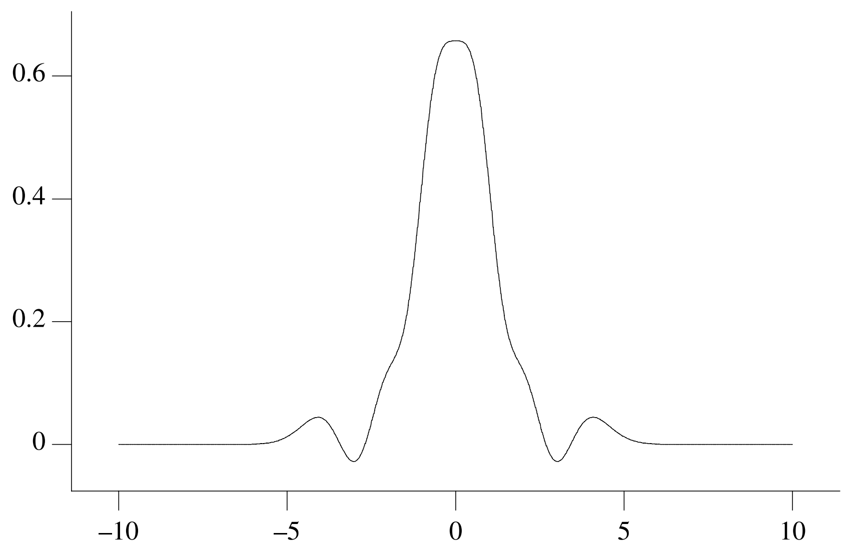
This is exactly of the same form as the expansions in the $g_{m\alpha,n\beta}$ we saw previously, with an integral over p, q replacing the sum over m, n. It follows that all h in $L^2(\mathbb{R})$ for which $\langle g, h\rangle = 1$ are *dual functions* for the $g_{p,q}$ in the sense that the above integral reproduces each f exactly. Among these, the particular choice $h = \|g\|^{-2}g$ is the one with minimal L^2-norm. One can easily imagine, however, if either g itself or its Fourier transform $\hat{g}$ is rather spread out, that other, more concentrated choices of h might be preferable. For instance, if $g = \frac{1}{\sqrt{2}}H_0 + \frac{1}{\sqrt{2}}H_k$, where H_n denotes the nth Hermite function, then the dual function with minimal L^2-norm is g itself, but the dual function G that minimizes $|||G|||^2 = \int |xG(x)|^2\, dx + \int |\xi\hat{G}(\xi)|^2 d\xi$ is different and more concentrated in both x and ξ. Writing $G = \sum_n \mu_n H_n$, one finds $|||G|||^2 = \sum_{n=0}^{\infty}(n+1)|\mu_n|^2$. Minimizing this under the constraint $\mu_0 + \mu_k = \sqrt{2}$ leads to $\mu_n = 0$ if $n \neq 0$, $n \neq k$ and to $\mu_0 = \sqrt{2}\,\frac{k+1}{k+2}$, $\mu_k = \sqrt{2}\,\frac{1}{k+2}$. Figure 1 shows a graph of both g and G for $k = 10$; G is clearly more concentrated and less oscillating than g. A different example, still in the limit case of vanishing α, β, is given by the choice

(a)



$\frac{1}{\sqrt{2}}\, H_0 + \frac{1}{\sqrt{2}}\, H_{10}$

(b)



$\sqrt{2}\left(\frac{11}{12}\, H_0 + \frac{1}{12}\, H_{10}\right)$

FIGURE 1

Graphs of (a) $g(x) = \frac{1}{\sqrt{2}}[H_0(x) + H_{10}(x)]$; (b) $G(x)$, the function dual to g for which $\int |xG(x)|^2\, dx + \int |\xi \hat{G}(\xi)|^2\, d\xi$ is minimal.

$g(x) = \pi^{-1}e^{-x}$ for $x \geq 0$, $g(x) = 0$ for $x < 0$, that is, the one-sided exponential. This function has a discontinuity, which leads to extensive spreading in the frequency domain. One can find dual functions that are much smoother; the dual function G that minimizes $\int (1 + |\xi|^2)|\hat{G}(\xi)|^2\, d\xi$, for instance, is given by $G(x) = \frac{1}{2}\left[e^{-|x|} + x(1 + \text{sign}(x))e^{-x}\right]$; both g and G are plotted in Figure 2.

A similar thing happens for the discrete families $g_{m\alpha,n\beta}$, although the analysis is a little less straightforward. Assume that the $g_{m\alpha,n\beta}$ constitute a frame and that we are interested in finding the dual function g^b for which $|||g^b||| = \|\Lambda g^b\|$ is minimal. (In the examples above, $\Lambda = (-d^2/dx^2 + x^2)^{1/2}$ and $\Lambda = (-d^2/dx^2 + 1)^{1/2}$, respectively.) Because of Proposition 3.8, this is equivalent to finding the function G with smallest L^2-norm that satisfies

$$T_{g;1/\beta,1/\alpha}\Lambda^{-1}G = \alpha\beta\; e_{0,0} \tag{7.1}$$

and then taking $g^b = \Lambda^{-1}G$. Note that we are implicitly assuming that Λ has a bounded inverse Λ^{-1}. This is not a severe restriction: we can assume $\Lambda \geq 0$ without loss of generality (otherwise, replace Λ by $(\Lambda^*\Lambda)^{1/2}$), and we can then add 1 to Λ, if necessary, without changing the nature of the smoothness or decay constraint. We shall systematically assume $\Lambda \geq \text{Id}$ in what follows. Finding the minimal L^2-solution G to (7.1) amounts to replacing $T_{g;1/\beta,1/\alpha}$ by $T_{g;1/\beta,1/\alpha}\Lambda^{-1}$ in the argument that led to the minimal norm solution $g^\#$ in §4. By this argument, we are therefore led to the choice

$$\alpha\beta\;\Lambda^{-1}\;T^*_{g;1/\beta,1/\alpha}\;(T_{g;1/\beta,1/\alpha}\;\Lambda^{-2}T^*_{g;1/\beta,1/\alpha})^{-1}\;e_{0,0}$$

for G. It is, however, not clear in what sense this should be understood, since the operator $T_{g;1/\beta,1/\alpha}\;\Lambda^{-2}T^*_{g;1/\beta,1/\alpha}$, although well defined and bounded on $\ell^2(\mathbb{Z}^2)$, is generally not invertible even if $T_{g;1/\beta,1/\alpha}\;T^*_{g;1/\beta,1/\alpha}$ is. The solution is to introduce an extra operator Ω on $\ell^2(\mathbb{Z}^2)$, typically also unbounded, which, in a sense that will become more precise below, complements on $\ell^2(\mathbb{Z}^2)$ the action of Λ on $L^2(\mathbb{R})$. Concretely, we have

7.1. Proposition

Assume that the $g_{m\alpha,n\beta}$ constitute a frame. Let Λ, Ω be (unbounded) operators on $L^2(\mathbb{R})$, $\ell^2(\mathbb{Z}^2)$, such that $\Lambda, \Omega \geq \text{Id}$ on their domains. Suppose that $e_{0,0}$ lies in the domain of Ω and that there exist $0 < A' \leq B' < \infty$ so that, for all c in $\ell^2(\mathbb{Z}^2)$,

$$A'\|\Omega^{-1}\;c\|^2 \leq \|\Lambda^{-1}\;T^*_{g;1/\beta,1/\alpha}\;c\|^2 \leq B'\|\Omega^{-1}\;c\|^2\;. \tag{7.2}$$

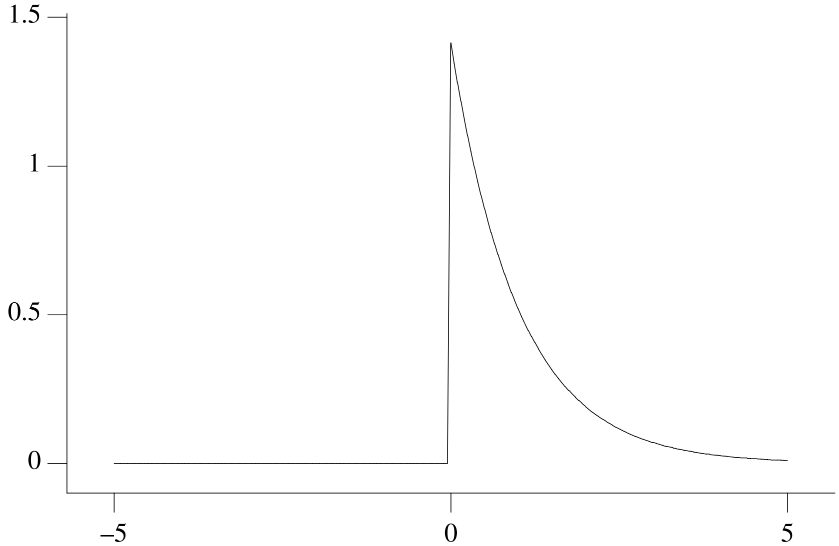
Then the function $\tilde{g}_\Lambda$ defined by

$$\tilde{g}_\Lambda = \alpha\beta\;\Lambda^{-2}\;T^*_{g;1/\beta,1/\alpha}\Omega\left(\Omega T_{g;1/\beta,1/\alpha}\;\Lambda^{-2}\;T^*_{g;1/\beta,1/\alpha}\Omega\right)^{-1}\Omega\;e_{0,0} \tag{7.3}$$

is a dual function for the frame $g_{m\alpha,n\beta}$; moreover, for any other dual function g^b, we have

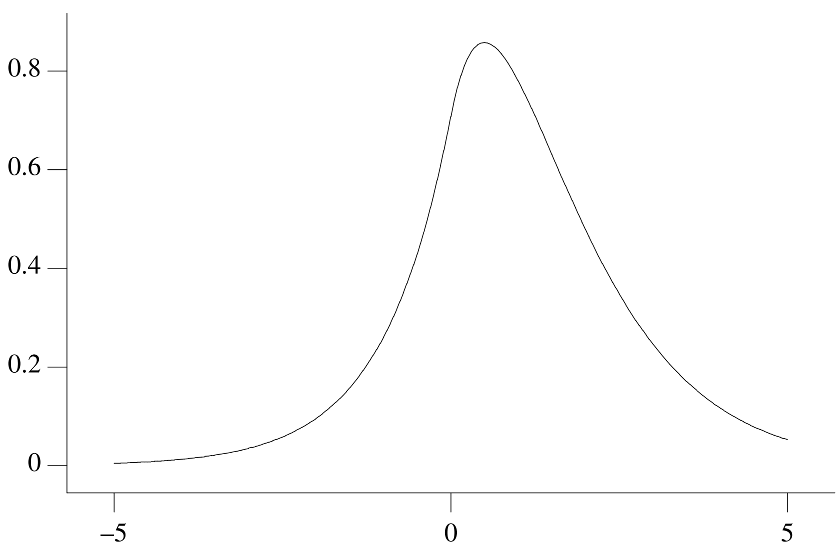
$$\|\Lambda\tilde{g}_\Lambda\| \leq \|\Lambda g^b\|\;.$$

(a)



$$\begin{cases} e^{-x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

(b)



$$\begin{cases} \frac{2}{e^{-|x|}} & x \leq 0 \\ \frac{2}{e^{-x}} + xe^{-x} & x \geq 0 \end{cases}$$

FIGURE 2

Graphs of (a) $g(x) = e^{-x}\chi_{[0,\infty)}(x)$; (b) $G(x)$, the function dual to g for which $\int [1 + |\xi|^2]\, |\hat{G}(\xi)|^2\, d\xi$ is minimal.

Proof. First of all, note that $\tilde{g}_\Lambda$ is well defined, since $\Omega T_{g;1/\beta,1/\alpha} \Lambda^{-2} T^*_{g;1/\beta,1/\alpha} \Omega$ is invertible by (7.2). Next, $T_{g;1/\beta,1/\alpha}\ \tilde{g}_\Lambda$ is obviously in the domain of Ω, and $\Omega T_{g;1/\beta,1/\alpha}\ \tilde{g}_\Lambda = \alpha\beta\Omega\ e_{0,0}$. Since Ω is invertible, it follows that $T_{g;1/\beta,1/\alpha}\ \tilde{g}_\Lambda = \alpha\beta\ e_{0,0}$, so that $\tilde{g}_\Lambda$ is a dual function for the frame $g_{m\alpha,n\beta}$. Finally, among all the functions that satisfy

$$\Omega T_{g;1/\beta,1/\alpha}\ \Lambda^{-1} \Lambda g^b = \alpha\beta\Omega\ e_{0,0}\ ,$$

the one for which $\|\Lambda g^b\|$ is minimal is given by projection onto [Ker $(\Omega T_{g;1/\beta,1/\alpha}\ \Lambda^{-1})]^\perp$, which leads to (7.3). □

This reduces the problem to identifying, for a given Λ, the appropriate operator Ω and proving bounds of type (7.2). We shall show here how this can be done for the two special choices $\Lambda = (1+x^2)^p$ and $\Lambda = (-d^2/dx^2+1)^q$, which correspond to trying to find more localized g^b or smoother g^b, respectively. In fact, we only need to analyze the first case since the second can be obtained from the first by Fourier transform. Let us assume, therefore, that Λ is a simple multiplication operator, $(\Lambda f)(x) = \lambda(x) f(x)$. We shall likewise restrict our attention to operators Ω of the form $(\Omega c)_{m,n} = \omega_n c_{m,n}$. The following proposition leads to estimates for A', B'.

7.2. Proposition

Let Λ, Ω be as above. Define

$$S = \sup_{n\in\mathbb{Z}, x\in\mathbb{R}} \sum_{\ell\in\mathbb{Z}} \omega_n^2 \lambda(x+\ell\beta)^{-2} \left| g\left(x+\ell\beta-\frac{n}{\alpha}\right)\right|^2 ,$$

$$I = \inf_{n\in\mathbb{Z}, x\in\mathbb{R}} \sum_{\ell\in\mathbb{Z}} \omega_n^2 \lambda(x+\ell\beta)^{-2} \left| g\left(x+\ell\beta-\frac{n}{\alpha}\right)\right|^2 ,$$

$$R = \sup_{n\in\mathbb{Z}, x\in\mathbb{R}} \sum_{\substack{k\in\mathbb{Z}\\ k\neq 0}} \omega_n\omega_{n+k} \left| \sum_\ell g(x+\ell\beta)\ \overline{g\left(x+\ell\beta-\frac{k}{\alpha}\right)}\ \lambda\left(x+\frac{n}{\alpha}+\ell\beta\right)^{-2} \right| .$$

If $S, R < \infty$ and $I > R$, then (7.2) *holds with $A' = (I-R)\beta$ and $B' = (S+R)\beta$.*

Proof. Renaming $\Omega^{-1}c = d$, we need to derive bounds on

$$\int_{-\infty}^{\infty} \lambda(x)^{-2} \left| \sum_{m,n} \omega_n d_{m,n}\ g_{m/\beta\ ,\ n/\alpha}(x) \right|^2 dx\ . \tag{7.4}$$

Define $d_n(x) = \beta^{-1/2} \sum_{m\in\mathbb{Z}} d_{m,n} \exp\left[-2\pi i \frac{m}{\beta}\ x\right]$; we have

$$\int_0^\beta |d_n(x)|^2\, dx = \sum_{m,n}\ |d_{m,n}|^2\ . \tag{7.5}$$

It follows that

$$
\begin{aligned}
(7.4) &= \int_{-\infty}^{\infty} \beta\lambda(x)^{-2} \left|\sum_n d_n(x)\omega_n g\left(x-\frac{n}{\alpha}\right)\right|^2 dx \\
&= \beta \sum_{\ell\in\mathbb{Z}} \int_0^{\beta} \left|\sum_n \omega_n d_n(x) g\left(x+\ell\beta-\frac{n}{\alpha}\right)\lambda(x+\ell\beta)^{-1}\right|^2 dx \\
&= \beta \sum_{\ell}\sum_{n,m} \omega_n\omega_m \int_0^{\beta} d_n(x)\,\overline{d_m(x)}\, g\left(x+\ell\beta-\frac{n}{\alpha}\right)\,\overline{g\left(x+\ell\beta-\frac{m}{\alpha}\right)}\,\lambda(x+\ell\beta)^{-2}\,dx \\
&= \beta\left\{\sum_n \int_0^{\beta} |d_n(x)|^2 \left[\sum_{\ell} \omega_n^2 \left|g\left(x+\ell\beta-\frac{n}{\alpha}\right)\right|^2 \lambda(x+\ell\beta)^{-2}\right] dx + \text{Rest}\right\} \qquad (7.6)
\end{aligned}
$$

with

$$
\begin{aligned}
|\text{Rest}| &\le \sum_{\substack{n,m\\ n\neq m}} \omega_n\omega_m \int_0^{\beta} |d_n(x)|\,|d_m(x)| \;\times \\
&\qquad \left|\sum_{\ell} g\left(x+\ell\beta-\frac{n}{\alpha}\right)\,\overline{g\left(x+\ell\beta-\frac{m}{\alpha}\right)}\,\lambda^{-2}(x+\ell\beta)\right| dx \\
&\le \left(\sum_{\substack{n,m\\ n\neq m}} \omega_n\omega_m \int_0^{\beta} |d_n(x)|^2 \left|\sum_{\ell} g\left(x+\ell\beta-\frac{n}{\alpha}\right)\,\overline{g\left(x+\ell\beta-\frac{m}{\alpha}\right)}\,\lambda^{-2}(x+\ell\beta)\right| dx\right)^{1/2} \times \\
&\qquad \left(\sum_{\substack{n,m\\ n\neq m}} \omega_n\omega_m \int_0^{\beta} |d_m(x)|^2 \left|\sum_{\ell} g\left(x+\ell\beta-\frac{n}{\alpha}\right)\,\overline{g\left(x+\ell\beta-\frac{m}{\alpha}\right)}\,\lambda^{-2}(x+\ell\beta)\right| dx\right)^{1/2} \\
&= \sum_{\substack{n,m\\ n\neq m}} \omega_n\omega_m \int_0^{\beta} |d_n(x)|^2 \left|\sum_{\ell} g\left(x+\ell\beta-\frac{n}{\alpha}\right)\,\overline{g\left(x+\ell\beta-\frac{m}{\alpha}\right)}\,\lambda^{-2}(x+\ell\beta)\right| dx \\
&= \sum_n \int_0^{\beta} |d_n(x)|^2 \sum_{k\neq 0} \omega_n\omega_{n+k} \left|\sum_{\ell} g\left(x+\ell\beta-\frac{n}{\alpha}\right)\,\overline{g\left(x+\ell\beta-\frac{n}{\alpha}-\frac{k}{\alpha}\right)}\,\lambda^{-2}(x+\ell\beta)\right| dx \\
&= \sum_n \int_0^{\beta} |d_n(x)|^2 \sum_{k\neq 0} \omega_n\omega_{n+k} \left|\sum_{\ell} g(x+\ell\beta)\,\overline{g\left(x+\ell\beta-\frac{k}{\alpha}\right)}\,\lambda\left(x+\ell\beta+\frac{n}{\alpha}\right)^{-2}\right| dx\,.
\end{aligned}
$$

(7.7)

Combining (7.5), (7.6), and (7.7) immediately leads to

$$\beta(I-R)\sum_{m,n}|d_{m,n}|^2 \le (7.3) \le \beta(S+R)\sum_{m,n}|d_{m,n}|^2 . \qquad \square$$

We have tried out this approach on a function h similar to the one-sided exponential of Figure 2a in order to find a dual function H with better frequency localization than h itself. The Fourier transform of the one-sided exponential is $(1+i\omega)^{-1}$; in order to regularize its behavior at ∞, we multiply this with a very wide Gaussian, defining

$$\hat{h}(\omega) = (1+i\omega)^{-1}\exp\left(-\frac{\omega^2}{50}\right) \quad \text{or} \quad h(x) = (5e^{-25x^2/2} * e)(x) ,$$

where $e(x) = \exp(-x)\chi_{[0,\infty)}$. (In this section, we shall systematically normalize the Fourier transform as $\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}}\int e^{-i\xi x} f(x)\,dx$.) We wish to consider the functions $h_{m\alpha,n\beta}(x) = e^{-2\pi max}$ $h(x-nb)$ and to find a dual function H with better frequency concentration. Translating all this into the Fourier domain and writing $g(x) = \hat{h}(x)$, we find that the lattice of interest is given by $g_{m\alpha,n\beta}(x)$, with $\beta = 2\pi a$, $\alpha = \frac{b}{2\pi}$, $g(x) = (1+ix)^{-1}\exp(-x^2/50)$. We choose $\lambda(x) = (1+x^2)^{1/4}$, and we define the ω_n by

$$\omega_n^2 = \left[\inf_{x\in\mathbb{R}} \sum_{\ell\in\mathbb{Z}} \lambda(x+\ell\beta)^{-2}\left|g\left(x+\ell\beta-\frac{n}{\alpha}\right)\right|^2\right]^{-1} .$$

It follows that $I = 1$ (where I is defined as in Proposition 7.2). For $a = b = 0.25$, numerical computation led to $S \simeq 1.0803$ and $R \simeq .0397$, so $\frac{B'}{A'} \simeq 1.16$. To determine $\tilde{g}_\Lambda$, we then follow the prescription (7.3). First, note that $\Omega\, e_{0,0} = \omega_0\, e_{0,0}$. Next, we compute the matrix for $L = \mathrm{Id} - \frac{2}{A'+B'}\,\Omega T_{g;1/\beta,1/\alpha}\,\Lambda^{-2}\,T^*_{g;1/\beta,1/\alpha}\,\Omega$:

$$(Lc)_{m,n} = \sum_{k,\ell} L_{m,n;k,\ell}\, c_{k,\ell}$$

with

$$L_{m,n;k,\ell} = \delta_{m,k}\delta_{n,\ell} - \frac{2}{A'+B'}\,\omega_n\omega_\ell\left\langle \Lambda^{-2} g_{\frac{k}{\beta},\ell/\alpha}\ ,\ g_{m/\beta\ ,\ n/\alpha}\right\rangle .$$

In practice, for our example, very few of these matrix elements are significant, except on or near the diagonal $m = k$, $n = \ell$. This makes it easy to compute the iterates $L^n e_{0,0}$; we then have

$$(\Omega T_{g;1/\beta,1/\alpha}\,\Lambda^{-2}\,T^*_{g;1/\beta,1/\alpha}\Omega)^{-1}\,\Omega e_{0,0} = \omega_0 d = \omega_0 \lim_{N\to\infty} d_N$$

where $d_0 = \frac{2}{A'+B'}\, e_{0,0}$ and d_N is defined recursively by $d_N = d_0 + Ld_{N-1}$. As in [3], the limit converges exponentially fast; the error made by truncating at step N is of the order $O\left[\left(\frac{B'}{A'} - 1\right)^N\right]$. Since $\frac{B'}{A'} - 1 = 0.16$ in our case, a few iterations suffice to obtain good accuracy.

Next, we use the $d_{m,n}$ to define the function

$$\alpha\beta\left[T^*_{g;1/\beta,1/\alpha}\Omega d\right](x) = \alpha\beta\sum_{m,n}\omega_n d_{m,n}\; g_{m/\beta,n/\alpha}(x)\ .$$

Finally, we multiply this by $(1+x^2)^{-1/2} = \lambda(x)^{-2}$ to obtain $\tilde{g}_\Lambda$. The inverse Fourier transform of $\tilde{g}_\Lambda$ is then the function H dual to the $h_{ma,nb}$ (with $a = b = 0.25$) that minimizes $\langle(1-\Delta)^{1/2}H, H\rangle$. Rather than computing this inverse Fourier transform numerically from $\tilde{g}_\Lambda$, we can use the integral representation

$$(1+\gamma^2)^{-1/2} = \frac{2}{\sqrt{\pi}}\int_0^\infty e^{-(1+\gamma^2)s^2}\, ds$$

to write

$$H(t) = \left[\left(1 - \frac{d^2}{dt^2}\right)^{-1/2} F\right](t)$$

$$= \frac{1}{\sqrt{\pi}}\int_0^\infty \frac{1}{s}\, e^{-1/s^2}\int_{-\infty}^\infty e^{-\frac{1}{4}s^2(t-t')^2} F(t')\, dt'\, ds$$

where

$$F(t) = \alpha\beta\sum_{m,n}\omega_n d_{m,n}\left(g_{m/\beta,n/\alpha}\right)^\vee(t)$$

$$= \alpha\beta\sum_{m,n}\omega_n d_{m,n}\; e^{i\ n/\alpha(x-2\pi\ m/\beta)} h\left(x - 2\pi\frac{m}{\beta}\right)\ .$$

Figure 3 illustrates this example. Figures 3a and 3b show h and the absolute value of its Fourier transform, $|g|$; Figures 3c and 3d plot the dual function H and the absolute value of its Fourier transform, $|\tilde{g}_\Lambda|$. For comparison, Figures 3e and 3f show the dual function $\tilde{h} = h^\#$ with minimal L^2-norm and the absolute value of its Fourier transform; clearly $\tilde{h}$ has sharper transitions than H.

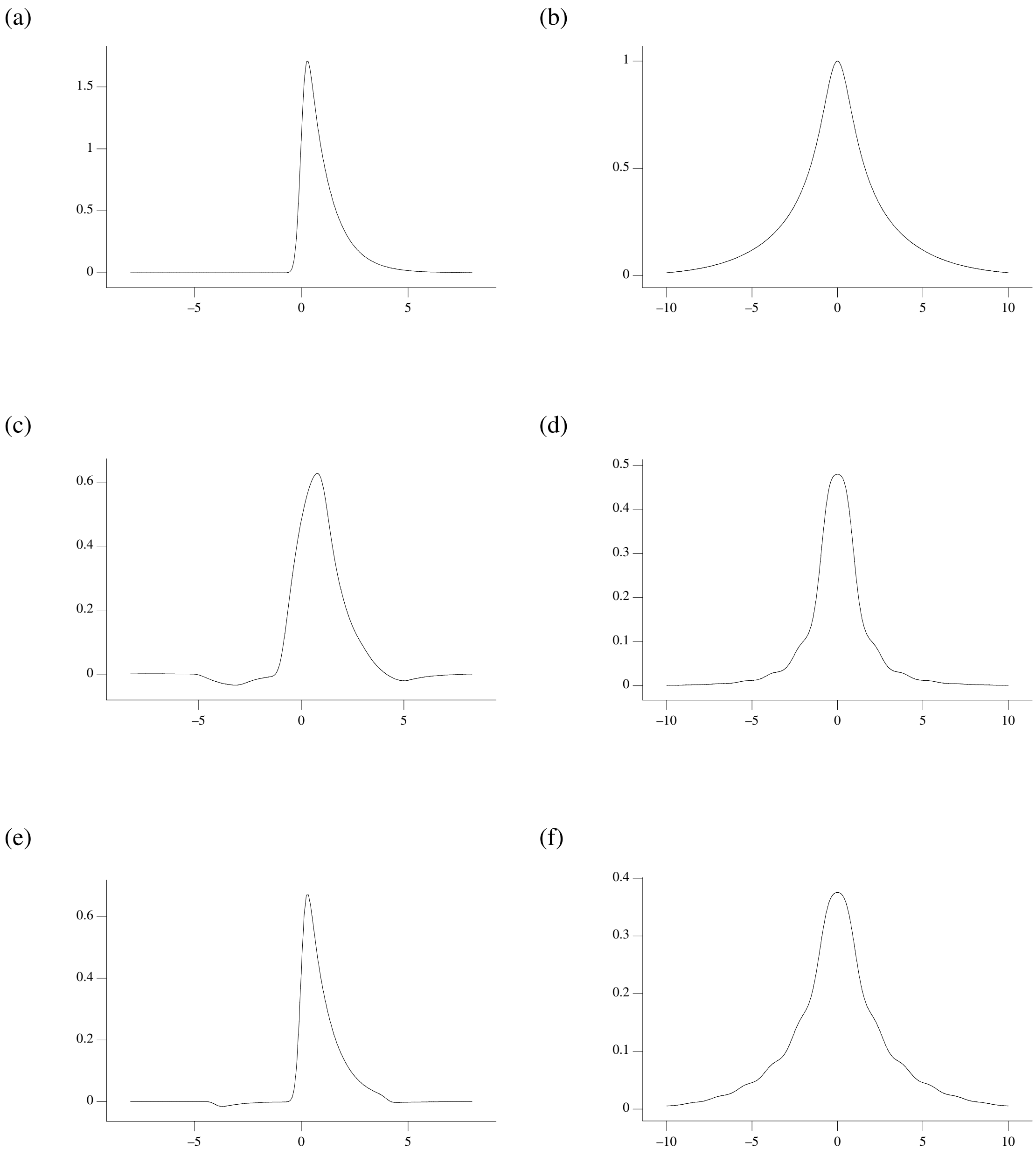


FIGURE 3

(a) The function h and (b) the absolute value $|\hat{h}|$ of its Fourier transform; (c) $(\alpha\beta)^{-1}H$ and (d) $(\alpha\beta)^{-1}|\hat{H}|$, where H is the dual function that minimizes $\|(1-\Delta)^{1/4}H\|$; (e) $(\alpha\beta)^{-1}\tilde{h}$ and (f) $(\alpha\beta)^{-1}|\widehat{\tilde{h}}|$, where $\tilde{h} = h^{\#}$ is the standard, L^2-minimal dual function.

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